Two-photon-exchange force between charged systems: Spinless particles

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We use the methods of dispersion theory to extend to the case of two charged particles earlier studies of two-photon-exchange forces between neutral particles and between a neutral and a charged particle. The two-photon-exchange amplitude for two charged particles is infrared divergent but when we subtract the amplitude obtained by iteration of the one-photon-exchange potential, which is necessary to avoid double counting, this divergence is canceled. The difference amplitude, regarded as an analytic function of the invariant squared momentum transfer t has a discontinuity which is less singular than t^{-1} for small t. We are thereby led to a convergent expression for the fourth-order potential $V^{(4)}$, given by the Laplace transform of this discontinuity. We give explicit gauge-invariant results for the long-range two-photon-exchange potential in the case of two spin-zero particles.

I. INTRODUCTION

The long-range potential between two systems that is induced by the exchange of photons between them can be effectively analyzed by the use of the methods of dispersion theory. These methods have been applied to the study of the two-photon-exchange potential between two neutral systems[1] and between a neutral and a charged system.[2] They have been used both to obtain a clearer insight into old results and to obtain new ones.

The use of the dispersion-theory approach to long-range forces has several advantages over other methods: (i) it is independent of any assumptions about the structure of the interacting systems, such as a nonrelativistic approximation for the internal wave functions; (ii) it avoids the calculation of ultraviolet-divergent integrals, which must then be carefully subtracted to obtain a finite potential; (iii) it allows the potential to be directly expressed in terms of observable quantities, the amplitudes for the scattering of photons by the two systems.

In this paper we continue the program of studying two-photon-exchange potentials $V_{2\gamma}$ by considering the two-photon-exchange corrections to the one-photon-exchange potential $V_{1\gamma}$ acting between two charges. For simplicity we consider point charges in this paper but, by combining the present results with our previous ones, it is straightforward to describe the potential acting between extended charges. We also confine our study to charges with zero spin; results for charges with spin $\frac{1}{2}$ will be given in a subsequent paper. Our formalism leads to results for the two-photon-exchange potential which are independent of the choice of gauge.

In the dispersion-theory approach the potential $V_{2\gamma}$ is expressed as a Laplace transform of the discontinuity in t of two-photon-exchange amplitude $M_{2\gamma}$; here t is the negative of the square of the four-momentum transfer but taken to be positive and hence in an unphysical region. It is not difficult to calculate this discontinuity for the case of two point charges, but once this is done two problems arise which did not occur for the cases that have been treated in earlier work.

(i) The t discontinuity of $M_{2\gamma}$ behaves as t^{-1} for small t. This signals the fact that $M_{2\gamma}$ is infrared (IR) divergent and hence the same would be true for the potential, if it were naively calculated directly from the discontinuity of $M_{2\gamma}$. The IR divergence of $M_{2\gamma}$ is well known from perturbation theory. If a nonzero photon mass μ is introduced to control the divergence, the imaginary part of the amplitude is found to behave as $\ln\mu$ for small μ. This divergence is unrelated to the emission of soft photons, but is instead a manifestation of the infinite phase in the scattering wave function that arises from the long-range character of the Coulomb interaction.

(ii) Unlike the case where one or both of the particles are neutral, the one-photon-exchange potential $V_{1\gamma}$ between two charged particles itself has a long-range part, the Coulomb interaction, to which two-photon exchange gives a correction. When the full two-photon-exchange amplitude $M_{2\gamma}$ is calculated by the dispersion theory method, or any other method for that matter, it includes an iteration piece M_I which corresponds to the contribution of $V_{1\gamma}$ in second-order time-independent perturbation theory. Furthermore, M_I also has an imaginary part which varies as $\ln\mu$. It would be incorrect to calculate a potential from $M_{2\gamma}$ and then add this to $V_{1\gamma}$ to obtain an improved potential to be used in a Schrödinger-type equation. Such a procedure would involve double counting of the effect of $V_{1\gamma}$ in all orders of perturbation theory beyond the first.

We see that for the purpose of obtaining a potential which may be used in a Schrödinger equation, for the study of bound states or other nonperturbative properties, it is necessary to subtract the second-order effect M_I of $V_{1\gamma}$ from $M_{2\gamma}$ and so to define an "irreducible" two-photon-exchange amplitude, $M_{2\gamma}^{\rm irr}=M_{2\gamma}-M_I$, from which a potential can be computed by the method outlined above. It turns out, providentially, that $M_{2\gamma}^{\rm irr}$ is infrared finite and so allows for the definition of a finite, meaningful potential $V_{2\gamma}$. That is, when we include a photon mass μ in order to define all amplitudes unambiguously and define $M_{2\gamma}^{\rm irr}$ by a subtraction with nonzero photon mass then both $M_{2\gamma}^{\rm irr}$ and $V_{2\gamma}$ remain finite even when μ is allowed to approach zero.

It is now possible to use $V=V_{1\gamma}+V_{2\gamma}$ as the potential in a relativistic Schrödinger equation. The scattering solutions to this equation will have several noteworthy features: (i) when computed to order e^4, they exactly reproduce the amplitude $M_{2\gamma}$; (ii) in each order beyond e^2, the amplitudes obtained from these solutions display the familiar IR divergence of any problem involving the scattering of two charged particles.[3] This means that when the scattering amplitude is calculated to all orders in e^2, it will contain a phase factor $e^{i\theta}$ with a phase θ proportional to $\ln\mu$, if a cutoff μ is introduced. This factor multiplies a finite-scattering amplitude which will include the effects of both one-photon and "irreducible" two-photon exchanges.

We now give an outline of the remainder of this paper. In Sec. II we deal with the precise definition of the potential we wish to calculate, by relating the field-theory amplitudes to those obtained from a relativistic Schrödinger-type equation, and define the lowest-order potential, both for a model field theory involving the Yukawa coupling of two-complex spin-0 fields to a neutral spin-0 field of mass μ and for scalar quantum electrodynamics (QED). We also review the dispersion theory approach to the calculation of potentials. In Sec. III we carry out the program sketched above for the scalar Yukawa theory. In Sec. IV we extend the calculations to scalar QED. The final Sec. V contains a summary and further discussion as well as comparison of our work with that of others. A number of related topics are treated in Appendixes.

II. POTENTIALS IN FIELD THEORY

In ordinary, nonrelativistic quantum mechanics there are several contexts in which it is useful to extract a potential from a scattering amplitude. For example, if the amplitude has been measured and fitted with a potential, this potential can be used in a Schrödinger equation to study possible bound states of the scattering particles. For composite systems such as atoms or molecules, where the interactions between the constituents may be regarded as known, an effective potential may be obtained, for example, from an approximately calculated scattering amplitude; the potential can then be used in a two-body Schrödinger equation to study aspects of the scattering that go beyond those manifest in the approximate amplitude from which the potential was obtained.

One would like to make similar uses of potentials for relativistic systems in the context of quantum field theory, where the quantity that is easiest to calculate is the scattering amplitude, which can be obtained in some approximation by Feynman-graph techniques. However, several problems arise in the use of potentials for such systems. One must decide on the equation in which such a potential is to be used. For many purposes an ordinary three-dimensional equation, analogous to the nonrelativistic Schrödinger equation, is the preferred way to treat such particles, and that is the approach we shall adopt. A second problem is how to express the potential for such an equation in terms of the scattering amplitude obtained from Feynman diagrams. Part of the problem is that, as indicated above, when the potential is expressed as a power series in the coupling constant it is necessary to remove the effect of iterating lower-order potentials when extracting higher-order potentials from higher-order scattering amplitudes. Another problematic aspect of extracting potentials from scattering amplitudes is that the matrix elements of the former are needed for all values of the momenta, while the latter are most conveniently obtained on the mass shell, especially in gauge theories such as QED.

In this section we discuss these problems of defining a relativistic potential in general terms. Much of the material described in this section is not new, but it is presented here in a form that is convenient for our purpose. Our treatment relies heavily on the use of dispersion relations, both as a calculational tool for the extraction of potentials from scattering amplitudes, and as a means of resolving some of the ambiguities mentioned above. Therefore, we review that approach, with special emphasis on long-range forces. After dealing with some kinematical preliminaries, we discuss the general concept of a two-body potential within the context of quantum field theory. Then we show how the dispersion approach can be used to construct potentials in each order of perturbation theory, given the scattering amplitude in that order and in lower orders. We conclude this section by calculating the second-order potentials for scalar Yukawa theory and for scalar electrodynamics.

A. Kinematical preliminaries

In any relativistic quantum field theory the S-matrix element S_{fi} for a transition from an initial state $|i\rangle$ to a final state $|f\rangle$ has the form

$$S_{fi}=\delta(f,i)-(2\pi)^4 i\delta(P_f-P_i)T_{fi} \ , \tag{2.1a}$$

where the P's denote total initial and final four-momenta. The quantity T_{fi} is the transition amplitude, related to the invariant Feynman amplitude M_{fi} by

$$T_{fi}=N_f M_{fi} N_i \ , \tag{2.1b}$$

where the N's are kinematical factors whose value depends on the normalization of one-particle states. We shall restrict our attention in this paper to the scattering of spin-0 particles. We denote by $|\mathbf{p}\rangle$ a spin-0 one-particle state of three-momentum $\mathbf{p}$, normalized according to

$$\langle \mathbf{p}' | \mathbf{p} \rangle = (2\pi)^3 \delta(\mathbf{p}' - \mathbf{p}) . \tag{2.2}$$

There is then a factor $(2E)^{-1/2}$ in T_{fi} for a particle of energy E present in either the initial or final state. For the case of the elastic scattering of particles A and B with initial three-momenta $\mathbf{p}_A$ and $\mathbf{p}_B$ and final three-momenta $\mathbf{p}'_A$ and $\mathbf{p}'_B$, respectively, symbolized by

$$A + B \rightarrow A' + B' , \tag{2.3}$$

the values of the N's are then

$$N_i = (4E_A E_B)^{-1/2}, \quad N_f = (4E'_A E'_B)^{-1/2} . \tag{2.4a}$$

Here the E's are one-particle energies,

$$E_A = (\mathbf{p}_A^2 + m_A^2)^{1/2}, \quad E_B = (\mathbf{p}_B^2 + m_B^2)^{1/2} , \tag{2.4b}$$

etc., with the m's denoting the particle masses. We denote the initial and final four-momenta by p_A, p_B, and p'_A, p'_B, respectively. We also define, partly for later use, the usual invariants s, t, and u by

$$s = (p_A + p_B)^2, \quad t = Q^2, \quad u = (p_A - p'_B)^2 , \tag{2.4c}$$

with

$$Q = p_A - p'_A = -p_B + p'_B , \tag{2.4d}$$

and recall that

$$s + t + u = 2m_A^2 + 2m_B^2 . \tag{2.4e}$$

In the c.m. system we write

$$p_A = (E_A, \mathbf{p}), \quad p_B = (E_B, -\mathbf{p}) , \tag{2.5a}$$

and

$$p'_A = (E'_A, \mathbf{p}'), \quad p'_B = (E'_B, -\mathbf{p}') , \tag{2.5b}$$

where $|\mathbf{p}'| = |\mathbf{p}|$ for elastic scattering and now

$$E_A = E'_A = (\mathbf{p}^2 + m_A^2)^{1/2}, \quad E_B = E'_B = (\mathbf{p}^2 + m_B^2)^{1/2} . \tag{2.5c}$$

In this system

$$s = W^2 , \tag{2.5d}$$

where $W = E_A + E_B$, and Q takes the form

$$Q = (0, \mathbf{Q}) , \tag{2.5e}$$

with $\mathbf{Q}$ the three-momentum transfer

$$\mathbf{Q} = \mathbf{p} - \mathbf{p}' , \tag{2.5f}$$

so that

$$t = -\mathbf{Q}^2 . \tag{2.5g}$$

B. Concept of two-body potential

We now seek to define a potential operator V with the following property. When V is added to an operator h_0 describing the free propagation of A and B, and a Schrödinger-type equation of the form

$$h\phi = E\phi , \tag{2.6a}$$

with

$$h = h_0 + V , \tag{2.6b}$$

is solved for scattering states, the resulting potential-theory transition amplitude T_{fi}^{pot} will coincide with the field-theory transition amplitude T_{fi}. Two-body equations of this and similar types have been used over the years by many previous authors in various contexts.[4]

We will restrict our attention to the c.m. system of the reaction (2.3) and make the natural choice

$$h_0 = E_A^{\text{op}} + E_B^{\text{op}} , \tag{2.7a}$$

where

$$E_A^{\text{op}} = (\mathbf{p}_{\text{op}}^2 + m_A^2)^{1/2}, \quad E_B^{\text{op}} = (\mathbf{p}_{\text{op}}^2 + m_B^2)^{1/2} , \tag{2.7b}$$

and $\mathbf{p}_{\text{op}}$ denotes the operator whose eigenvalues give the momentum of A in the c.m. system. We make this choice in order to follow as closely as possible the description of two-particle systems which is standard in nonrelativistic quantum mechanics, without, however, making any nonrelativistic approximations; at the same time we do not wish to allow the appearance of negative energies at the zeroth-order level, i.e., in the absence of interaction between the particles. As a result of this choice, pair effects will show up only in the calculation of V rather than explicitly in the solution of (2.6a). For A and B both spin-0 particles, the wave function ϕ in (2.6a) is just a function of the relative coordinate $\mathbf{r}$ in $\mathbf{r}$ space or of the relative momentum $\mathbf{p}$ in $\mathbf{p}$ space.

We shall refer to the operator V as the "potential," with the understanding that in general it may be nonlocal and energy dependent. Also, if the energy is high enough to create new particles, V may be non-Hermitian. In the c.m. system, the amplitude T_{fi}^{pot} for a transition from an initial product plane-wave state ϕ_i, with three-momentum $\mathbf{p}$ and $-\mathbf{p}$ for A and B, respectively, to a final state ϕ_f, with momenta $\mathbf{p}'$ and $-\mathbf{p}'$, generated by this potential is given by

$$T_{fi}^{\text{pot}} = \langle \mathbf{p}' | V + V(W - h + i\epsilon)^{-1} V | \mathbf{p} \rangle , \tag{2.8}$$

where $|\mathbf{p}\rangle$ and $|\mathbf{p}'\rangle$ denote the initial and final states, respectively, and W denotes the total energy in the c.m. system:

$$W = E_A(\mathbf{p}) + E_B(\mathbf{p}) = E_A(\mathbf{p}') + E_B(\mathbf{p}') . \tag{2.9}$$

Our requirement on V then becomes

$$T_{fi}^{\text{pot}} = T_{fi} , \tag{2.10a}$$

where T_{fi} is also to be evaluated in the c.m. system. In view of (2.1b), (2.4a), and (2.5c) we have, in that system,

$$T_{fi} = [4E_A(\mathbf{p}) E_B(\mathbf{p})]^{-1} M_{fi} . \tag{2.10b}$$

In this paper we deal only with Feynman graphs involving the exchange of one or more quanta between A and B; i.e., we do not consider radiative corrections of the vertex or self-energy type. Thus the graphs we include are proportional to an even power of a mean coupling strength g, defined by

$$g^2 = g_A g_B \ , \tag{2.11}$$

where g_A and g_B denote the coupling strength for the emission of a single quantum by A and B, respectively [a minus sign is to be understood on the right-hand side of (2.11) if g_A and g_B have opposite sign]. It follows that M_{fi} has the form

$$M_{fi} = M_{fi}^{(2)} + M_{fi}^{(4)} + \cdots \ , \tag{2.12}$$

where the first term is of order g^2, the second of order g^4, and so on. We will assume that V and $T_{fi}^{\rm pot}$ can also be expanded in a power series in g^2,

$$V = V^{(2)} + V^{(4)} + \cdots \tag{2.13}$$

and

$$T_{fi}^{\rm pot} = T_{fi}^{\rm pot(2)} + T_{fi}^{\rm pot(4)} + \cdots \ . \tag{2.14}$$

It follows from (2.8) and (2.13) that

$$T_{fi}^{\rm pot(2)} = \langle \mathbf{p}' | V^{(2)} | \mathbf{p} \rangle \ , \tag{2.15a}$$

$$T_{fi}^{\rm pot(4)} = \langle \mathbf{p}' | V^{(2)} (W - h_0 + i\epsilon)^{-1} V^{(2)} | \mathbf{p} \rangle + \langle \mathbf{p}' | V^{(4)} | \mathbf{p} \rangle \ , \tag{2.15b}$$

and so on. The requirement (2.10) then reduces to a sequence of equations:

$$\langle \mathbf{p}' | V^{(2)} | \mathbf{p} \rangle = (4E_A E_B)^{-1} M_{fi}^{(2)} \ , \tag{2.16a}$$

$$\langle \mathbf{p}' | V^{(4)} | \mathbf{p} \rangle = (4E_A E_B)^{-1} M_{fi}^{(4)} - \langle \mathbf{p}' | V^{(2)} (W - h_0 + i\epsilon)^{-1} V^{(2)} | \mathbf{p} \rangle \ , \tag{2.16b}$$

and so on.

