Symmetries of the massless Dirac equation in Minkowski space

Stéphane Durand, Jean-Marc Lina, and Luc Vinet
Laboratoire de Physique Nucléaire, Université de Montréal, C.P. 6128, Succ. A, Montréal, Québec, Canada H3C 3J7
(Received 31 August 1988)

All symmetries of the massless Dirac equation in Minkowski space are obtained by explicitly solving the conformal versions of the Killing, Penrose-Floyd, and Yano equations. In an arbitrary curved four-space, it is also shown that the dual of a conformal Yano tensor is a conformal Killing vector and that the set of conformal Penrose-Floyd tensors is stable under the dual map.

The symmetries of the massless Dirac equation in an arbitrary curved background have recently been investigated by Kamran and McLenaghan.[1] They found that the first-order differential operators which transform solutions into solutions can be expressed in terms of one vector B and of two completely antisymmetric tensors, D of rank 2 and E of rank 3, which, respectively, satisfy the conformal versions of the Killing, Penrose-Floyd, and Yano equations.[2] The purpose of this paper is to supplement their analysis with an explicit and complete solution of these equations in Minkowski space.

Let M_4 denote a four-dimensional pseudo-Riemannian manifold endowed with a metric g. The Dirac operator on M_4 is given by

$$\Delta = i\gamma^\mu \mathcal{D}_\mu \ , \tag{1}$$

where $\mathcal{D}_\mu = \partial_\mu + \Gamma_\mu$ stands for the Lorentz-covariant derivative. The γ matrices satisfy as usual $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$. We shall also use $\gamma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$ and $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$. By a symmetry of the massless Dirac equation we mean a first-order differential operator $\mathcal{K}$ satisfying

$$[\mathcal{K}, \Delta] = \lambda\Delta \tag{2}$$

for some arbitrary λ. It has been shown in Ref. 1 that the most general $\mathcal{K}$ can be written in the form

$$\mathcal{K} = \alpha(x)\Delta + F^\alpha(x)\mathcal{D}_\alpha + G(x) \ , \tag{3}$$

where

$$F^\alpha(x) = B^\alpha(x) + iD^\alpha{}_\beta(x)\gamma_5\gamma^\beta + E^\alpha{}_{\beta\gamma}(x)\gamma^{\beta\gamma} \ , \tag{4a}$$

$$G(x) = lP_L + rP_R + \tfrac{3}{8}B^\alpha{}_{;\alpha} - \frac{3i}{4}\,{}^*E^\alpha{}_{;\alpha}\gamma_5 + \tfrac{1}{3}\,{}^*D^\alpha{}_{\mu;\alpha}\gamma^\mu + \frac{i}{3}D^\alpha{}_{\mu;\alpha}\gamma_5\gamma^\mu + \tfrac{1}{4}(E^\alpha{}_{\mu\nu;\alpha} - B_{\mu;\nu})\gamma^{\mu\nu} \tag{4b}$$

with $P_{R,L} = (1 \pm \gamma_5)/2$, $\alpha(x)$ an arbitrary function, and l and r arbitrary constants. The dual tensors ${}^*D$ and ${}^*E$ are defined as usual with the help of the reparametrization-invariant completely antisymmetric ϵ symbol: ${}^*D_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}D^{\alpha\beta}$ and ${}^*E_\mu = \frac{1}{6}\epsilon_{\mu\alpha\beta\gamma}E^{\alpha\beta\gamma}$. For $\mathcal{K}$ to be a symmetry of the massless Dirac equation, the vectors B_μ and the completely antisymmetric tensors $D_{\mu\nu}$ and $E_{\mu\nu\rho}$ must satisfy

$$B_{(\mu;\nu)} = \tfrac{1}{4}g_{\mu\nu}B^\alpha{}_{;\alpha} \quad \text{(conformal Killing)} \ , \tag{5a}$$

$$D_{\mu(\nu;\rho)} = -\tfrac{1}{3}D^\alpha{}_{\mu;\alpha}g_{\nu\rho} + \tfrac{1}{3}D^\alpha{}_{(\nu;|\alpha|}g_{\rho)\mu} \quad \text{(conformal Penrose-Floyd)} \ , \tag{5b}$$

$$E_{\mu\nu(\rho;\sigma)} = \tfrac{1}{2}E^\alpha{}_{\mu\nu;\alpha}g_{\rho\sigma} + \tfrac{1}{2}E_{\nu(\rho|}{}^\alpha{}_{;\alpha|}g_{\sigma)\mu} - \tfrac{1}{2}E_{\mu(\rho|}{}^\alpha{}_{;\alpha|}g_{\sigma)\nu} \quad \text{(conformal Yano)} \ . \tag{5c}$$

We have kept the conventions of Ref. 1 according to which indices between curly brackets and outside double bars are to be symmetrized. In the following, it will also be understood that indices between square brackets are to be antisymmetrized.