These equations serve to define $V^{(n)}$, but only schematically. One calculates the quantity $T_{fi}^{(2)}$ from field theory and uses this to calculate $V^{(2)}$. This $V^{(2)}$ is used to compute the "iteration term" on the right-hand side (RHS) of Eq. (2.15b) and this term is then subtracted from the field-theory amplitude $T_{fi}^{(4)}$; the result is used to compute $V^{(4)}$. The process may be continued indefinitely.

However, several problems arise in determining $V^{(n)}$ by this process. Consider Eq. (2.16a), which essentially equates the Fourier transform of $V^{(2)}$ to a scattering amplitude obtained from field-theory graphs. In order to extract the operator $V^{(2)}$ from this, it is necessary to invert the Fourier transform. However, this requires knowing $T_{fi}^{(2)}$ for all values of the three-momenta, whereas the scattering amplitude is given only on the energy shell, for $\mathbf{p}^2 = \mathbf{p}'^2$. It is not convenient to use the scattering amplitude off the energy shell directly, as that quantity can be considerably more difficult to calculate and in the case of a gauge theory will in general be gauge dependent.

Another problem, which sometimes arises from $V^{(2)}$ but which always arises for $V^{(4)}$, is that the field-theory scattering amplitudes, as well as the iteration terms, are energy dependent. There are several possible ways of dealing with this energy dependence. The most straightforward, which we adopt here, is to take the energy as a parameter and to allow the potential to depend both on the relative coordinate and this energy parameter. However, we will require that V should be essentially local, so that apart from certain kinematic factors which will be explicitly displayed, V will not depend on the momentum operator. To specify more precisely our procedure for defining the potential we use a method that has often been used in the past and which we next review, the method of dispersion relations.

C. The dispersion-theory approach to potentials

Before proceeding, we note that the energy factors E_A and E_B in equations such as (2.16a) and (2.16b) are of purely kinematical origin. It is convenient to obtain equations in which such factors are replaced by masses, especially for the transition to the nonrelativistic limit. This may be done in a variety of ways. The method most convenient for theories involving the nonderivative coupling of spin-0 particles, such as a scalar Yukawa-type theory, is to introduce a potential operator U related to V via

$$V = y_{\rm op} U y_{\rm op} \ , \tag{2.17a}$$

where

$$y_{\rm op} = y(\mathbf{p}_{\rm op}) \tag{2.17b}$$

and

$$y(\mathbf{p}) = [m_A m_B / E_A(\mathbf{p}) E_B(\mathbf{p})] \ . \tag{2.17c}$$

With an expansion for U analogous to that for V, i.e.,

$$U = U^{(2)} + U^{(4)} + \cdots \ , \tag{2.18}$$

we then have as the counterpart of (2.16a) and (2.16b),

$$\langle \mathbf{p}' | U^{(2)} | \mathbf{p} \rangle = (4m_A m_B)^{-1} M_{fi}^{(2)} \tag{2.19a}$$

and

$$\langle \mathbf{p}' | U^{(4)} | \mathbf{p} \rangle = (4m_A m_B)^{-1} (M_{fi}^{(4)} - M_I) \ , \tag{2.19b}$$

where M_I is an iteration amplitude defined by

$$M_I = (4m_A^2 m_B^2) \langle \mathbf{p}' | U^{(2)} (E_A^{\rm op} E_B^{\rm op})^{-1} \times (W - h_0 + i\epsilon)^{-1} U^{(2)} | \mathbf{p} \rangle \ . \tag{2.20}$$

In order to use equations similar to (2.19a) and (2.19b) in any order n to determine $U^{(n)}$, we introduce an s-dependent but local operator $U^{(n)}(r;s)$ such that

$$\langle \mathbf{p}' | U^{(n)} | \mathbf{p} \rangle = \int d\mathbf{r} \exp(i\mathbf{Q}\cdot\mathbf{r}) U^{(n)}(r;s) \ , \tag{2.21}$$

with $\mathbf{Q} = \mathbf{p} - \mathbf{p}'$. To determine $U^{(n)}(r;s)$ from (2.21), it is necessary that the left-hand side of this equation be defined for all values of $\mathbf{Q}$. However, constraints such as (2.19a) and (2.19b) only define the left-hand side of (2.21) in the physical region $0 < \mathbf{Q}^2 < 4p^2$. We now use the fact that the amplitudes $M^{(n)}(s,t)$ and $M_I^{(n)}(s,t)$ are analytic functions of t, so that their domain of definition may be extended outside the physical region by analytic continuation and hence to all values of $\mathbf{Q}^2=-t$; we then require that the constraints be satisfied also in this extended range. The potentials $U^{(n)}(r;s)$ are then uniquely determined by inverse Fourier transformation. The use of dispersion relations for the M's allows the U's to be expressed as linear combinations of Laplace transforms of their associated discontinuities.[1]

Let S denote a set of Feynman diagrams for two-particle elastic scattering and let M_S denote the corresponding contribution to the invariant amplitude M. Suppose that, for fixed s, $M_S=M_S(s,t)$ is an analytic function of t, whose only singularities are branch points on the real t axis and which vanishes for large $|t|$. Let t_0 and $\bar{t}_0$ denote the position of the nearest right-hand and left-hand singularities, respectively. We may then write, by virtue of Cauchy's theorem,

$$M_S=M_S^{(R)}+M_S^{(L)} , \tag{2.22}$$

where the first and second terms represent the contribution from the right-hand and left-hand cuts, respectively,

$$M_S^{(R)}=\pi^{-1}\int_{t_0}^{\infty}dt'(t'-t)^{-1}\rho_S(s,t') , \tag{2.23a}$$

$$M_S^{(L)}=\pi^{-1}\int_{-\infty}^{\bar{t}_0}dt'(t'-t)^{-1}\rho_S(s,t') . \tag{2.23b}$$

Here $\rho_S(s,t)$ is the spectral function, defined by

$$\rho_S(s,t)=[M_S]_t/2i , \tag{2.24a}$$

and $[M_S]_t$ is the discontinuity of M_S across the branch cuts,

$$[M_S]_t=M(s,t+i0)-M(s,t-i0) . \tag{2.24b}$$

Corresponding to the decomposition (2.22) of M_S, we also write U_S as a sum of two terms,

$$U_S=U_S^{(R)}+U_S^{(L)} , \tag{2.25}$$

and require that

$$\langle \mathbf{p}'|U_S^{(R)}|\mathbf{p}\rangle=(4m_Am_B)^{-1}M_S^{(R)} , \tag{2.26a}$$

$$\langle \mathbf{p}'|U_S^{(L)}|\mathbf{p}\rangle=(4m_Am_B)^{-1}M_S^{(L)} . \tag{2.26b}$$

We discuss these two equations separately.

1. $U_S^{(R)}$, *the potential from the right-hand cut*

We consider a form for $U_S^{(R)}$ which is local and spherically symmetric but may depend parametrically on s; i.e., we write

$$U_S^{(R)}=U_S^{(R)}(r;s) . \tag{2.27}$$

The left-hand side of (2.26) then becomes a Fourier transform

$$\int d\mathbf{r}\, e^{i\mathbf{Q}\cdot\mathbf{r}}U_S^{(R)}(r;s) ,$$

formally defined for all values of the real vector $\mathbf{Q}$ and hence for *all* negative values of $t=-\mathbf{Q}^2$, not just in the physical region $-4p^2<t<0$. Now $M_S^{(R)}$ is uniquely defined by (2.23a) for all $t<t_0$ and, *a fortiori*, for all negative t. Hence, if we require the equality (2.26a) to hold also outside the physical region this transform may be inverted to give

$$U_S^{(R)}(r;s)=(4m_Am_B)^{-1}(2\pi)^{-3}\int d\mathbf{Q}\, e^{-i\mathbf{Q}\cdot\mathbf{r}}M_S(s,-\mathbf{Q}^2) . \tag{2.28a}$$

Thus, on use of the spectral representation (2.23a) we get

$$U_S^{(R)}(r;s)=(16\pi^2m_Am_Br)^{-1}\int_{t_0}^{\infty}dt\,\rho_S(s,t)e^{-t^{1/2}r} . \tag{2.28b}$$

Equation (2.28b), together with the usual dispersion theory rules for calculating $\rho_S(s,t)$ from Feynman graphs and the recurrence formulas (2.19a) and (2.19b), along with their higher-order analogs, constitutes our rule for defining a potential from field theory. For $t_0=0$, $U_S^{(R)}$ contains all the long-range part coming from S.

So far our discussion has been quite general. For later use it will be convenient to have available the form taken by $U_S^{(R)}$ in the special case where $t_0=0$ and where, furthermore, the function $t^{1/2}\rho_S(s,t)$ can be expanded in a power series in $t^{1/2}$ in a neighborhood of $t=0$. On writing

$$\rho_S(s,t)=a_2(s)t^{-1/2}+a_3(s)+a_4(s)t^{1/2}+\cdots , \tag{2.29a}$$

substituting this form of $\rho_S(s,t)$ into (2.28b), and using the relation

$$\int_0^{\infty}dt\,t^ne^{-t^{1/2}r}=2(2n+1)!r^{-2n-2} , \tag{2.29b}$$

we get a corresponding series for $U_S^{(R)}$:

$$U_S^{(R)}=c_2(s)r^{-2}+c_3(s)r^{-3}+c_4(s)r^{-4}+\cdots , \tag{2.29c}$$

where

$$c_n(s)=(n-2)!a_n(s)/8\pi^2m_Am_B \quad (n=2,3,\ldots) . \tag{2.29d}$$

2. $U_S^{(L)}$, *the potential from the left-hand cut*

The potential arising from the left-hand cut will turn out to be short range and therefore is not of importance for this paper.

The first remark to be made about $U_S^{(L)}$ is that even if it is required to be local and rotationally invariant, extension of the physical region equality (2.26b) to all $t<0$ does not determine $U_S^{(L)}$ uniquely. This is because $M_S^{(L)}$ itself is not uniquely defined for all negative t, only for $t>\bar{t}_0$, the nearest left-hand branch point. For $t<t_0$ we have, of course, the two obvious choices $M_{S;\pm}=M_S(s,t\pm i0)$ to which we may associate potentials $U_{S;\pm}$. A more symmetrical choice is to define a potential $\bar{U}_S^{(L)}$ as the average of these two; this leads to

$$\overline{U}_S^{(L)}(r;s)=(16\pi^2 m_A m_B r)^{-1}\int_{\infty}^{\overline{t}_0} dt\,\rho_S(s,t)\cos[(-t)^{1/2}r]\ . \tag{2.30}$$

The second remark is that for the purposes of this paper we need not be greatly concerned with the question of the "best" choice, because for the theories under consideration, even for zero mass of the quanta, all of these potentials correspond to short-range interactions.

To see this we note, for example, that the amplitude M_L associated with the two-rung ladder graph, shown in Fig. 3(a), has for fixed s, an ordinary threshold singularity at $t=t_0=4\mu^2$ and no left-hand t singularity. But it also has, for fixed t, an ordinary threshold singularity at $s=s_0=(m_A+m_B)^2$. Now the amplitude M_X associated with the crossed-ladder graph [Fig. 3(b)], is obtained by making the replacement $u\to s$ in M_L:

$$M_X(s,t)=M_L(u,t)\ . \tag{2.31}$$

Thus M_X not only has the $t=t_0$ singularity of M_L but also a t singularity at $t=\overline{t}(s)$, where $\overline{t}(s)$ is the value of t such that $u=s_0$, i.e.,

$$\overline{t}_0(s)=2m_A^2+2m_B^2-s-s_0\ . \tag{2.32a}$$

It follows that for $s>s_0$,

$$\overline{t}_0(s)<-4m_A m_B\ . \tag{2.32b}$$

The contribution to $M^{(4)}$ of the discontinuity across the left-hand cut is of the form, with $\rho^{(4)}$ the fourth-order spectral function

$$M_{\text{left}}^{(4)}=(-1/\pi)\int_{-\infty}^{\overline{t}_0} dt'\rho^{(4)}(s,t')/(t'-t)\ . \tag{2.33}$$

For values of $|t|$ which are small compared to $m_A m_B$, the right-hand side of (2.33) varies very slowly with t and its Fourier transform may be approximated by a distribution proportional to $\delta(\mathbf{r})$, which is obviously short range. We will disregard such left-hand cut contributions in what follows.

D. Some second-order potentials

We now consider the second-order potentials which arise from two different field theories. The first is a Yukawa-type theory with an interaction Lagrangian density of the form

$$L_1=-G_A\phi_A^\dagger(x)\phi_A(x)\phi(x)+(A\to B)\ , \tag{2.34}$$

where ϕ_A and ϕ_B denote complex scalar fields and ϕ is a real scalar field, associated with spin-0 particles of mass m_A, m_B, and μ, respectively. We present this model in order to bring out the most important points clearly, with a minimum of complications; we will be mainly interested in the limit of zero value for μ. This will serve as a warm-up for the second theory, scalar QED, which has the complications associated with the vector character of the transmitted quantum. The Lagrangian density for scalar QED is

$$L_2=[-ie_A(\phi_A^\dagger\partial_\mu\phi_A A^\mu-\text{H.c.})+e_A^2\phi_A^\dagger\phi_A A_\mu A^\mu]+[A\to B]\ , \tag{2.35}$$

where now ϕ_A and ϕ_B denote scalar fields associated with spin-0 particles of charge e_A and e_B, respectively, and $A^\mu(x)$ denotes the electromagnetic field. We will study the lowest-order amplitude $M^{(2)}$ for the elastic scattering process (2.3) for each of these theories.

1. Scalar Yukawa theory

The lowest-order Feynman graph, shown in Fig. 1, corresponds to an amplitude $M^{(2)}$ given by

$$M^{(2)}=G^2/(t-\mu^2)\ , \tag{2.36}$$

where G^2 is shorthand for the product of the G's:

$$G^2=G_A G_B\ . \tag{2.37}$$

We also introduce dimensionless coupling constants g_A and g_B via

$$G_A=2m_A g_A,\quad G_B=2m_B g_B\ , \tag{2.38a}$$

and their geometric mean g via

$$g^2=g_A g_B\ , \tag{2.38b}$$

so that

$$G^2=4m_A m_B g^2\ . \tag{2.38c}$$

Then Eq. (2.19a) takes the form

$$\langle \mathbf{p}'|U^{(2)}|\mathbf{p}\rangle=-g^2/(\mathbf{Q}^2+\mu^2)\ . \tag{2.39}$$

A choice for $U^{(2)}$ which is both local in $\mathbf{r}$ space and Hermitian is

$$U^{(2)}(r)=-g^2 e^{-\mu r}/4\pi r\ , \tag{2.40a}$$

which is the familiar static Yukawa potential. With the convention (2.17a) we then have

$$V^{(2)}=y_{\text{op}}U^{(2)}y_{\text{op}}\ . \tag{2.40b}$$

The factors y_{op} thus take into account corrections required by relativistic invariance.

The dispersion-theory approach described in the previous subsections leads (uniquely) to the same result. To

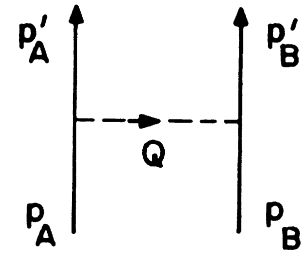

FIG. 1. Lowest-order Feynman graph describing one-meson exchange in scalar Yukawa theory. The solid lines represent spin-0 particles A and B, and the dashed line represents a spin-0 meson of mass μ.

see this, note that the "discontinuity across the pole" of $M^{(2)}$ at $t=\mu^2$ is given by

$$[M^{(2)}]_t = G^2\left[\frac{1}{t+i\epsilon-\mu^2}-\frac{1}{t-i\epsilon-\mu^2}\right]$$
$$=-2\pi i\delta(t-\mu^2)G^2\ , \tag{2.41a}$$

so that the spectral function is

$$\rho^{(2)}=-\pi G^2\delta(t-\mu^2)\ . \tag{2.41b}$$

According to Eq. (2.28b) the associated potential is given by

$$U^{(2)}=(16\pi^2 m_A m_B r)^{-1}(-\pi G^2 e^{-\mu r})$$
$$=-(g^2/4\pi r)e^{-\mu r}\ , \tag{2.41c}$$

in agreement with (2.40).

This takes care of the second-order constraint (2.19a). We will study the problem of finding $U^{(4)}$ from the fourth-order constraint (2.19b) for the Yukawa-type theory in Sec. III.

2. *Scalar QED*

The lowest-order Feynman graph, shown in Fig. 2, corresponds to a contribution

$$M^{(2)}=-e_A e_B P_A\cdot P_B/t\ , \tag{2.42}$$

where

$$P_A=p_A+p_A',\quad P_B=p_B+p_B'\ . \tag{2.43a}$$

Since

$$P_A\cdot P_B=s-u=2a+t\ , \tag{2.43b}$$

where the energy-dependent quantity a is defined by

$$a=a(s)\equiv s-m_A^2-m_B^2=2(\mathbf{p}^2+E_AE_B)\ , \tag{2.43c}$$

Eq. (2.42) has the form

$$M^{(2)}=-e_Ae_B[(2a/t)+1]\ . \tag{2.44}$$

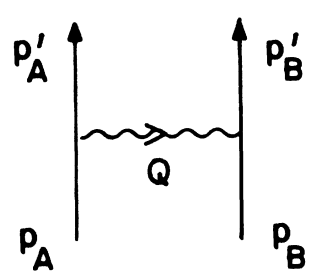

FIG. 2. Lowest-order Feynman graph describing one-photon exchange in scalar QED. The solid lines represent charged spin-0 particles and the wavy line a photon.

We note at once that the problem of defining a potential associated with $M^{(2)}$ is more delicate than that encountered in the scalar Yukawa theory, for two reasons. The first is that although $M^{(2)}$ is again mermorphic in the complex t plane, the residue at the pole, now at $t=0$, now is no longer a constant, independent of s. The second, more significant difference is that $M^{(2)}$ no longer vanishes as $|t|\to\infty$, so that a spectral representation such as (2.23a) is not valid without modifications. Let us first consider the problem from a more familiar point of view, involving Fourier transforms.

We write $V^{(2)}$ as a sum:

$$V^{(2)}=V_a^{(2)}+V_b^{(2)}\ , \tag{2.45}$$

with the first term in (2.45) designed to give the $1/t$ term. The condition (2.19a) then requires that

$$\langle \mathbf{p}'|V_a^{(2)}|\mathbf{p}\rangle=-e_Ae_Bz^2(\mathbf{p})/t \tag{2.46a}$$

and

$$\langle \mathbf{p}'|V_b^{(2)}|\mathbf{p}\rangle=-e_Ae_B/4E_AE_B\ , \tag{2.46b}$$

where

$$z(\mathbf{p})=(a/2E_AE_B)^{1/2}\ . \tag{2.47}$$

In this case we may define energy-independent local potentials $U_a^{(2)}$ and $U_b^{(2)}$ by writing

$$V_a^{(2)}=z_{\rm op}U_a^{(2)}z_{\rm op} \tag{2.48a}$$

and

$$V_b^{(2)}=y_{\rm op}U_b^{(2)}y_{\rm op}\ , \tag{2.48b}$$

where $y_{\rm op}$ is given by (2.17b) and $z_{\rm op}$ by

$$z_{\rm op}=z(\mathbf{p}_{\rm op})\ . \tag{2.48c}$$

The conditions (2.46a) and (2.46b) then become

$$\langle \mathbf{p}'|U_a^{(2)}|\mathbf{p}\rangle=-e^2/t \tag{2.49a}$$

and

$$\langle \mathbf{p}'|U_b^{(2)}|\mathbf{p}\rangle=-e_Ae_B/4m_Am_B\ . \tag{2.49b}$$

As local and Hermitian solutions of the constraints (2.37a) and (2.37b) we may take

$$U_a^{(2)}(r)=U_C(r)\ , \tag{2.50a}$$

where $U_C(r)$ is the Coulomb potential,

$$U_C(r)=e_Ae_B/4\pi r \tag{2.50b}$$

and

$$U_b^{(2)}(r)=-e_Ae_B\delta(\mathbf{r})/4m_Am_B\ , \tag{2.50c}$$

which is a contact term, with no classical analog.

It should be noted that the presence of the two factors of $z_{\rm op}$ connecting $V_a^{(2)}$ and $U_a^{(2)}$ does have a simple physical interpretation. To see this, note that from the definition (2.43c) and (2.47) it follows that

$$z(\mathbf{p})=[1+(\mathbf{p}^2/E_AE_B)]^{1/2}\ , \tag{2.51a}$$

which may also be written in the form

$$z(\mathbf{p})=(1-\mathbf{v}_A\cdot\mathbf{v}_B)^{1/2} , \tag{2.51b}$$

where $\mathbf{v}_A$ and $\mathbf{v}_B$, defined by

$$\mathbf{v}_A=\mathbf{p}/E_A, \quad \mathbf{v}_B=-\mathbf{p}/E_B , \tag{2.51c}$$

are the velocities of A and B in the c.m. system. Since

$$z^2(\mathbf{p})=1-\mathbf{v}_A\cdot\mathbf{v}_B \tag{2.51d}$$

we see that in the classical limit the Coulomb potential is just modified by the retardation factor familiar from classical electrodynamics.

The dispersion theory approach leads to similar results; these depend slightly on the way in which the energy dependence of the Born term is handled. One approach is to define $U^{(2)}$ as in the Yukawa case, via (2.17a). Then the formulas of the preceding section apply. The spectral function associated with $M^{(2)}$ is now

$$\rho^{(2)}(s;t)=2\pi a e_A e_B \delta(t) \tag{2.52a}$$

and Eq. (2.23a) yields, for the long-range part $U_{\rm LR}^{(2)}(r;s)$ of $U^{(2)}$, the result

$$U_{\rm LR}^{(2)}(r;s)=\left[\frac{a(s)}{2m_A m_B}\right]U_C(r) . \tag{2.52b}$$

However, because $M^{(2)}$ does not vanish at $t=\infty$, the spectral representation for $M^{(2)}$ has an additive constant

$$M^{(2)}=-e_A e_B+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dt'}{t'-t}\rho^{(2)}(s,t') . \tag{2.53}$$

To reproduce this constant one must add a short-range term $U_{\rm SR}^{(2)}$ such that

$$\langle \mathbf{p}'|U_{\rm SR}^{(2)}|\mathbf{p}\rangle=(2m_A m_B)^{-1}(-e_A e_B) \tag{2.54}$$

which coincides with the condition (2.49) and leads to the (highly) local and s-independent result (2.50c).

Another approach, the one we adopt in this paper, gives a $U_{\rm LR}^{(2)}$ which is not only local but also s independent, as in the case of the scalar Yukawa theory. We write

$$V=z_{\rm op}Uz_{\rm op} . \tag{2.55}$$

Then the constraint (2.19a) becomes

$$\langle \mathbf{p}'|U^{(2)}|\mathbf{p}\rangle=\left[\frac{2m_A m_B}{a(s)}\right](4m_A m_B)^{-1}M^{(2)}$$

$$=\frac{1}{2a(s)}M^{(2)}=-e_A e_B\left[\frac{1}{t}+\frac{1}{2a(s)}\right] . \tag{2.56}$$

This yields

$$U_{\rm LR}^{(2)}=U_C(r) \tag{2.57a}$$

and

$$U_{\rm SR}^{(2)}=\frac{-e_A e_B}{2a(s)}\delta(\mathbf{r}) . \tag{2.57b}$$

Although these results are almost as simple as those obtained in the scalar Yukawa theory, their extension to fourth order is complicated by the fact that, on the one hand, the iteration of $V^{(2)}$ leads to both an infrared divergence (from $V_a^{(2)}$) and ultraviolet divergence (from $V_b^{(2)}$) and, on the other, that the fourth-order Feynman diagrams have such divergences also. We may imagine that this is dealt with provisionally by the introduction of an infrared cutoff (small photon mass μ) and ultraviolet cutoff in the photon propagator. If this is done, the fourth-order constraint takes a form not very different from that for the scalar Yukawa theory. We will deal with these matters further in Sec. IV, where we study two-photon exchange in scalar QED. Finally, we note that it is also possible to choose the second-order potential so that no ultraviolet divergences occur in the computation of M_r [Eq. (2.20)], provided that we allow derivative operators to appear in $U^{(2)}$. This alternative possibility is discussed in Appendix F.

III. TWO-QUANTUM EXCHANGE: SCALAR YUKAWA THEORY

A. Fourth-order amplitude

1. Preliminaries

The fourth-order diagrams involving two-quantum exchange are shown in Figs. 3(a) and 3(b). The corresponding contribution $M^{(4)}$ to the fourth-order invariant amplitude is

$$M^{(4)}=M_L+M_X , \tag{3.1}$$

where M_L and M_X represent the contributions of the two-rung ladder graph (3a) and of the two-rung crossed-ladder graph (3b), respectively. These are given by

$$M_L=iG^4\int[d^4k/(2\pi)^4][D_A D_B D(k)D(k')]^{-1} \tag{3.2a}$$

and

$$M_X=iG^4\int[d^4k/(2\pi)^4][D_A D_B' D(k)D(k')]^{-1} , \tag{3.2b}$$

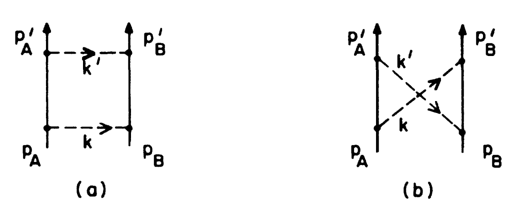

FIG. 3. Fourth-order Feynman graphs describing two-meson exchange in scalar Yukawa theory. (a) The two-rung ladder or box graph. (b) The two-rung crossed-ladder or crossed-box graph.

where

$$D_A=(p_A-k)^2-m_A^2+i\epsilon\ ,$$
$$D_B=(p_B+k)^2-m_B^2+i\epsilon\ ; \quad (3.3a)$$

$$D'_A=(p_A-k')^2-m_A^2+i\epsilon\ ,$$
$$D'_B=(p_B+k')^2-m_B^2+i\epsilon\ , \quad (3.3b)$$

and

$$D(k)=k^2-\mu^2\ , \quad (3.3c)$$

with k' related to k via

$$k+k'=Q\ , \quad (3.3d)$$

where Q is the four-momentum transfer defined by (2.20b). The sum (3.1) may be conveniently written in a compact form which is symmetric between k and k':

$$M^{(4)}=i(G^4/2)\int d^4k\, d^4k' \times\delta(Q-k-k')A_+B_+/(2\pi)^4D(k)D(k'), \quad (3.4)$$

where

$$A_\pm=D_A^{-1}\pm D_A'^{-1},\quad B_\pm=D_B^{-1}\pm D_B'^{-1}\ . \quad (3.5)$$

The amplitude $M^{(4)}$ has analyticity properties with respect to t of the type assumed for the amplitude M_S considered in Sec. II C. From the discussion there it follows that, as experience has shown, an efficient way to obtain the long-range potential associated with $M^{(4)}$ is to compute the discontinuity of $M^{(4)}$ with respect to t across the right-hand cut.

2. *Discontinuity calculation*

The amplitude $M^{(4)}$ is an analytic function of t with a nearest singularity at $t=t_0$, where t_0 is the threshold for the reaction

$$A+\overline{A}\to\phi+\phi'\ ,$$

with ϕ and ϕ' representing on-shell mesons with four-momenta k and k', respectively. Thus

$$t_0=4\mu^2\ . \quad (3.6)$$

We shall imagine that s is initially fixed at a negative value and later make an analytic continuation to $s>s_0$. The discontinuity $[M^{(4)}]_t$ of $M^{(4)}$ across a cut extending from t_0 to plus infinity along the real t axis is given by generalized unitarity, i.e., by making the replacement

$$D^{-1}(k)\to-(2\pi i)\delta(k^2-\mu^2)\theta(k^0) \quad (3.7)$$

and a similar replacement for $D^{-1}(k')$ in (3.4). We now work in the c.m. system of the *crossed reaction*

$$A+\overline{A}\to\overline{B}+B'\ , \quad (3.8)$$

where Q has the form $Q=(t^{1/2},0)$. We may then write

$$k=(\omega,\mathbf{k}),\quad k'=(\omega,-\mathbf{k})\ , \quad (3.9a)$$

with

$$\omega=t^{1/2}/2,\quad |\mathbf{k}|=(t-t_0)^{1/2}\ . \quad (3.9b)$$

The phase-space volume element $d\Phi$ in the expression for the discontinuity, defined by

$$d\Phi=d^4k\, d^4k'\delta^{(4)}(Q-k-k') \times\delta(k^2-\mu^2)\delta(k'^2-\mu^2)\theta(k^0)\theta(k'^0)\ ,$$

then reduces, on elimination of the delta functions, to

$$(|\mathbf{k}|/8w)d\Omega\ ,$$

where $d\Omega$ is an element of solid angle about the direction $\hat{\mathbf{k}}$ of $\mathbf{k}$. We then get

$$[M^{(4)}]_t=-i(G^4/16\pi)[(t-t_0)/t]^{1/2}\langle A_+B_+\rangle\ , \quad (3.10)$$

where the angular brackets denote an angular average, i.e., integration with a factor $d\Omega/4\pi$, of the quantity inside the brackets. Following the procedure of Ref. 1 we next write

$$p_A=(\omega,\mathbf{p}),\quad p_{\overline{A}}=-p'_A=(\omega,-\mathbf{p})\ , \quad (3.11a)$$

$$p'_B=(\omega,\mathbf{p}'),\quad p_{\overline{B}}=-p_B=(\omega,-\mathbf{p}')\ , \quad (3.11b)$$

where

$$\mathbf{p}=i\xi_A m_A\hat{\mathbf{p}},\quad \mathbf{p}'=i\xi_B m_B\hat{\mathbf{p}}'\ , \quad (3.11c)$$

with

$$\xi_A=[1-(t/4m_A^2)]^{1/2},\quad \xi_B=[1-(t/4m_B^2)]^{1/2}\ . \quad (3.11d)$$

Here $\hat{\mathbf{p}}$, and $\hat{\mathbf{p}}'$ are complex unit vectors, so that the external four-momenta continue to be on the mass shell. With these choices one finds

$$D_A=[(t-t_0)^{1/2}m_A\xi_A](-\tau_A-ix_A)\ , \quad (3.12a)$$

and

$$D_B=[(t-t_0)^{1/2}m_B\xi_B](-\tau_B+ix_B)\ , \quad (3.12b)$$

with

$$\tau_A=(t-2\mu^2)/2(t-t_0)^{1/2}\xi_A m_A,\quad x_A=\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}\ , \quad (3.12c)$$

and

$$\tau_B=(t-2\mu^2)/2(t-t_0)^{1/2}\xi_B m_B,\quad x_B=\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}'\ . \quad (3.12d)$$