The corresponding expression for λ in $[\mathcal{K}, \Delta] = \lambda\Delta$ is given by

$$\lambda = -\tfrac{1}{4}B^\mu{}_{;\mu} + (r - l)\gamma_5 - i\alpha_{,\mu}\gamma^\mu - \frac{i}{3}D^\mu{}_{\nu;\mu}\gamma_5\gamma^\nu - \tfrac{1}{2}E_{\mu\nu}{}^\rho{}_{;\rho}\gamma^{\mu\nu} \ . \tag{6}$$

The symmetries of the massive Dirac equation are similarly obtained by analyzing the condition $[\mathcal{K}, \Delta] = \lambda(\Delta + m)$. It is easily shown that this is tantamount to solving $[\mathcal{K}, \Delta] = 0$, a special case of (2). The corresponding conditions on B, D, and E are obtained by setting the right-hand sides of Eqs. (5) equal to zero; they thus read

$$B_{(\mu;\nu)} = 0, \quad D_{\mu(\nu;\rho)} = 0, \quad E_{\mu\nu(\rho;\sigma)} = 0 \ . \tag{7}$$

In the following we shall provide a complete solution of Eqs. (5) when g is the Lorentzian flat metric. The solutions of Eqs. (7) have been constructed in Ref. 3 for this choice of metric: ten Killing vectors, ten Penrose-Floyd tensors, and five Yano tensors were found. In solving Eqs. (5) we shall, of course, recover these generators but shall find, in addition, the conformal symmetries that are specific to the massless equation.[4] All in all we shall have the 15 well-known conformal Killing vectors, 20 conformal Penrose-Floyd tensors, and 15 conformal Yano tensors. If one includes the identity and γ_5 this gives a set of 52 symmetries (of first order in the derivatives) for the massless Dirac equation in Minkowski space.

Let us first enunciate two simple lemmas that prove useful in solving Eqs. (5). Note that these lemmas are valid in *general curved-space backgrounds*.

Lemma 1. The tensor $E_{\mu\nu\rho}$ obeys the conformal Yano equation (5c) if and only if its dual ${}^*E_\mu$ is a conformal Killing vector.

(A special case of this result was derived in Ref. 6.) The proof goes as follows. Contract the left-hand side of (5c) with $\epsilon^{\mu\nu\rho\kappa}$ to obtain

$$\epsilon^{\mu\nu\rho\kappa}E_{\mu\nu(\rho;\sigma)} = -4\,{}^*E^{\kappa}{}_{;\sigma} + \delta^{\kappa}_{\sigma}\,{}^*E^{\mu}{}_{;\mu} \ .$$

Perform the same operation on the right-hand side of (5c) to find $-2\,{}^*E^{\kappa}{}_{;\sigma}+2\,{}^*E_{\sigma}{}^{;\kappa}$. Equate the two results and get

$${}^*E_{(\mu;\nu)} = \tfrac{1}{4}g_{\mu\nu}\,{}^*E^{\alpha}{}_{;\alpha}$$

showing that ${}^*E$ is a conformal Killing vector. The converse is proven by reversing all the steps. In the same way, one can prove the following.

Lemma 2. $D_{\mu\nu}$ is a conformal Penrose-Floyd tensor if and only if its dual ${}^*D_{\mu\nu}$ is a conformal Penrose-Floyd tensor.

This implies in particular that ${}^*D_{\mu\nu}$ will generally satisfy (5b) even if $D_{\mu\nu}$ verifies only $D_{\mu(\nu;\rho)}=0$.