The corresponding primed quantities are obtained by replacing x_A and x_B by their negatives

$$D'_A=D_A(x_A\to-x_A),\quad D'_B=D_B(x_B\to-x_B)\ . \quad (3.12e)$$

It follows that

$$A_+=[(t-t_0)^{1/2}m_A\xi_A]^{-1}(-2\tau_A)/(\tau_A^2+x_A^2) \quad (3.13a)$$

and

$$B_+=[(t-t_0)^{1/2}m_B\xi_B]^{-1}(-2\tau_B)/(\tau_B^2+x_B^2) \quad (3.13b)$$

so that (3.10) reduces to

$$[M^{(4)}]_t=-i(G^4/16\pi)[(t-t_0)t]^{-1/2} \times(4\tau_A\tau_B/m_Am_B\xi_A\xi_B)I_0\ , \quad (3.14)$$

where I_0 is defined by

$$I_0=\langle 1/d_Ad_B\rangle \quad (3.15a)$$

with

$$d_A=\tau_A^2+x_A^2,\quad d_B=\tau_B^2+x_B^2\ . \quad (3.15b)$$

The integral I_0 has been evaluated previously,[5] with the result that

$$I_0=(2\tau_A\tau_B)^{-1}(F_+ +\pi N_+^{-1})\ , \quad (3.15c)$$

where

$$F_\pm(s,t)=\pm N_-^{-1}\arctan(N_-/D_+) -N_\mp^{-1}\arctan(N_+/D_-)\ , \quad (3.15d)$$

with

$$N_\pm(s,t)=(\tau_A^2+\tau_B^2+1-y^2\pm 2\tau_A\tau_By)^{1/2}\ , \quad (3.15e)$$

$$D_\pm(s,t)=y\pm\tau_A\tau_B\ , \quad (3.15f)$$

and

$$y=\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}'\ . \quad (3.15g)$$

The square root in (3.15e) is defined so that if the quantity inside the parentheses becomes negative the root with a negative imaginary part is to be taken. The quantity y may be expressed in terms of the invariants s and t:

$$y=y(s,t)=(s-u)/4b =(2s+t-2m_A^2-2m_B^2)/4b\ , \quad (3.15h)$$

where b is defined by

$$b=m_A\xi_Am_B\xi_B\ . \quad (3.15i)$$

3. $\mu=0$ limit

Since we are studying this model primarily as preparation for the case of QED, we focus on the value of the discontinuity in the limit of zero mass for the exchanged mesons. In this limit $t_0=0$ and

$$\tau_A=t^{1/2}/2m_A\xi_A,\quad \tau_B=t^{1/2}/2m_B\xi_B\ , \quad (3.16)$$

so that

$$N_\pm(s,t)=\{(t/4)[(m_A\xi_A)^{-2}+(m_B\xi_B)^{-2}] +1-y^2\pm(yt/2b)\}^{1/2} \quad (3.17a)$$

and

$$D_\pm(s,t)=y\pm(t/4b)\ . \quad (3.17b)$$

On substitution of the expression (3.15h) for $y(s,t)$ into (3.17a) one finds, after some algebra and use of the relation

$$p=\{[s-(m_A+m_B)^2][s-(m_A-m_B)^2]/4s\}^{1/2} \quad (3.18)$$

for the magnitude of the c.m. momentum in the physical region of the direct channel, that

$$N_+(s,t)=N(s)/\xi_A\xi_B\ , \quad (3.19a)$$

where

$$N(s)=-ips^{1/2}/m_Am_B \quad (3.19b)$$

and

$$N_-(s,t)=-ib^{-1}(p^2s+byt)^{1/2}\ . \quad (3.19c)$$

Note that while the functions N_+ and N_- are imaginary upon analytic continuation to the region $s>0$, the functions F_+ and F_- are real. We now substitute (3.15c) into (3.14), both with $\mu=0$, and separate the contribution of the π/N_+ term from the arctangent terms to get (dropping the superscript 4 on M now, to ease the notation)

$$[M]_t=[M]_t^{(0)}+[M]_t^{(1)}\ , \quad (3.20)$$

where

$$[M]_t^{(0)}=(G^4/8ps^{1/2})t^{-1} \quad (3.21)$$

and

$$[M]_t^{(1)}=-i(G^4/8\pi b)F_+(s,t)t^{-1}\ . \quad (3.22)$$

Substitution of $[M]_t^{(0)}$ into a spectral representation such as (2.23a) for M would lead to an integral which is logarithmically divergent for small t, whereas $[M]_t^{(1)}$ gives a finite contribution. One expects such a divergence, since the Feynman integrals for M_L and M_X are infrared divergent if $\mu=0$ and this divergence remains in their sum. However, as we shall see below, this divergence cancels when we consider the difference of the discontinuity of M and that of M_I, defined by (2.20).

Before turning to this, let us consider the nature of the residual term (3.22). Inspection shows that the function $F_+(s,t)$ is analytic in the neighborhood of $t=0$. Moreover, since $N_+=N_-$ and $D_+=D_-$ at $t=0$, i.e.,

$$N_\pm(s,0)=N(s) \quad (3.23a)$$

and

$$D_\pm(s,0)=D(s)\ , \quad (3.23b)$$

with

$$D(s)=y(s,0)=(s-m_A^2-m_B^2)/2m_Am_B\ , \quad (3.24)$$

the function $F_+(s,t)$ vanishes at $t=0$. It follows that

$$[M]_t^{(1)}=-i(G^4/8\pi m_Am_B)F'_+ +O(t/\overline{m}^2)\ , \quad (3.25)$$

where $\overline{m}=\overline{m}(s)$ is a quantity with the dimensions of a mass, which is not zero at $p=0$, and

$$F'_+\equiv F'_+(s,0)=\lim_{t\to 0}(F_+(s,t)/t) \quad (3.26)$$

is just the derivative of F_+ with respect to t at $t=0$. Since F'_+ is finite, substitution of $[M]_t^{(1)}$ into (2.23a) leads to an integral which converges at small t. Computation gives

$$F'_+=[2m_A m_B D^2(s)]^{-1} \times\{f^{-3}[\arctan f-f(1+f^2)^{-1}]-(1+f^2)^{-1}\} , \tag{3.27a}$$

where

$$f=N(s)/D(s)=-2ips^{1/2}/(s-m_A^2-m_B^2) . \tag{3.27b}$$

For later use we note that from (3.15d) and (3.23a) and (3.23b) it follows that

$$F_-(s,0)=-2N^{-1}(s)\arctan[N(s)/D(s)] . \tag{3.28}$$

For $p\to 0$, $f\to 0$ also and $D(s)\to 1$; in this limit one finds that F'_+ has a finite value F'_0 given by

$$F'_0\equiv F'_+(s_0,0)=-1/6m_A m_B , \tag{3.29a}$$

whereas for $F_-(s,0)$ we have

$$F_-(s_0,0)=-2 . \tag{3.29b}$$

On recalling that $G^2=4m_A m_B g^2$, we see that for small p and small t

$$[M]_t^{(1)}=i(g^4/3\pi)+O(t/\overline{m}^2)+O(p^2/\overline{m}^2) . \tag{3.30}$$

We now turn to the study of M_I.

B. Iteration of second-order potential

1. Preliminaries

The relativistic amplitude M_I arising from the iteration of the second-order potential $V^{(2)}$ defined by (2.40a) and (2.40b), is given, on use of (2.20) and insertion of a complete set of plane-wave intermediate states $|l\rangle$, by

$$M_I=G^4(2\pi)^{-3}\int dl[4E_A(l)E_B(l)\mathcal{D}(\mathbf{q}^2+\mu^2) \times(\mathbf{q}'^2+\mu^2)]^{-1} , \tag{3.31}$$

where

$$\mathcal{D}=W(\mathbf{p})-W(l)+i\epsilon \tag{3.32}$$

with $W(\mathbf{p})$, defined by (2.9), the total incident energy in the c.m. system, and

$$\mathbf{q}=\mathbf{p}-l,\quad \mathbf{q}'=-\mathbf{p}'+l . \tag{3.33a}$$

The total three-momentum transfer in the c.m. system is

$$\mathbf{Q}=\mathbf{q}+\mathbf{q}'=\mathbf{p}-\mathbf{p}' . \tag{3.33b}$$

In terms of the invariants s and t, defined by (2.4c) and (2.4d), the physical region is given by

$$s>s_0=(m_A+m_B)^2,\quad -4p^2<t<0 . \tag{3.34}$$

To study the behavior of M_I as a function of t, with p fixed and positive or equivalently with s fixed and larger than s_0, we first combine the denominator factors $(\mathbf{q}^2+\mu^2)$ and $(\mathbf{q}'^2+\mu^2)$ with a Feynman parameter α. The angular integration in (3.31) is then readily performed, with the result that

$$M_I=(G^4/8\pi^2)\int_0^\infty dl\, l^2[E_A(l)E_B(l)\mathcal{D}]^{-1} \times\int_0^1 d\alpha\, J^{-1} , \tag{3.35}$$

where $l=|l|$ and

$$J=(p^2-l^2)^2+2\mu^2(p^2+l^2)+\mu^4 -4\alpha(1-\alpha)l^2t . \tag{3.36}$$

From (3.35) and (3.36) we see that, with p fixed and t initially in the physical region, M_I can be extended to an analytic function in a cut t plane, with singularities only on the positive t axis. Since J does not vanish at the end points of either the α or the l integrations, the condition for a singularity is that $J=0$ and $\partial J/\partial\alpha=0$, $\partial J/\partial l=0$. For $t\neq 0$, the last two conditions require that $\alpha=\frac{1}{2}$, $l^2=p^2+(t/2)-\mu^2$ and for these values

$$J\to(t-t_0)(t+4p^2)/4 ,$$

where

$$t_0=4\mu^2 . \tag{3.37}$$

Thus, the only t singularity of M_I is a branch point at $t=t_0$. This is of course just what one expects, by analogy with the corresponding Feynman box diagram.

2. Discontinuity calculation

For later purposes it is convenient to consider separately the dispersive part D_I and absorptive part A_I of M_I in the energy variable, associated with the principal-value part and delta-function part of the factor $1/\mathcal{D}$ in (3.35), respectively. Thus with

$$1/\mathcal{D}=P[W(\mathbf{p})-W(l)]^{-1}-i\pi\delta[W(\mathbf{p})-W(l)] \tag{3.38}$$

we have

$$M_I=D_I+iA_I , \tag{3.39}$$

where

$$D_I=(G^4/8\pi^2)P\int_0^\infty dl\, l^2C(p,l)\int_0^1 d\alpha\, J^{-1} \tag{3.40}$$

with

$$C(p,l)=\{E_A(l)E_B(l)[W(p)-W(l)]\}^{-1} \tag{3.41}$$

and

$$A_I=-(G^4/8\pi)\int_0^\infty dl\, l^2[E_A(l)E_B(l)]^{-1} \times\delta(W(p)-W(l))\int_0^1 d\alpha\, J^{-1} . \tag{3.42}$$

Note that since both D_I and A_I are manifestly real in the physical region, in that region this separation also corresponds to a separation into real and imaginary parts. Since the absorptive part, which arises from real intermediate states, is simpler than the dispersive part we study it first.

(a) Absorptive part. On carrying out the integration over l in (3.42) we get

$$A_I = -(G^4/8\pi)[p/W(p)]\int_0^1 d\alpha\, J_0^{-1} , \quad (3.43a)$$

where J_0 is the value of J at $l = p$,

$$J_0 = 4\mu^2 p^2 + \mu^4 - 4\alpha(1-\alpha)p^2 t . \quad (3.43b)$$

From (3.43) it is clear that A_I it itself an analytic function in a cut t plane with a branch-point singularity at a value t_1 determined by the conditions $J_0 = 0$ and $\partial J_0/\partial\alpha = 0$. These yield $\alpha = \frac{1}{2}$, as before, and

$$t_1 = 4\mu^2 + (\mu^4/p^2) . \quad (3.44)$$

The discontinuity $[A_I]_t$ of A_I across a cut extending from t_1 to plus infinity may be obtained by replacing the factor J_0^{-1} by the difference Δ between its values at $t + i\epsilon$ and $t - i\epsilon$ in the limit where $\epsilon \to 0$; this is given by

$$\Delta = 2\pi i\,\delta(J_0) . \quad (3.45)$$

On carrying out the now trivial integration over α we find

$$[A_I]_t = -i(G^4/8p)[st(t-t_1)]^{-1/2} . \quad (3.46)$$

In the limit of vanishing mass μ, (3.46) becomes

$$[A_I]_t = -iG^4/8ps^{1/2}t \quad (\mu = 0) , \quad (3.47a)$$

and, correspondingly, we get a contribution $[M_I]_t^{(0)}$ to $[M_I]_t$ given by

$$[M_I]_t^{(0)} = G^4/8ps^{1/2}t . \quad (3.47b)$$

This is precisely the leading term found for $[M]_t$ in Sec. III A so that the logarithmic divergence cancels in the difference amplitude, as promised.

(b) Dispersive part. Since M_I and A_I are analytic in the cut t plane, with branch points at $t = t_0$ and $t = t_1$, respectively, it follows that D_I is analytic in the cut t plane with nearest singularity at $t = t_0$. The discontinuity of D_I across a cut starting at t_0 and extending to plus infinity may again be found by use of (3.45). This yields

$$[D_I]_t = (G^4/8\pi^2)2\pi i K , \quad (3.48)$$

where

$$K = P\int_0^\infty dl\, l^2 C(p,l)\int_0^1 d\alpha\,\delta(J) \quad (3.49)$$

with J given by (3.36) and C by (3.41). A short calculation gives

$$\int_0^1 d\alpha\,\delta(J) = (2lt^{1/2})^{-1}\theta(L)L^{-1/2} ,$$

where $-L$ is the value of J at $\alpha = \frac{1}{2}$,

$$-L = (p^2 - l^2)^2 + 2\mu^2(p^2 + l^2) + \mu^4 - l^2 t .$$

The roots of $-L$ considered as a function of $y = l^2$ are at

$$y_\pm = (a' \pm b')/2 ,$$

where

$$a' = t + 2p^2 - 2\mu^2, \quad b' = [(t - 4\mu^2)(t + 4p^2)]^{1/2} . \quad (3.50)$$

Thus

$$L = (y_+ - y)(y - y_-)$$

and L is positive only for $t > t_0$ and for

$$y_- < y < y_+ .$$

On introduction of a new integration variable x via

$$l^2 = y = (b'x + a')/2 , \quad (3.51)$$

L assumes the form $L = b'^2(1-x^2)/4$ and, with $l\,dl = dy/2 = b'dx/4$, the integral for K reduces to

$$K = (1/4t^{1/2})P\int_{-1}^1 dx\,(1-x^2)^{-1/2}C(p,l(x)) . \quad (3.52)$$

To proceed further we separate C into a part C_1 which, like C, is singular at $p^2 = l^2$ and a nonsingular remainder C_2,

$$C = C_1 + C_2 . \quad (3.53)$$

The residue of C at $p^2 = l^2$ is given by $2/W(p)$ so that we define

$$C_1 = [2/W(p)](p^2 - l^2)^{-1} . \quad (3.54a)$$

The remainder C_2 is then found, after considerable algebra, to be

$$C_2 = (E_A' E_B' W)^{-1}[(p^2 + l^2 + m_A^2 + m_B^2)(E_A E_B + E_A' E_B')^{-1} + W'/(W + W')] , \quad (3.54b)$$

where $E_A' = E_A(l)$, $E_B' = E_B(l)$, and $W' = W(l)$.

Corresponding to the decomposition (3.53) of C we have

$$K = K_1 + K_2 , \quad (3.55)$$

where

$$K_i = (1/4t^{1/2})P\int_{-1}^1 dx\,(1-x^2)^{-1/2}C_i(p,l(x)) \quad (i = 1,2) . \quad (3.56)$$

The symbol P is needed only for K_1 since C_2 is nonsingular. We consider the two parts of K in turn.

(i) Study of K_1. From (3.51), (3.54a), and (3.56) we get

$$K_1 = (ts)^{-1/2}(b')^{-1}R(x_0) , \quad (3.57)$$

where

$$R(x_0) = P\int_{-1}^1 dx\,(1-x^2)^{-1/2}(x_0 - x)^{-1} \quad (3.58a)$$

and

$$x_0=(2p^2-a')/b'$$
$$=(2\mu^2-t)/[(t-t_0)(t+4p^2)]^{1/2} . \qquad (3.58b)$$

The function $R(x_0)$ may be evaluated exactly with the result

$$R(x_0)=\pi(x_0^2-1)^{-1/2}\theta(x_0^2-1)\epsilon(x_0) . \qquad (3.59)$$

With $t>t_0$, the condition that $x_0^2>1$ requires that

$$t<t_1 .$$

Thus we get

$$K_1=-(\pi/2)p^{-1}[ts(t_1-t)]^{-1/2} \qquad (t_0<t<t_1) \qquad (3.60)$$

and $K_1=0$ otherwise. The corresponding contribution to the discontinuity of D_I for this range of t is then

$$[D_{I;1}]_t=-i(G^4/8\pi^2)p^{-1}[ts(t_1-t)]^{-1/2} . \qquad (3.61)$$

Note that this part of $[D_I]_t$ is obtainable from $[A_I]_t$ by simply replacing $(t-t_1)^{1/2}$ by $(t_1-t)^{1/2}$ in (3.46).

We are primarily interested in the limit $\mu\to 0$. In this limit K_1 makes no contribution to D_I, as we now show. First note that if it were permissible to simply set $\mu=0$ inside the integral sign in (3.58a) we could replace x_0 by x_0', its value for $\mu=0$, viz.,

$$x_0'=-[t/(t+4p^2)]^{1/2} .$$

This is manifestly less than unity in magnitude for $t>0$ and p^2 positive, so that according to (3.57) and (3.59) we have $K_1=0$. However, in view of the additional limiting process implicit in the principal-value prescription it behooves us to be less cavalier. We shall therefore verify directly that K_1 makes no contribution to D_I in the $\mu=0$ limit. From (3.57) we infer that, apart from factors which are independent of μ and finite for $p\neq 0$, the contribution $D_{I;1}$ associated with K_1 is proportional to

$$H=\int_{t_0}^{t_1}dt'[(t'-t)t'^{1/2}(t_1-t')^{1/2}]^{-1} .$$

On setting $t'=4\mu^2 v$ we have

$$H=\int_1^{v_1}dv[(4\mu^2 v-t)v^{1/2}(v_1-v)^{1/2}]^{-1} ,$$

where $v_1=1+\mu^2/4p^2$. For $\mu\ll p$ the integration interval shrinks to zero and, on replacing v by unity in the first two factors inside the square brackets we see that for μ small enough H is proportional to

$$\int_1^{v_1}dv\,(v_1-v)^{-1/2}=\mu/p .$$

Thus, there is no contribution to D_I from K_1 in the limit $\mu\to 0$.

(ii) Study of K_2. From (3.56) we have

$$K_2=(1/4t^{1/2})\int_{-1}^{1}dx(1-x^2)^{-1/2}C_2(p,l) , \qquad (3.62)$$

where C_2 is defined by (3.54b). In this case we may pass to the limit $\mu=0$ right away, since the dependence of C_2 on μ, via its dependence on l^2, is very mild. Since we are interested in the region where $|t|$ is much less than either m_A or m_B, we may get a good approximation to K_2 by replacing $C_2(p,l)$ in (3.62) by its value for $t=0$. It is straightforward to find the leading term, $K_2^{(0)}$, in an expansion of K_2 in powers of t/m_A^2 and t/m_B^2. Note that from (3.50) and (3.51) we can infer that, for $\mu=0$ and $t=0$,

$$l^2=y=p^2 . \qquad (3.63)$$

It follows that, for $\mu=0$,

$$C_2(p,l)=C_2(p)+O(t) , \qquad (3.64a)$$

where

$$C_2(p)\equiv C_2(p,p)=(E_A^2+E_AE_B+E_B^2)/2E_A^2E_B^2W , \qquad (3.64b)$$

and on using this in (3.62) we get

$$K_2=K_2^{(0)}+O(t^{1/2}/\overline{m}) , \qquad (3.65a)$$

with

$$K_2^{(0)}=\pi C_2(p)/4t^{1/2} , \qquad (3.65b)$$

the factor of π arising from the integral over x. The corresponding contribution to the discontinuity defined by (3.48) is thus

$$[D_{I;2}]_t^{(0)}=(2\pi i)(G^4/8\pi^2)[\pi C_2(p)/4t^{1/2}] \qquad (\mu=0) \qquad (3.66a)$$

and the total discontinuity of the dispersive part D_I of M_I has the form

$$[D_I]_t=[D_{I;2}]_t^{(0)}+O(t^{1/2}/\overline{m}) . \qquad (3.66b)$$

Although (3.66b) is only a good approximation for small t, the associated spectral integral converges if extended to infinity. The leading term in an expansion of D_I in powers of t is therefore also proportional to $t^{-1/2}$. We now consider the potential arising from two-quantum exchange in more detail.

C. Fourth-order potential

The constraint on the fourth-order potential $V^{(4)}$, written in the form (2.17a), viz.,

$$V^{(4)}=y_{\rm op}U^{(4)}y_{\rm op} \qquad (3.67)$$

is given by (2.19b). Following the discussion of Sec. II C we may write the long-range part $U_{\rm LR}^{(4)}$ of $U^{(4)}$ in the form

$$U_{\rm LR}^{(4)}(r;s)=(16\pi^2 m_Am_Br)^{-1}\int_0^\infty dt\,\rho_{\rm diff}^{(4)}(s,t)e^{-t^{1/2}r} , \qquad (3.68a)$$

where $\rho_{\rm diff}^{(4)}$ is the difference spectra function:

$$\rho_{\rm diff}^{(4)}=([M^{(4)}]_t-[M_I]_t)/2i . \qquad (3.68b)$$

From Eqs. (3.20), (3.21), and (3.22) for the discontinuity of $M^{(4)}$ and Eqs. (3.47) and (3.48) for that of M_I we get, on recalling that K_1 contributes zero, the relation

$$\rho_{\rm diff}^{(4)}=-(G^4/16\pi b)(F_+/t)-(G^4/8\pi)K_2 \ , \tag{3.69}$$

where F_+ is defined by (3.15d) and K_2 by (3.62). Note that this spectral function is purely real, at least for small p^2; the discontinuities of the absorptive parts of $M^{(4)}$ and M_I cancel for all $t>0$. In (3.69) no expansion in t has been made.

From (3.68a) and (3.69) it is straightforward to obtain an expansion of $U_{\rm LR}^{(4)}$ in inverse powers of r, from an expansion of $\rho_{\rm diff}^{(4)}$ in powers of $t^{1/2}$. We give below the explicit forms of the first two terms in such an expansion. From (3.26) we have

$$F_+(s,t)/t=F'_+(s,0)+O(t/\overline{m}^2) \ , \tag{3.70}$$

where F'_+ is given by (3.27). On writing, as in Eq. (2.29a),

$$\rho_{\rm diff}^{(4)}=a_2(s)t^{-1/2}+a_3(s)+O(t^{1/2}/\overline{m}) \tag{3.71}$$

we have, from (3.65), (3.66), (3.70), and (3.69),

$$\begin{aligned} a_2(s)&=-G^4C_2(p)/32 \ , \\ a_3(s)&=-G^4F'_+(s,0)/16\pi m_A m_B \ . \end{aligned} \tag{3.72}$$

On using (2.29c) and (2.29d) we then find that

$$U_{\rm LR}^{(4)}(r;s)=c_2(s)r^{-2}+c_3(s)r^{-3}+O(r^{-4}\overline{m}^{-3}) \ , \tag{3.73a}$$

where

$$c_2(s)=a_2(s)/8\pi^2 m_A m_B, \quad c_3(s)=a_3(s)/8\pi^2 m_A m_B \ . \tag{3.73b}$$

At low energies we may approximate the functions $C_2(p)$ and $F'_+(s,0)$ by their threshold values, given by setting $p=0$ in (3.63) and by (3.29a), respectively; this yields

$$U_{\rm LR}^{(4)}(r;s_0)=c_2(s_0)r^{-2}+c_3(s_0)r^{-3}+O(r^{-4}\overline{m}^{-3}) \ , \tag{3.74a}$$

with

$$c_2(s_0)=-(g^2/4\pi)^2(m_A^2+m_A m_B+m_B^2)/2m_A m_B(m_A+m_B) \tag{3.74b}$$

and

$$c_3(s_0)=-(g^2/4\pi)^2/3\pi m_A m_B \ . \tag{3.74c}$$

Note that if we let either mass tend to infinity, the r^{-3} term in (3.74a) vanishes but the r^{-2} term survives. In particular, we have, for, e.g., $m_B\to\infty$,

$$U_{\rm LR}^{(4)}(r;s_0)\sim\frac{-(g^2/4\pi^2)^2}{2m_A r^2} \ . \tag{3.74d}$$

We study the infinite-mass limit in more detail in the next and final subsection of this long section.

D. Infinite-mass limit

It is instructive and will serve as a partial check on our calculation to consider the amplitudes and associated potentials studied above in the limit where one of the masses m_A or m_B becomes infinite. To be precise we study the limit $m_B\to\infty$, with p^2, t, and $g^2=G^2/4m_A m_B$ fixed. From Eq. (2.28b) it follows that the computation of these potentials will require the evaluation of the limit of the ratio $\rho_{\rm diff}^{(4)}/m_B\to\infty$.

Note first that, since $\tau_B\to 0$ for $m_B\to\infty$, the functions N_+ and D_+, defined by (3.17a) and (3.17b) coincide with N_- and D_-, respectively, in this limit. It then follows from (3.15d) that

$$\lim_{m_B\to\infty} F_+(s,t)=0 \ . \tag{3.75}$$

On dividing (3.21) and (3.22) by m_B we see that the limit of $[M^{(4)}]_t/m_B$ as $m_B\to\infty$ is proportional to the limit of $F_+(s,t)$ as $m_B\to\infty$, so that using (3.75) we get

$$\lim_{m_B\to\infty}[M^{(4)}]_t^{(1)}/m_B=0 \ . \tag{3.76}$$

It follows that

$$\lim_{m_B\to\infty}[M^{(4)}]_t/m_B=\lim_{m_B\to\infty}[M]_t^{(0)}/m_B=\frac{2g^4m_A^2}{pt} \ . \tag{3.77}$$

Turning now to $[M_I]_t=[D_I]_t+[iA_I]_t$, since, according to (3.47), $[iA_I]_t=[M]_t^{(0)}$ for any value of m_B, we also have

$$\lim_{m_B\to\infty}\frac{[iA_I]_t}{m_B}=\frac{2g^4m_A^2}{pt} \ , \tag{3.78}$$

which therefore cancels against the $m_B\to\infty$ limit of $[M]_t^{(0)}/m_B$, as before.

To find the discontinuity of the dispersive part in this limit, we use the form (3.54b) of $C_2(p,l)$ to find that

$$\lim_{m_B\to\infty} m_B C_2(p,l)=\frac{1}{E'_A(E_A+E'_A)} \ . \tag{3.79}$$

It then follows, from (3.48), (3.62), and the vanishing of K_1, that

$$\lim_{m_B\to\infty}\frac{[D_I]_t}{m_B}=\frac{ig^4f_A(p^2,t)}{2\sqrt{t}} \ , \tag{3.80a}$$

where

$$f_A(p^2,t)=\frac{2m_A^2}{\pi}\int_{-1}^{1}\frac{dx\,(1-x^2)^{-1/2}}{E'_A(E_A+E'_A)} \ . \tag{3.80b}$$

Here $E'_A=E_A(l)$ is expressed in terms of x through Eqs. (3.50) and (3.51), which involve t parametrically. On combining the results (3.77), (3.78), and (3.80) we see that the difference spectral function $\rho_{\rm diff}(s,t)$, defined by (3.68b), satisfies

$$\lim_{m_B\to\infty}\frac{\rho_{\rm diff}^{(4)}(s,t)}{m_B}=\frac{-g^4}{4\sqrt{t}}f_A(p^2,t) \ . \tag{3.81}$$

On using the definition (3.68) for $U_{\rm LR}^{(4)}(r;s)$ we get, with p^2 fixed,

$$U_A^{(4)}(r;p^2) \equiv \lim_{m_B\to\infty} U_{\rm LR}^{(4)}(r;s) = \frac{-g^4}{16\pi^2 m_A r}\int_0^\infty \frac{dt}{4\sqrt{t}} e^{-t^{1/2}r} f_A(p^2,t) \ . \quad (3.82)$$

Thus in the $m_B \to \infty$ limit the long-range potential in order g^4 arises solely from the iteration terms. This is still a complicated function of r and p^2, but an expansion of $U_A^{(4)}$ in inverse powers of r is readily obtained. The leading term in $U_A^{(4)}(r;p^2)$ at large r is given by setting $t=0$ in the function f_A, so that

$$U_A^{(4)}(r;p^2) \sim \frac{-(g^2/4\pi)^2}{2m_A r^2} f_A(p^2,0) \ , \quad (3.83)$$

where

$$f_A(p^2,0) = m_A^2/E_A^2(p^2) \ . \quad (3.84)$$

It follows that

$$U_A^{(4)}(r;0) \sim \frac{-(g^2/4\pi)^2}{2m_A r^2} \ . \quad (3.85)$$

On comparison with (3.74d) we see that this coincides with the $m_B \to \infty$ limit of $U_{\rm LR}^{(4)}(r;s_0)$. In Appendix A we provide a further check on our method by comparing (3.85) with the result expected from use of the Klein-Gordon equation, extended to include interaction with an external scalar field.

We now turn to scalar QED, a case of greater physical interest. As we will see, there is then no term of order r^{-2}, in the limit where either one of the particle masses goes to infinity.

IV. TWO-QUANTUM-EXCHANGE: SCALAR QED

A. Fourth-order amplitude

1. Preliminaries

There are five Feynman diagrams which describe the scattering in order $e_A^2 e_B^2$, shown in Fig. 4. Figures 4(a) and 4(b) show the box and crossed-box graphs, analogs of those encountered in the scalar Yukawa theory, Figs. 4(c) and 4(d) show single-seagull graphs, and Fig. 4(e) shows the double-seagull graph. The sum $M^{(4)}$ of the contributions $M_a^{(4)}, M_b^{(4)}, \ldots$, associated with these diagrams can be written in a compact form, analogous to (3.4),

$$M^{(4)} = i(e^4/2)\int [d^4k/(2\pi)^4] L_A{:}L_B/k^2k'^2 \ , \quad (4.1)$$

where $e^4 = e_A^2 e_B^2$, the tensor $L_A^{\mu\nu} = L_A^{\mu\nu}(P_A;k,k')$ describes the emission of two virtual photons by A, and $L_B^{\mu\nu} = L_B^{\mu\nu}(P_B;-k,-k')$ describes the absorption of two photons by B. Thus

$$L_A^{\mu\nu} = [(P_A^\nu - k^\nu)(P_A^\mu + k'^\mu)/D_A] + [(P_A^\mu - k'^\mu)(P_A^\nu + k^\nu)/D'_A] - 2g^{\mu\nu} \ , \quad (4.2a)$$

$$L_B^{\mu\nu} = [(P_B^\nu - k^\nu)(P_B^\mu + k'^\mu)/D_B] + [(P_B^\mu - k'^\mu)(P_B^\nu + k^\nu)/D'_B] - 2g^{\mu\nu} \ , \quad (4.2b)$$

where P_A and P_B are defined by (2.43a) and the denominators $D_A, D'_A, \ldots$ by (3.3a) and (3.3b). The colon in (4.1) denotes a contraction of tensor indices. This contraction produces the integrands of the amplitudes associated with Figs. 4(a)–4(d) twice; the factor $\frac{1}{2}$ in (4.1) corrects for double counting. The integrand of the double-seagull graph is produced only once; the factor $\frac{1}{2}$ here is the correct symmetry factor associated with such a diagram.

While our calculation will be done using the Feynman gauge propagators, the results for $M^{(2)}$, $M^{(4)}$, and their discontinuities are gauge invariant, as must be the case for physical scattering amplitudes. Thus the potential derived from the M's through Eqs. (2.19b), (2.23a), and (2.28b) will also be gauge invariant.

Strictly speaking, to make the integral in (4.1) well defined, the photon propagator factors $1/k^2$ and $1/k'^2$ in (4.1) should be cut off both at low and high momenta, e.g., by replacing $1/k^2$ by $(k^2-\mu^2)^{-1}-(k^2-\Lambda^2)^{-1}$; this will control the infrared (IR) and ultraviolet (UV) divergences associated with each of the graphs of Fig. 4 (Ref. 6). These divergences are similar in nature to those which occur in the iteration of the one-quantum exchange potential, mentioned in Sec. II C 2. However, the discontinuity $[M^{(4)}]_t$ is the only part of $M^{(4)}$ needed for our purpose and this part is both IR and UV finite, i.e., finite in the limit $\mu \to 0$ and $\Lambda \to \infty$; this feature is a significant advantage of the dispersion-theoretic approach to the problem at hand. We can therefore omit these cutoffs and proceed directly with the calculation of the discontinuity.