We now come to solving the generalized conformal Killing equations (5). Before going to flat Minkowski space, let us use lemma 1 to determine the E contribution to $\mathcal{K}$ in any curved-space background. Set $D=0$ in $\mathcal{K}$, let ξ and ζ be two conformal Killing vectors, and write B and ${}^*E$, conformal Killing vectors themselves, as $B=(i/2)(\zeta+\xi)$ and ${}^*E=\frac{1}{4}(\zeta-\xi)$. We then obtain for $\mathcal{K}$ the expression

$$\mathcal{K}_{(D=0)} = (B^{\alpha}-2i\,{}^*E_{\mu}\gamma_5\gamma^{\alpha\mu})\mathcal{D}_{\alpha} + \tfrac{3}{8}(B^{\alpha}{}_{;\alpha}-2i\,{}^*E^{\alpha}{}_{;\alpha}\gamma_5)$$
$$-\tfrac{1}{4}(B_{\nu;\mu}-2i\,{}^*E_{\nu;\mu}\gamma_5)\gamma^{\nu\mu} \qquad (8a)$$

which can be rewritten (using $\gamma^{\alpha\mu}\mathcal{D}_{\alpha}=-i\gamma^{\mu}\Delta+\mathcal{D}^{\mu}$) as

$$\mathcal{K}_{(D=0)} = iP_L\mathcal{L}_{\xi}+iP_R\mathcal{L}_{\zeta}+f(x)\Delta \ , \qquad (8b)$$

where $f(x)=-2\,{}^*E_{\mu}\gamma_5\gamma^{\mu}$ and $\mathcal{L}_X$ is the Lie derivative along X acting on spinors:

$$\mathcal{L}_X = X\cdot\mathcal{D}+\tfrac{3}{8}X^{\mu}{}_{;\mu}-\tfrac{1}{4}X_{\alpha;\beta}\gamma^{\alpha\beta} \ . \qquad (9)$$

We therefore see that the Yano and Killing terms (E and B) yield two independent sets of conformal symmetries acting, respectively, on the spaces of right-handed and left-handed massless fermions.

Let us finally focus on the Penrose-Floyd contributions to $\mathcal{K}$. Note that these do not preserve chirality. Here we shall restrict ourselves to Minkowski space. Let us point out that one could start from the known solutions to $D_{\mu(\nu,\rho)}=0$ (Ref. 3), namely,

$$D_{\mu\nu}=a_{[\mu\nu]}+\epsilon_{\mu\nu\rho\sigma}b^{\rho}x^{\sigma} \ , \qquad (10)$$

and use the dual map (see lemma 2) to generate more solutions to the conformal Penrose-Floyd equation (5b). Some solutions will be specific to the massless case since the divergence ${}^*D^{\mu}{}_{\alpha,\mu}$ will be equal to some constant parameters, say c_{α}. More generally, we can show[7] from (5b) that in Minkowski space $D^{\alpha}{}_{\mu,\alpha}$ satisfies the Killing equation

$$D^{\alpha}{}_{(\mu,\nu),\alpha}=0 \ . \qquad (11)$$

The general expression for the divergence of the conformal Penrose-Floyd tensor is thus

$$D^{\alpha}{}_{\mu,\alpha} = -\tfrac{3}{2}c_{\mu}+\tfrac{3}{2}d_{[\mu\nu]}x^{\nu} \ , \qquad (12)$$

where the c_{μ} and the antisymmetric $d_{\mu\nu}$ are constant. The numerical factors have been introduced for convenience. The general solution of (5b) is then found to be the following 20-parameter tensor:

$$D_{\mu\nu}=a_{[\mu\nu]}+\epsilon_{\mu\nu\rho\sigma}b^{\rho}x^{\sigma}+c_{[\mu}x_{\nu]}$$
$$+(x^{\rho}d_{\rho[\mu}x_{\nu]}+\tfrac{1}{4}x^2d_{[\mu\nu]}) \ . \qquad (13)$$

In addition to the symmetries $(a_{[\mu\nu]},b_{\mu})$ that are present in the massive case, it leads to ten new generators; those associated with the parameters c_{μ} are easily seen (using the equation of motion) to be equal to γ_5 times the operators corresponding to the parameters b_{μ}, while

$$\mathcal{K}_{(d_{[\mu\nu]})}=\gamma_5[\gamma^{\rho}x_{\rho}x_{[\mu}\partial_{\nu]}-\tfrac{1}{2}x^2\gamma_{[\mu}\partial_{\nu]}$$
$$+\gamma_{[\mu}x_{\nu]}(x^{\rho}\partial_{\rho}+1)]+\frac{i}{2}\epsilon_{\mu\nu\rho\sigma}\gamma^{\rho}x^{\sigma} \qquad (14)$$

are the generators that the six $d_{[\mu\nu]}$ give rise to. Of these, three can be expressed as γ_5 times the remaining three. The same is true with the six $\mathcal{K}_{(a_{[\mu\nu]})}$.