2. Discontinuity calculation

Following the procedures of Sec. III we have, in analogy with (3.10), but with μ already set equal to zero,

$$[M^{(4)}]_t = -i(e^4/16\pi)\langle L_A{:}L_B\rangle \ , \quad (4.3)$$

where now the contraction is carried out on the photon

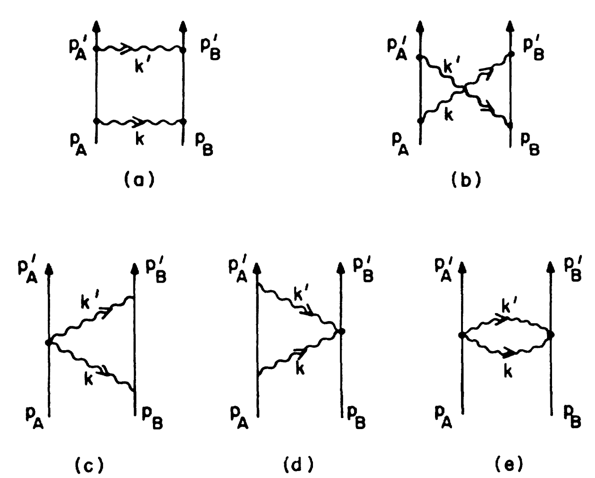

FIG. 4. Fourth-order graphs describing two-photon exchange between charged particles in scalar QED (a) and (b): box and crossed-box graphs; (c) and (d) single-seagull graphs; (e) double-seagull graph.

mass shell, $k^2=k'^2=0$, and the angular brackets denote an angular average over the direction $\hat{\mathbf{k}}$ of $\mathbf{k}$, in the c.m. system of the crossed channel, as before. To carry out the computations involved it is convenient to write the tensors L_A and L_B in the form

$$L_A^{\mu\nu}=(P_A^\nu P_A^\mu - k^\nu k'^\mu)A_+ + (P_A^\nu k'^\mu - P_A^\mu k^\nu)A_- - 2g^{\mu\nu}\ , \tag{4.4a}$$

$$L_B^{\mu\nu}=(P_B^\nu P_B^\mu - k^\nu k'^\mu)B_+ - (P_B^\nu k'^\mu - P_B^\mu k^\nu)B_- - 2g^{\mu\nu}\ , \tag{4.4b}$$

where $A_\pm$ and $B_\pm$ are the sums and differences of particle propagators defined by (3.5). Further, we separate the quantity $L_A{:}L_B$ into a part $(L_A{:}L_B)_1$ coming from the terms in (4.4a) and (4.4b) not involving $g_{\mu\nu}$'s, which arise from the ladder and crossed-ladder graphs, and a part $(L_A{:}L_B)_2$ involving one or two $g_{\mu\nu}$'s, which arises from the seagull graphs. We then find

$$(L_A{:}L_B)_1=[(P_A\cdot P_B)^2 - P_A\cdot k P_A\cdot k' - P_B\cdot k P_B\cdot k']A_+B_+ + \{[(P_A\cdot P_B)(P_A\cdot k - P_A\cdot k')A_+B_-] - [A\to B]\} + (P_A\cdot k P_B\cdot k' + P_A\cdot k' P_B\cdot k)A_-B_- \tag{4.5}$$

and

$$(L_A{:}L_B)_2 = -2[(P_A^2 - k\cdot k')A_+ + (P_A\cdot k' - P_A\cdot k)A_- + (P_B^2 - k\cdot k')B_+ - (P_B\cdot k' - P_B\cdot k)B_- - 8]\ . \tag{4.6}$$

On using the coordinate system defined by (3.9) and (3.11), with $\mu=0$, we have

$$k=(t^{1/2}/2)(1,\hat{\mathbf{k}}),\quad k'=(t^{1/2}/2)(1,-\hat{\mathbf{k}})\ , \tag{4.7a}$$

$$P_A=(0,2\mathbf{p}),\quad P_B=(0,2\mathbf{p}')\ , \tag{4.7b}$$

so that, using (3.11c) and (3.11d) and the definitions (3.12c) and (3.12d) for x_A and x_B, we get

$$P_A\cdot k = -P_A\cdot k' = -it^{1/2}m_A\xi_A x_A,\quad P_B\cdot k = -P_B\cdot k' = -it^{1/2}m_B\xi_B x_B\ , \tag{4.7c}$$

and

$$D_A = -(t/2) + it^{1/2}\xi_A m_A x_A,\quad D_A' = -(t/2) - it^{1/2}\xi_A m_A x_A\ , \tag{4.7d}$$

$$D_B = -(t/2) - it^{1/2}\xi_B m_A x_B,\quad D_B' = -(t/2) + it^{1/2}\xi_B m_B x_B\ . \tag{4.7e}$$

It follows that

$$A_+ = -t/D_AD_A',\quad A_- = -2it^{1/2}\xi_A m_A x_A/D_AD_A',\quad B_+ = -t/D_BD_B',\quad B_- = 2it^{1/2}\xi_B m_B x_B/D_BD_B'\ ,$$

and hence that

$$A_+B_+ = t^2/Y,\quad A_-B_- = 4tbx_Ax_B/Y,\quad A_+B_- = -2it^{3/2}\xi_B m_B x_B/Y,\quad A_-B_+ = 2it^{3/2}\xi_A m_A x_A/Y\ , \tag{4.7f}$$

with Y defined by

$$Y = D_AD_A'D_BD_B'\ . \tag{4.7g}$$

On using the relations (4.7a)–(4.7g) we get, with $b=m_Am_B\xi_A\xi_B$, as in (3.15i),

$$(L_A{:}L_B)_1 = [(P_A\cdot P_B)^2 - (8P_A\cdot P_B b)x_Ax_B - (t\xi_A^2 m_A^2)x_A^2 - (t\xi_B^2 m_B^2)x_B^2 + (8b^2)x_A^2x_B^2](t^2/Y)\ . \tag{4.8}$$

Similarly we find that (4.6) reduces to

$$(L_A{:}L_B)_2 = -2(\{-t[P_A^2 - (t/2)] + 4t\xi_A^2 m_A^2 x_A^2\}/D_AD_A' + \{-t[P_B^2 - (t/2)] + 4t\xi_B^2 m_B^2 x_B^2\}/D_BD_B') + 16\ . \tag{4.9}$$

Since

$$D_AD_A' = t^2/4 + t\xi_A^2 m_A^2 x_A^2\ ,\qquad D_BD_B' = t^2/4 + t\xi_B^2 m_B^2 x_B^2\ , \tag{4.10}$$

Eq. (4.9) may be written in the more compact form

$$(L_A{:}L_B)_2 = 2t\{[P_A^2 + (t/2)]/D_AD_A' + [P_B^2 + (t/2)]D_BD_B'\}\ , \tag{4.11}$$

the contribution "16" of the double-seagull graph having canceled with part of that from the single-seagull graphs. The denominator Y in (4.5) has the form

$$Y = t^2b^2d_Ad_B\ , \tag{4.12}$$

where $d_A=\tau_A^2+x_A^2$, $d_B=\tau_A^2+x_B^2$, as in (3.15), but now with the τ's already evaluated at $\mu=0$, i.e., given by (3.16). The evaluation of the angular average $\langle (L_A{:}L_B)_1\rangle$ is therefore immediate, in terms of integrals

I_j defined by

$$I_0=\langle 1/d_A d_B\rangle,\quad I_1=\langle x_A x_B/d_A d_B\rangle\ , \tag{4.13a}$$

and

$$I_2=\langle x_A^2/d_A d_B\rangle\ ,\quad I_3=\langle x_B^2/d_A d_B\rangle\ ,\quad I_4=\langle x_A^2 x_B^2/d_A d_B\rangle\ , \tag{4.13b}$$

evaluated in an earlier paper.[5] The integral I_0 was already given in Eq. (3.15c) and I_1 can be expressed in a similar way,

$$I_1=(\tfrac{1}{2})[F_-(s,t)+(\pi/N_+)]\ , \tag{4.14a}$$

with $F_-(s,t)$ defined by (3.15d). The remaining I_j are simple combinations of arctangent functions and I_0,

$$\begin{aligned}I_2&=\tau_B^{-1}\arctan\tau_B^{-1}-\tau_A^2 I_0\ ,\\ I_3&=\tau_A^{-1}\arctan\tau_A^{-1}-\tau_B^2 I_0\ ,\\ I_4&=1-\tau_A\arctan\tau_A^{-1}-\tau_B\arctan\tau_B^{-1}+\tau_A^2\tau_B^2 I_0\ .\end{aligned} \tag{4.14b}$$

On using (4.8), (4.12), and (4.13a) and (4.13b) we get, finally,

$$\begin{aligned}\langle (L_A{:}L_B)_1\rangle=b^{-2}[&(P_A\cdot P_B)^2 I_0-(8P_A\cdot P_B b)I_1\\ &-(t\xi_A^2 m_A^2)I_2-(t\xi_B^2 m_B^2)I_3\\ &+(8b^2)I_4]\end{aligned} \tag{4.15}$$

and from (4.11) and the relations

$$\langle 1/d_A\rangle=\tau_A^{-1}\arctan\tau_A^{-1}\ ,\quad \langle 1/d_B\rangle=\tau_B^{-1}\arctan\tau_B^{-1}\ , \tag{4.16}$$

we get

$$\langle (L_A{:}L_B)_2\rangle=\{(4/\xi_A m_A t^{1/2})[P_A^2+(t/2)]\times\arctan(2\xi_A m_A/t^{1/2})\}+(A\to B)\ . \tag{4.17}$$

Note that since

$$P_A^2=4m_A^2-t,\quad P_B^2=4m_B^2-t\ , \tag{4.18}$$

the right-hand side of (4.17) is independent of s, as is necessary for seagull contributions, whereas the right-hand side of (4.15) does depend on s, not only via the quantity $P_A\cdot P_B$, but also through the dependence of the I_j on the quantity $y(s,t)$, given by (3.15h). We now turn to the task of simplifying and analyzing these results.

3. *Analysis of results*

The total discontinuity of the fourth-order amplitude is given by (4.3) as

$$[M^{(4)}]_t=-i(e^4/16\pi)[\langle (L_A{:}L_B)_1\rangle+\langle (L_A{:}L_B)_2\rangle]\ . \tag{4.19}$$

On substituting the expressions (3.15c) and (4.14) for the I_j in (4.15), and upon recalling that N_+ is imaginary while the F_+ are real, we find that the first term in the last set of square brackets in (4.19) may be separated into an imaginary part S_1 which comes from the π/N_+ terms present in each of the I_j and a real remainder R_1. After some algebra one gets

$$S_1=i(8\pi a^2/ps^{1/2})t^{-1} \tag{4.20a}$$

and

$$\begin{aligned}R_1=b^{-2}\{&[(P_A\cdot P_B)^2+t^2]F_+/2\tau_A\tau_B-8P_A\cdot P_B bF_-/2-t\xi_A^2 m_A^2\arctan\tau_B^{-1}/\tau_B-t\xi_B^2 m_B^2\arctan\tau_A^{-1}/\tau_A\\ &+8b^2(1-\tau_A\arctan\tau_A^{-1}-\tau_B\arctan\tau_B^{-1})\}\ .\end{aligned} \tag{4.20b}$$

The contribution of S_1 to $[M^{(4)}]_t$ is then given by (we again drop the superscripts on $M^{(4)}$)

$$[M]_t^{(0)}=(e^4a^2/2ps^{1/2})t^{-1}\ , \tag{4.21}$$

i.e., by a simple pole in t; this term is the analog of (3.21) and, like that term, will cancel in the difference amplitude with a corresponding contribution from M_I.

Turning now to the remainder R_1, we note first that, as inspection shows, only the terms involving arctangents are not analytic in t at $t=0$ and the nonanalytic part is easily split off by use of the relation

$$\arctan\tau^{-1}=\pi/2-\arctan\tau\ . \tag{4.22}$$

This yields a term

$$R_1'=-3\pi t^{1/2}[(\xi_A m_A)^{-1}+(\xi_B m_B)^{-1}] \tag{4.23a}$$

and a term which is analytic at $t=0$, defined by

$$R_1''=R_1-R_1'\ , \tag{4.23b}$$

which consists of the F_+ and F_- terms and the terms left-over after use of (4.22). It follows that the function R_1 can be expanded in a power series in $t^{1/2}$ near $t=0$; for example, the first term in this expression is given by

$$R_1(s,t)=R_1(s,0)+O(t^{1/2}/\overline{m}) , \quad (4.24a)$$

where, with $F'_+(s,0)$ and $F_-(s,0)$ given by (3.27) and (3.28), respectively,

$$R_1(s,0)=(8/m_A m_B)[a^2F'_+(s,0)-aF_-(s,0) +m_A m_B] . \quad (4.24b)$$

The seagull-graph contribution may similarly be split into a part S_2 which diverges at small t, but only as $t^{-1/2}$ instead of t^{-1}, and a remainder R_2; on use of (4.17), (4.18), and (4.22) we have, without approximation,

$$\langle (L_A:L_B)_2\rangle = S_2+R_2 , \quad (4.25)$$

where

$$S_2=\pi[(8m_A^2-t)/\xi_A m_A+(8m_B^2-t)/\xi_B m_B]t^{-1/2} \quad (4.26a)$$

and

$$R_2=-2[(8m_A^2-t)/\xi_A m_A t^{1/2}] \times\arctan(t^{1/2}/2\xi_A m_A)+(A\to B) . \quad (4.26b)$$

For small t we get

$$S_2=8\pi(m_A+m_B)t^{-1/2}+O(t^{1/2}/\overline{m}) \quad (4.27a)$$

and

$$R_2=-16+O(t/\overline{m}^2) . \quad (4.27b)$$

Following the notation of Eq. (3.20) we write the total discontinuity in the form

$$[M]_t=[M]_t^{(0)}+[M]_t^{(1)} , \quad (4.28)$$

where the first term is defined by (4.21) and the second term is the contribution from R_1 (4.20b), S_2 (4.26a), and R_2 (4.26b). Thus we get, without approximation,

$$[M]_t^{(1)}=-i(e^4/16\pi)(R_1+S_2+R_2) . \quad (4.29)$$

On use of (4.24) and (4.27), Eq. (4.29) yields

$$[M]_t^{(1)}=-ie^4\{[(m_A+m_B)/2]t^{-1/2} +[R_1(s,0)-16]/16\pi\} +O(t^{1/2}/\overline{m}) . \quad (4.30)$$

Note that unlike the case of the scalar Yukawa theory, in scalar QED there is now a $t^{-1/2}$ term in $[M]_t$. However, as we shall see, the coefficient of $t^{-1/2}$ will be greatly modified by the subtraction of $[M_I]_t$.

B. Iteration of second-order potential

The fourth-order transition amplitude generated by use of $V^{(2)}$ is formally given by

$$T^{(4)}=(2\pi)^{-3} \times\int dl\langle \mathbf{p}'|V^{(2)}|l\rangle\langle l|V^{(2)}|\mathbf{p}\rangle/[W(\mathbf{p})-W(l)+i\epsilon] , \quad (4.31)$$

where, with the convention (2.55),

$$V^{(2)}=V_a^{(2)}+V_b^{(2)}=z_{\text{op}}(U_a^{(2)}+U_b^{(2)})z_{\text{op}} \quad (4.32)$$

with $U_a^{(2)}$ defined by (2.57a) and $U_b^{(2)}$ by (2.57b), rather than (2.50a) and (2.50b). Corresponding to the separation (4.32), we have, in an obvious notation, with the superscript 4 dropped henceforth for ease of writing,

$$T=T_{aa}+T_{ab}+T_{ba}+T_{bb} . \quad (4.33)$$

The second and third terms in (4.33) are UV finite and independent of t. If the divergent integral for the fourth term is cut off at large $|l|$, this term is also t independent. In order to calculate the long-range potential we need only the discontinuities of these integrals, so that in the spirit of the preceding subsection we focus on T_{aa} alone, with

$$[T]_t=[T_{aa}]_t . \quad (4.34)$$

On use of (4.31), (4.32), (2.48a), (2.49a), and (2.51b), we get

$$T_{aa}=[e^4/(2\pi)^3](1+p^2/E_AE_B) \times\int dl\,\tilde{C}(p,l)(\mathbf{q}^2\mathbf{q}'^2)^{-1} , \quad (4.35a)$$

where

$$\tilde{C}(p,l)=(E'_AE'_B+l^2)C(p,l) , \quad (4.35b)$$

$E_A=E_A(p)$, $E'_A=E_A(l)$, etc., and $C(p,l)$ is defined by (3.41). On using the identity

$$E'_AE'_B+l^2=E_AE_B+p^2+[W^2(l)-W^2(p)]/2 \quad (4.36)$$

in the integrand of (4.35a) and the relation (2.43c) for a, we get

$$M_I=[e^4/(2\pi)^3]a^2\int dl\,C(p,l)(\mathbf{q}^2\mathbf{q}'^2)^{-1}-[e^4/(2\pi)^3]a\int dl(E'_AE'_B)^{-1}[W(p)+W(l)](\mathbf{q}^2\mathbf{q}'^2)^{-1} , \quad (4.37)$$

where M_I is related to T_{aa} by

$$M_I = 4E_A E_B T_{aa} , \tag{4.38}$$

consistent with the normalization conventions established in Sec. II. The angular integral in both lines of (4.37) coincides with the one already encountered in Eq. (3.31), if there we set $\mu=0$. We have not bothered to introduce such an infrared cutoff in (4.37) because, as the analysis of Sec. III already shows, the t discontinuity of the integrals over l are finite in the $\mu \to 0$ limit. Following the methods of Sec. III B, we readily find that with

$$M_I = D_I + iA_I , \tag{4.39}$$

so that

$$[M_I]_t = [D_I]_t + i[A_I]_t , \tag{4.40}$$

the discontinuity of the absorptive part A_I is given by

$$[A_I]_t = -i(e^4 a^2/2ps^{1/2})t^{-1} \tag{4.41a}$$

and that of the dispersive part D_I by

$$[D_I]_t = i(2e^4 a/\pi)\tilde{K} . \tag{4.41b}$$

Here $\tilde{K}$, defined in analogy with (3.52), is given by

$$\tilde{K} = (1/4t^{1/2})\int_{-1}^{1} dx\,(1-x^2)^{-1/2}\{(aC_2(p,l)/2) - [(W+W')/2E'_A E'_B]\} \tag{4.42a}$$

with $l^2=y$ related to x via (3.50) and (3.51), and C_2 defined by (3.54b). The analog of the K_1 term [Eq. (3.56)], which was found to make no contribution, has already been removed in (4.42a).

As in Sec. III, we can find the behavior of $\tilde{K}$ for small values of t, with the result that

$$\tilde{K} = X(p)(\pi/4)t^{-1/2} + O(t^{1/2}/\overline{m}) , \tag{4.42b}$$

where, with $a(s)$ defined by (2.43c) and $C_2(p)$ by (3.64b),

$$X(p) = aC_2(p)/2 - W(p)/E_A E_B . \tag{4.42c}$$

The corresponding expression for $[D_I]_t$ is

$$[D_I]_t = ie^4(aX/2)t^{-1/2} + O(t^{1/2}/\overline{m}) . \tag{4.43}$$

Expansion of the coefficient function $d(p^2)=aX/2$ in powers of p^2 yields

$$d(p^2) \equiv aX(p)/2 = d(0) + d'(0)p^2 + \cdots , \tag{4.44a}$$

where

$$d(0) = -(m_A + m_B + m_{AB})/2 , \tag{4.44b}$$

with m_{AB} the reduced mass

$$m_{AB} = m_A m_B/(m_A + m_B) , \tag{4.44c}$$

and $d'(0)$ is given by

$$d'(0) = -[(m_A+m_B)^{-1} + 2m_{AB}^{-1}]/4 . \tag{4.44d}$$

Thus, for small t and small p^2, Eq. (4.43) takes the form

$$[D_I]_t = ie^4 d(0)t^{-1/2} + ie^4 d'(0)p^2 t^{-1/2} + O(p^4 t^{-1/2}/\overline{m}^3) + O(t^{1/2}/\overline{m}) . \tag{4.45}$$

C. The two-photon-exchange potential at large r

We are now equipped to study the spectral function $\rho_{\text{diff}}^{(4)}$ associated with the difference amplitude $M - M_I$,

$$\rho_{\text{diff}}^{(4)} = ([M]_t - [M_I]_t)/2i . \tag{4.46}$$

On use of Eqs. (4.28) and (4.21) for $[M]_t$ and Eqs. (4.41a) and (4.41b) for $[M_I]_t$ we see that the discontinuity of $M^{(0)}$ and that of the absorptive part A_I of M_I cancel for all s, as in the case of the scalar Yukawa theory, so that

$$\rho_{\text{diff}}^{(4)} = ([M]_t^{(1)} - [D_I]_t)/2i , \tag{4.47}$$

where $[M]_t^{(1)}$ is given by (4.29) and $[D_I]_t$ by (4.41b) and (4.42a).

The long-range potential $U_{\text{LR}}^{(4)}$ can then be obtained by substituting (4.47) into (2.28b). The result is complicated and not very enlightening. To get a more useful result we consider the leading terms in an expansion of $U_{\text{LR}}^{(4)}$ in powers of r^{-1}. On use of the small-t expansions (4.30) and (4.43) we find that $\rho_{\text{diff}}^{(4)}$ assumes the form (3.71), viz.,

$$\rho_{\text{diff}}^{(4)} = a_2(s)t^{-1/2} + a_3(s) + O(t^{1/2}\overline{m}) , \tag{4.48}$$

where now however

$$a_2(s) = -e^4[m_A + m_B + 2d(p^2)]/4 \tag{4.49a}$$

and

$$a_3(s) = -e^4[R_1(s,0) - 16]/32\pi . \tag{4.49b}$$

On using Eqs. (3.73a) and (3.73b) to compute the long-range potential $U_{\text{LR}}^{(4)}$ associated with this spectral function we get

$$U_{\text{LR}}^{(4)} = c_2(s)r^{-2} + c_3(s)r^{-3} + O(r^{-4}\overline{m}^{-3}) , \tag{4.50a}$$

where now

$$c_2(s) = -(e^2/4\pi^2)^2[m_A + m_B + 2d(p^2)]/2m_A m_B \tag{4.50b}$$

and

$$c_3(s) = -(e^2/4\pi^2)^2[R_1(s,0) - 16]/16\pi m_A m_B . \tag{4.50c}$$

For small p^2 we get, using (4.44a)–(4.44c),

$$c_2(s) = c_2(s_0) + c_2'(s_0)p^2 + \cdots , \tag{4.51}$$

where

$$c_2(s_0) = (e^2/4\pi)^2/2(m_A + m_B) \tag{4.52a}$$

and $c_2'(s_0)$, defined as the derivative of $c_2(s)$ with respect to p^2 at $p^2=0$, is given by

$$c_2'(s_0)=(e^2/4\pi)^2[(m_A+m_B)^{-1}+2m_{AB}^{-1}]/4m_Am_B\ . \tag{4.52b}$$

We note that the $1/r^2$ potential is repulsive, whatever the signs of the individual charges may be.

To evaluate $c_3(s)$ for $p^2\to 0$, we use the definition (4.24b) of $R_1(s,0)$ and Eqs. (3.29a) and (3.29b). We then find that

$$R_1(s_0,0)=\tfrac{104}{3}\ , \tag{4.53}$$

so that from (4.50c) we get

$$c_3(s_0)=-(e^2/4\pi)^2(7/6\pi)(m_Am_B)^{-1}\ . \tag{4.54}$$

Note that both the r^{-2} and r^{-3} terms in $U_{\rm LR}^{(4)}$ vanish when either m_A or m_B goes to infinity. In an atom made of a spin-0 proton and a spin-0 electron the r^{-2} term would give rise to a level shift of order (m_e/m_p) times the fine structure, of order $\alpha^4 m_e$, so that this term represents a recoil correction to the fine structure. This is in contrast to the case of the zero-mass scalar Yukawa theory where the coefficient of r^{-2} vanishes only if both of the masses become large; for that theory, in a hydrogenic bound state the contribution from two-quantum exchange would then be as large as the fine structure. We shall discuss these differences further in Sec. V. In the following subsection we consider the $m_B\to\infty$ limit of scalar QED in more detail.

D. Infinite-mass limit

It is useful to consider the limit $m_B\to\infty$, with p^2 and t fixed, as we did for the scalar Yukawa theory in Sec. III D. From Eq. (4.29) we have

$$\frac{[M]_t^{(1)}}{m_B}=-i(e^4/16\pi)\left[\frac{R_1}{m_B}+\frac{S_2}{m_B}+\frac{R_2}{m_B}\right]. \tag{4.55}$$

From inspection of the definition (4.16) and for S_2 and R_2 one readily finds that

$$\lim_{m_B\to\infty}\frac{S_2}{m_B}=8\pi t^{-1/2} \tag{4.56a}$$

and

$$\lim_{m_B\to\infty}\frac{R_2}{m_B}=0\ . \tag{4.56b}$$

With regard to the limit of R_1/m_B as $m_B\to\infty$, since the factor $b^{-2}=(\xi_Am_A\xi_Bm_B)^{-2}$ varies as m_B^{-2} for large m_B, the terms not involving F_+ and F_- in (4.20b) make zero contribution to this limit. The same holds for the term involving $F_+(s,t)$ since, according to (3.75), F_+ vanishes in this limit. Finally, from the definition (3.15d) one finds that $F_-(s,t)$ is finite in the limit $m_B\to\infty$, so that the contribution of the F_- term in (4.20b) is also zero in this limit. Thus

$$\lim_{m_B\to\infty}\frac{R_1}{m_B}=0 \tag{4.56c}$$

and (4.55) yields

$$\lim_{m_B\to\infty}\frac{[M]_t^{(1)}}{m_B}=\frac{-ie^4}{2t^{1/2}}\ . \tag{4.57}$$

Turning now to $[D_I]_t$, defined by (4.41b), we note first that since $a=2p^2+2E_AE_B$,

$$\lim_{m_B\to\infty}\frac{a}{m_B}=2E_A\ . \tag{4.58}$$

Combining this with Eq. (3.79) and the definition (4.42a) of $\tilde{K}$ we find, after some algebra, that

$$\lim_{m_B\to\infty}\tilde{K}=\frac{-\pi\tilde{f}_A(p^2,t)}{8E_At^{1/2}}\ ; \tag{4.59}$$

here

$$\tilde{f}_A(p^2,t)=\frac{2E_A}{\pi}\int_{-1}^{1}\frac{dx\,(1-x^2)^{-1/2}}{(E_A+E_A')} \tag{4.60}$$

is the counterpart of the function $f_A(p^2,t)$ introduced in the discussion of the $m_B\to\infty$ limit of the scalar Yukawa theory, defined by (3.80b). It follows from (4.41b), (4.58), and (4.59) that

$$\lim_{m_B\to\infty}\frac{[D_I]_t}{m_B}=\frac{-ie^4}{2t^{1/2}}\tilde{f}_A(p^2,t)\ . \tag{4.61}$$

This corresponds to Eq. (3.80a). Note the minus sign in (4.61) and the fact that $\tilde{f}_A$ has the value unity at $t=0$ for any value of p^2:

$$\tilde{f}_A(p^2,0)=1\ . \tag{4.62}$$

For the difference spectral function $\rho_{\rm diff}^{(4)}(s,t)$, defined by (4.47), we get

$$\lim_{m_B\to\infty}\frac{\rho_{\rm diff}^{(4)}}{m_B}=\frac{e^4}{4t^{1/2}}[\tilde{f}_A(p^2,t)-1] \tag{4.63}$$

as the counterpart of (3.81). The quantity $U_A^{(4)}(r;p^2)$, defined as the $m_B\to\infty$ limit of the long-range part of the two-photon-exchange potential, is therefore given by

$$U_A^{(4)}(r,p^2)=\lim_{m_B\to\infty}U_{\rm LR}^{(4)}=\frac{e^4}{16\pi^2m_Ar}\int_0^\infty\frac{dt}{4t^{1/2}}[\tilde{f}_A(p^2,t)-1]e^{-t^{1/2}r}\ . \tag{4.64}$$

So for scalar QED both $M^{(4)}$ and M_I contribute to the long-range potential in the infinite-mass limit.

From the relation (4.62), it follows that the integrand of (4.64) behaves at $t^{1/2}$ rather than $t^{-1/2}$ for small t. To find the precise behavior of $U_A^{(4)}$ at large r, we need the behavior of $\tilde{f}_A(p^2,t)$ for small t. To this end we note first that the factor $(E_A+E_A')^{-1}$ in the integrand of (4.60) may be expanded in powers of $(p^2-l^2)/(2E_A)^2$, most simply by use of the identity

$$\frac{1}{E_A+E_A'}=\frac{1}{2E_A}\left[1+\frac{p^2-l^2}{(E_A+E_A')^2}\right]. \tag{4.65a}$$

Iteration of this equation yields

$$\frac{1}{E_A+E'_A}=\frac{1}{2E_A}\left[1+\frac{p^2-l^2}{(2E_A)^2}+2\left[\frac{p^2-l^2}{(2E_A)^2}\right]^2+\cdots\right] . \tag{4.65}$$

From (3.51) and (3.50) we infer that, for $\mu=0$,

$$p^2-l^2=-\{[t(t+4p^2)]^{1/2}x+t\}/2 . \tag{4.65c}$$

On substituting (4.65c) into (4.65b) we get

$$\frac{2E_A}{E_A+E'_A}=1-\frac{t}{8E_A^2}+2\frac{t(t+4p^2)x^2+t^2}{16E_A^4}+\cdots$$
$$+2\frac{2t^{3/4}(t+4p^2)^{1/2}x}{16E_A^4}+\cdots . \tag{4.65d}$$

Here the ellipsis in the first line represents monomials of the form $t^n(p^2)^m$, with n and m non-negative integers such that $n+m\geq 3$, multiplied by *even* powers of x, whereas the ellipsis in the second line represents higher-order terms of the type shown, multiplied by *odd* powers of x. Although the function $2E_A/(E_A+E'_A)$ for fixed p^2 and $x\neq 0$ is not analytic in t at $t=0$ (except when $p^2=0$), the integral (4.60) *is* analytic in t at $t=0$, as well as continuous in p^2 at $p^2=0$, because the terms odd in x make no contribution to the integral. Since

$$\frac{1}{\pi}\int_{-1}^{1}dx(1-x^2)^{-1/2}=1 ,$$
$$\frac{1}{\pi}\int_{-1}^{1}dx(1-x^2)^{-1/2}x^2=\tfrac{1}{2}$$

we get, from (4.60) and (4.65d),

$$\tilde{f}_A(p^2,t)=1-\frac{t}{8E_A^2}+\frac{p^2t}{4E_A^4}+\frac{3t^2}{16E_A^4}+\cdots . \tag{4.66}$$

It follows from (4.64) and (4.66) that

$$U_A^{(4)}(r,p^2)=\left[\frac{m_A}{E_A}\right]^2\left[1-\frac{2p^2}{E_A^2}\right]U_A^{(4)}(r,0)+O\left[\frac{1}{r^5}\right] , \tag{4.67}$$

where

$$U_A^{(4)}(r,0)=-\frac{(e^2/4\pi)^2}{8m_A^3r^4} . \tag{4.68}$$

So for scalar QED in the $m_B\to\infty$ limit the asymptotic fourth-order potential is much smaller than the corresponding potential in the scalar Yukawa theory. In Appendix A we show that (4.68) is also obtained from the Klein-Gordon equation for a particle of charge e_A and mass m_A, moving in a static electric field with potential $A^0(\mathbf{x})=e_B/4\pi r$, as one would expect.

V. SUMMARY AND DISCUSSION

In this section we summarize our results and comment on their significance and on some differences between our results and those of other workers. We then discuss some questions that have come up in the course of our analysis, such as the elimination of the infrared divergence in the fourth-order potential and the use of our results for the calculation of bound-state energies.

A. Summary

In Sec. II of this paper we introduced a general formalism for describing the scattering or bound states of a two-body system. This formalism is based on a relativistic two-body equation which for spinless particles has the form

$$(E_A^{\rm op}+E_B^{\rm op}+V)\phi=E\phi , \tag{5.1}$$

where $E_i^{\rm op}=(\mathbf{p}_{\rm op}^2+m_i^2)^{1/2}$ and $\mathbf{p}_{\rm op}$ is the c.m. system momentum operator. The potential V is calculated from the Feynman diagrams, both irreducible and reducible, associated with the underlying field theory, as a power series in the coupling constant. In this paper we have computed the potential arising from the exchange of one and two quanta for two different field theories. In both cases it is convenient to extract some kinematical factors which are unity in the nonrelativistic limit and to write V in the form

$$V=N_{\rm op}UN_{\rm op} , \tag{5.2a}$$

where $U=U(r;s)$ is a local, energy-dependent potential given as a power series in the coupling constant:

$$U(r;s)=U^{(2)}+U^{(4)}+\cdots . \tag{5.2b}$$

The first theory is the scalar Yukawa theory with interaction Lagrangian density given by (2.34). In this case we choose

$$N_{\rm op}=(m_Am_B/E_A^{\rm op}E_B^{\rm op})^{1/2} . \tag{5.3}$$

Then $U^{(2)}$ is just the Yukawa potential:

$$U^{(2)}=-(g^2/4\pi)\exp(-\mu r)/r , \tag{5.4}$$

with $g^2=G_AG_B/4m_Am_B$ and μ the mass of the neutral boson. For the fourth-order term, $U^{(4)}$, we have emphasized the calculation of its long-range part $U_{\rm LR}^{(4)}$ in the limit $\mu=0$. This is given by

$$U_{\rm LR}^{(4)}=(16\pi^2m_Am_Br)^{-1}\int_0^\infty dt\,\rho_{\rm diff}^{(4)}(s,t)\exp(-t^{1/2}r) . \tag{5.5}$$

Here $\rho_{\rm diff}^{(4)}$ is the difference fourth-order spectral function given by (3.69). The K term in (3.69), defined by (3.62), involves a transcendental integral which can be expanded in powers of $t^{1/2}$. The corresponding expression for $U_{\rm LR}^{(4)}$ is an expansion in inverse powers of r, of the form

$$U_{\rm LR}^{(4)}=c_2(s)r^{-2}+c_3(s)r^{-3}+\cdots , \tag{5.6}$$

where $c_2(s)$ and $c_3(s)$ are given by (3.73b) and (3.72). In the limit $p\to 0$,

$$c_2(s)\to-(g^2/4\pi)^2(m_A^2+m_Am_B+m_B^2)/2m_Am_B(m_A+m_B) \tag{5.7a}$$

and

$$c_3(s)\to-(g^2/4\pi)^2/3\pi m_Am_B . \tag{5.7b}$$

The second theory we have considered is scalar QED, with Lagrangian density defined by (2.35). We now choose

$$N_{\rm op}=[1+(\mathbf{p}_{\rm op}^2/E_A^{\rm op}E_B^{\rm op}]^{1/2} \ . \quad (5.8)$$