Let us record for easy reference the general solutions of Eqs. (5a) and (5c):

$$B_{\mu}=a_{\mu}+\omega_{[\mu\nu]}x^{\nu}+(2k_{\nu}x^{\nu}x_{\mu}-x^2k_{\mu})+dx_{\mu} \ , \qquad (15)$$

$$E_{\mu\nu\rho}=\epsilon_{\mu\nu\rho\sigma}[e^{\sigma}+f^{[\sigma\delta]}x_{\delta}+(2g^{\delta}x_{\delta}x^{\sigma}-x^2g^{\sigma})+hx^{\sigma}] \ . \qquad (16)$$

From the corresponding explicit form of λ [see (6)],

$$\lambda=-(2k_{\mu}x^{\mu}+d)+(r-l)\gamma_5-i\alpha_{,\mu}\gamma^{\mu}$$
$$+\frac{i}{2}(d_{[\mu\nu]}x^{\nu}-c_{\mu})\gamma_5\gamma^{\mu}$$
$$-\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}(4g^{\rho}x^{\sigma}-f^{[\rho\sigma]})\gamma^{\mu\nu} \ , \qquad (17)$$

it is straightforward to determine the symmetries that remain in the presence of a mass term since λ must be equal to zero in this case; the 26 symmetries (including the identity) of the massive Dirac equation in Minkowski space are simply obtained by setting $d=(r-l)=0$, $k_{\mu}=c_{\mu}=g_{\mu}=0$, $\forall\mu$ and $d_{[\mu\nu]}=f_{[\mu\nu]}=0$, $\forall\mu,\nu$ in our expressions. (Note that the function α is irrelevant.)

In concluding let us say that the results presented here should be relevant to the classification of the separable coordinate systems for the massless Dirac equation.[8] Let us also mention that symmetries of Dirac equations can be used to identify constants of motion for nonrelativistic supersymmetric quantum Hamiltonians.[9]

We want to thank Haris Panagopoulos and Martin Légaré for useful discussions. One of us (J.-M.L.) is grateful to the Laboratoire de Physique Nucléaire and the Faculté des Etudes Supérieures de l'Université de Montréal for financial support. The other (S.D.) acknowledges the financial support of the Natural Sciences and Engineering Research Council (NSERC) of Canada and the Fonds Formation de Chercheurs et l'Adie a la Recherche of the Ministère de l'Education du Québec.

---

[1]N. Kamran and R. G. McLenaghan, Phys. Rev. D **30**, 357 (1984); Lett. Math. Phys. **7**, 381 (1983).

[2]K. Yano and S. Bochner, *Curvature and Betti Numbers* (Annals of Mathematics Studies No. 32) (Princeton University Press, Princeton, NJ, 1953); S. R. Penrose, Ann. N. Y. Acad. Sci. **224**, 125 (1973). See also B. Carter, Phys. Rev. D **16**, 3395 (1977); B. Carter and R. G. McLenaghan, *ibid.* **19**, 1093 (1979).

[3]R. G. McLenaghan and P. Spindel, Bull. Soc. Math. Belg. **31**, 65 (1979); R. G. McLenaghan and P. Spindel, Phys. Rev. D **20**, 409 (1979).

[4]The authors of Ref. 5 also claim to have all these symmetries. They have not however explicitly obtained the six generators given in Eq. (14). Let us also point out that their approach significantly differs from ours in that they use a noncovariant formalism.

[5]W. I. Fushchich and A. G. Nikitin, J. Phys. A **20**, 537 (1987).

[6]P. Spindel, Gen. Relativ. Gravit. **11**, 419 (1979).

[7]This argument was suggested to us by H. Panagopoulos.

[8]N. Kamran, M. Légaré, R. G. McLenaghan, and P. Winternitz, J. Math. Phys. **29**, 403 (1988).

[9]S. Durand, J.-M. Lina, and L. Vinet, UdeM-LPN-TH02, 1988 (unpublished).