Then the long-range part $U_{\rm LR}^{(2)}$ of the lowest-order term $U^{(2)}$ is the Coulomb potential

$$U_{\rm LR}^{(2)}=U_C\equiv e_A e_B/4\pi r \ . \quad (5.9)$$

There is also a short-range part $U_{\rm SR}^{(2)}$, given by (2.57b). The fourth-order term $U^{(4)}(r;s)$ has a long-range part $U_{\rm LR}^{(4)}$ given by (5.5), where now the spectral function $\rho_{\rm diff}^{(4)}$ is given by (4.47). The coefficients $c_2(s)$ and $c_3(s)$ in (5.6) are then given by (4.50b) and (4.50c). For small p^2 we have

$$c_2(s)=(e^2/4\pi)^2/2(m_A+m_B) +(e^2/4\pi)^2(p^2/4m_Am_B)[(m_A+m_B)^{-1} +2m_{AB}^{-1}]+\cdots \quad (5.10a)$$

with m_{AB} the reduced mass and, for $p^2=0$,

$$c_3(s)\to-(e^2/4\pi)^2(7/6\pi)/m_Am_B \ . \quad (5.10b)$$

For $m_B\to\infty$ both $c_2(s)$ and $c_3(s)$ vanish and $U_{\rm LR}^{(4)}$ is given by (4.64); the leading terms for large r are given by (4.67), which for $p^2=0$ reduces to

$$U_A^{(4)}(r,0)=-(e^2/4\pi)^2/8m_A^3r^4 \ . \quad (5.11)$$

B. Discussion

If the masses of the two particles are both finite, then in each theory the potential $U^{(4)}$ has a leading asymptotic term[7] which, for small momentum, behaves as r^{-2} for large r and is repulsive. This term, but none of the other terms in (5.6), has previously been obtained in a quantum-field theoretic context by Gupta and Radford;[8] however it is not clear in what two-body equation these authors mean to use their result. An earlier calculation of the r^{-2} term by Iwasaki[9] gave zero; this author, who uses Coulomb gauge in a nongauge-invariant formalism, also does not specify the two-body equation he has in mind. There are additional terms in $U^{(4)}$ proportional to powers of p^2 and to higher inverse powers of r. Thus for general masses, the leading correction to the Coulomb potential in scalar QED is a potential that behaves asymptotically as r^{-2}. Note that this asymptotic correction potential contains five less inverse powers of r than the retarded two-photon-exchange potential, first calculated by Casimir and Polder, that acts between neutral systems. It also contains two less inverse powers of r than the well-known r^{-4} potential that acts between a charge and a nondegenerate neutral system. The reason for this much slower decrease of our $U^{(4)}$ has to do with the fact that in the present case there are states arbitrarily near in energy to the initial state of the two charges, which are linked to that state by photon emission. Therefore the behavior of the two-photon-exchange potential between two charges resembles that of a charge and a degenerate neutral system, where we have previously found a similar behavior, i.e., an anomalously small inverse power in $V_{2\gamma}$.

It should be stressed that the form of the correction potential is specific to the use of Eq. (5.1), and to our method for extracting potentials from scattering amplitudes. In this formalism, the potentials $V^{(2)}$ and $V^{(4)}$ are uniquely determined, and have the indicated forms. In particular, while we have used the Feynman gauge in our explicit calculations, our results are manifestly gauge independent.

In other approaches to calculating corrections to the Coulomb potential, such as that of Austen and DeSwart,[10] a different wave equation is used, which does not have the Hamiltonian form of Eq. (5.1), and the potentials have a gauge ambiguity. These authors make use of this ambiguity to arrange that the entire correction to the Coulomb potential is an energy-dependent potential that is of order e^2, not e^4. While we believe that their potential does not have as straightforward a meaning as ours, the ultimate advantages of either approach depend on the ease of calculating specific physical quantities, and that matter remains to be decided.

It is instructive to consider the situation in which one mass is much larger than the other ($m_B\gg m_A$), the so-called "external field approximation." In this circumstance, the behavior of the two theories we have analyzed differs considerably. In the Yukawa theory, the r^{-2} term survives in this limit, while in scalar QED it does not. This difference can be understood by using an old result, which we discuss in Appendix D. This result states that, with radiative corrections neglected, in the limit of infinite mass of one particle, the quantum-field-theory amplitudes approach the amplitudes generated by solving the Klein-Gordon equation with only the Coulomb potential included, rather than those calculated from Eq. (5.1). These two amplitudes differ by certain pair terms that are contained in the Klein-Gordon equation and not in Eq. (5.1) when only $V^{(2)}$ is included there. Thus the terms in $V^{(4)}$ that survive the limit $m_B\to\infty$ are these "missing" pair terms, and as shown in Appendix A, these have different forms in the two theories that we consider.

With these results we have completed our study, using dispersion relation methods, of the two-photon-exchange potential that acts between spinless systems. In previous papers we have considered the form that this potential takes between two neutral systems, and between a charged system and a neutral system which may or may not possess degenerate ground states.[11,12] In a future paper we will study the corresponding two-photon-exchange potential between spin-$\frac{1}{2}$ systems. We conclude this paper with some remarks about issues related to Eq. (5.1) and the potential $V^{(4)}$.

C. The Coulomb phase

An important property of Eq. (5.1), with V defined by (5.2a), (5.2b), and (5.5), is that the scattering cross sections calculated from this equation are free of certain infrared singularities that are contained in the field theory amplitude $M^{(4)}$, and in higher-order field-theory amplitudes. These singularities sum up to give an infinite phase factor, which does not affect the cross sections. This phase factor is well known in nonrelativistic Coulomb scattering, where it has been discussed for the case in which there is an additional, short-range potential acting as well.[13] It has also been discussed in the context of relativistic Coulomb scattering.[14]

Let us first verify that the solution of (5.1) yields a finite-scattering amplitude. One can show (see Appendix B) that the asymptotic form of a scattering solution of (5.1), associated with an incoming plane wave $e^{i\mathbf{k}\cdot\mathbf{r}}=e^{ikz}$ is given by

$$\phi_{\text{rel}}=e^{i(kz+N_{\text{rel}})}+\frac{f_{\text{rel}}}{r}e^{i(kr-N_{\text{rel}})}+O\left[\frac{1}{r^2}\right], \quad (5.12a)$$

where

$$N_{\text{rel}}=a_{\text{rel}}\ln k(r-z)+b_{\text{rel}} \quad (5.12b)$$

with

$$a_{\text{rel}}(k)=\eta(k)a(k), \quad b_{\text{rel}}(k)=\arg\Gamma(1+ia_{\text{rel}}), \quad (5.12c)$$

and

$$a(k)=\frac{e_Ae_B}{4\pi}\frac{m_{\text{AB}}}{k}. \quad (5.12d)$$

The factor $\eta(k)$ is a correction factor arising from the relativistic kinematics, given by

$$\eta(k)=\frac{E_A(k)E_B(k)}{E_A(k)+E_B(k)}\frac{z^2(k)}{m_{AB}}, \quad (5.12e)$$

with m_{AB} the reduced mass. The scattering amplitude f_{rel} is given by

$$f_{\text{rel}}=(f_C^{\text{rel}}+\delta f_{\text{SR}}^{\text{rel}}). \quad (5.12f)$$

Here f_C^{rel} is the relativistic counterpart of f_C, the nonrelativistic amplitude for a pure Coulomb potential, viz.,

$$f_C^{\text{rel}}=\eta(k)f_C, \quad (5.12g)$$

with

$$f_C=\frac{a(k)}{k(1-\cos\theta)}, \quad (5.12h)$$

and $\delta f_{\text{SR}}^{\text{rel}}$ is a correction to f_C^{rel} arising from a residual interaction, of order e^2, associated with the relativistic kinematics, and from $V^{(4)}$, both of which fall off more rapidly than $1/r$. It follows, as in the nonrelativistic case, that the differential scattering cross section is given by $|f_{\text{rel}}|^2$ and so is finite for $\theta>0$.

It is interesting to compare this situation with the nonrelativistic case where, as is well known, the Born approximation gives the exact answer

$$f_C^{\text{Born}}=f_C. \quad (5.13)$$

In scalar QED, the Born approximation is no longer exact, but it does appear to give exactly the most singular part of the amplitude, for any values of the masses. To verify this we note that with relativistic kinematics the c.m. scattering amplitude f is related to the c.m. transition amplitude T by

$$f=\frac{-E_A(k)E_B(k)}{2\pi W(k)}T \quad (5.14a)$$

and hence to the Feynman amplitude $M=4E_AE_BT$ by

$$f=\frac{-1}{8\pi W(k)}M. \quad (5.14b)$$

From (2.44) we then find that

$$f_{\text{sing}}^{\text{Born}}=\frac{-1}{8\pi W(k)}M_{\text{sing}}^{(2)}=f_C^{\text{rel}}. \quad (5.14c)$$

Returning now to the question of the divergent phase, in analogy with the usual field-theoretic treatment where the IR divergence is controlled by replacing $1/k^2$ by $1/(k^2-\mu^2)$, we introduce a cutoff on $U_C(r)$ in (5.9). For simplicity we replace $U_C(r)$ by $F(\mu r)U_C(r)$, where $F(\mu r)=1$ for $r\le\mu^{-1}$ and $F(\mu r)=0$ for $r>\mu^{-1}$. The asymptotic form of ϕ for $r\gg\mu^{-1}$ is then $e^{ikz}+f[\mu]e^{ikr}/r$. By an extension to (5.1) of the method, involving analysis of the partial wave amplitudes, described for the nonrelativistic problem by, e.g., Goldberger and Watson,[13] one can establish that $f[\mu]$ has the form

$$f[\mu]=e^{i\Phi_{\text{rel}}[\mu]}f_{\text{rel}}[\mu]+\delta f'. \quad (5.15a)$$

Here

$$\Phi_{\text{rel}}[\mu]=-a_{\text{rel}}(k)\ln(k\mu^{-1}) \quad (5.15b)$$

is a relativistic analog of the phase $\Phi[\mu]=-a(k)\ln(k\mu^{-1})$ found in the nonrelativistic pure Coulomb problem, obtained by setting $r=\mu^{-1}$ in the asymptotic form of the outgoing spherical wave, and $f_{\text{rel}}[\mu]$ has the limit f_{rel} [Eq. (5.12f)]. The quantity $\delta f'$ oscillates rapidly as $\mu\to 0$ and therefore does not contribute to the physical cross section.

We therefore find, as expected, that in the presence of the Coulomb interaction and shorter-range interactions, there is a μ-dependent phase factor that multiplies the finite-scattering amplitude $f_{\text{rel}}[\mu=0]$. The shorter-range potentials contained in $V^{(4)}$ do not contribute to this phase. They do of course affect the scattering amplitude $f_{\text{rel}}[\mu=0]$.

If the phase factor $\exp(i\Phi_{\text{rel}})$ is expanded in powers of e^2, it generates terms in M_I which diverge as $\mu\to 0$, such as the one, e.g., which would arise from A_I given in (4.41a). The divergent dependence on the cutoff μ in Eq. (5.15b), which arises from the Coulomb-exchange graphs, has a quite different origin from the usual infrared divergence arising from radiative correction graphs. The latter divergence cancels, order by order in total cross sections, when added to the divergence arising from the emission of soft photons. When calculated to order n in e^2, the Coulomb-exchange divergence occurs with a phase differing by $(n-1)\pi/2$ from the phase of the soft-photon infrared divergence. As a result, when the corrections in different orders are added together and "exponentiated," the Coulomb divergence appears as an imaginary exponent such as the one in Eq. (5.15b), whereas the soft-photon divergence and the radiative correction infrared divergences contribute *real* exponential factors, whose divergent parts cancel, leaving a finite factor which depends on the energy resolution of the detectors. Since these results are well known, and have been treated in full detail, for example, by Yennie *et al.*[3] we do not review them here any further.

D. Calculation of bound-state energies

Equation (5.1) may be used for the computation of bound-state energies, by looking for normalizable solutions in the region $E < m_A + m_B = \sqrt{s_0}$. In practice one would use an iterative procedure, first approximating $U(r,s)$ by $U(r,s_0)$ and finding an approximate energy $E_b^{(0)} = \sqrt{s_b^{(0)}}$, then replacing $U(r,s_0)$ by $U(r,s_b^{(0)})$ and finding an improved energy $E_b^{(1)} = \sqrt{s_b^{(1)}}$, and so on.

The association of discrete eigenvalues of $h_{\rm op}$ with bound-state energies is justifiable despite the parametric dependence of V on s, at least if this dependence is analytic in s and not too strong; for clarity we now write $V[s]$ instead of V, to make the dependence of V on s explicit. Suppose that the associated transition amplitude $T(s,t)$ can be continued analytically below s and that one finds a pole at a value $s = s_b = E_b^2$. From the viewpoint of S-matrix theory, such a pole corresponds to the existence of a bound-state of total energy E_b. We argue that such a pole is associated with the existence of a discrete eigenvalue of $h_{\rm op}$, with

$$h_{\rm op} = h_{\rm op}[s] = E_A^{\rm op} + E_B^{\rm op} + V[s] , \quad (5.16)$$

as follows. Recall first that the associated transition operator $T_{\rm op} = T_{\rm op}[s]$ may be written in the form, with $E = \sqrt{s}$,

$$T_{\rm op}[s] = V[s] + V[s]\frac{1}{E - h_{\rm op}[s] + i\epsilon}V[s] . \quad (5.17)$$

Suppose now that there exists a normalizable function ϕ_b such that

$$h_{\rm op}[s_b]\phi_b = E_b\phi_b . \quad (5.18)$$

Let us assume, as seems reasonable, that for s near to but not equal to s_b, the operator $h_{\rm op}[s]$ has a corresponding eigenfunction $\phi_b[s]$ and associated eigenvalue $E_b[s](\neq\sqrt{s}\,)$, which approach ϕ_b and $E_b = \sqrt{s_b}$, respectively, as $s \to s_b$:

$$\phi_b[s] \to \phi_b[s_b] = \phi_b , \quad (5.19a)$$

$$E_b[s] \to E_b[s_b] = E_b . \quad (5.19b)$$

With $h_{\rm op}[s]$ Hermitian in the neighborhood of s_b, there will be for a fixed value of s a complete set of orthogonal eigenfunctions, $\{\phi_\beta[s]\}$ and one of these functions may be chosen to be $\phi_b[s]$. We may then write

$$1_{\rm op} = P_b[s] + Q_b[s] , \quad (5.19c)$$

where $P_b[s]$ is the projection operator onto $\phi_b[s]$ and $Q_b[s]$ that for the orthogonal complement:

$$P_b[s] = |\phi_b[s]\rangle\langle\phi_b[s]| , \quad (5.19d)$$

$$Q_b[s] = \sum_{\beta\neq b} |\phi_\beta[s]\rangle\langle\phi_\beta[s]| . \quad (5.19c)$$

For simplicity we shall assume that $\phi_b[s]$ corresponds to a nondegenerate S state. On inserting (5.19c) into (5.17) and taking the matrix element of (5.17) with plane wave states $|\mathbf{p}\rangle$ and $|\mathbf{p}'\rangle$ we then get

$$T(s,t) = \langle\mathbf{p}'|V[s]|\mathbf{p}\rangle + \frac{N_b^*[s]N_b[s]}{E - E_b[s]} + T_{\rm rem}(s,t) , \quad (5.20a)$$

where

$$N_b[s] = \langle\phi_b[s]|V[s]|\mathbf{p}\rangle \quad (5.20b)$$

and $T_{\rm rem}(s,t)$ denotes the contribution from $Q_b[s]$. We see that unless $N_b[s_b]$ vanishes the second term in (5.20a) has a pole at $s = s_b$. In the absence of (unlikely) cancellations, $T(s,t)$ will therefore have a pole at $s = s_b$. The argument is readily extended to bound states of any angular momentum and degree of degeneracy.

From the definition (5.2a) of $V[s]$, we see that $V[s]$ will be Hermitian as long as $U(r;s)$ is real. From the representation (5.5) we see that a sufficient condition for the reality of $U_{\rm LR}(r;s)$ is that of the difference spectral function $\rho_{\rm diff}(s,t)$. This function is real through fourth order, as our calculations show, even for $\mu=0$. In higher orders $\rho_{\rm diff}(s,t)$ will certainly become complex if $s > s_1 = \sqrt{s_0} + \mu$, the threshold for the production processes $A + B \to A' + B' + \phi$; the resultant non-Hermiticity of $V[s]$ is then required to satisfy unitarity. For $s < s_1$, and in particular for $s < s_0$, the precise character of $V[s]$ in higher orders with regard to Hermiticity requires further investigation. It should be noted that the Hermiticity of $V[s]$ is not necessary for the existence of real eigenvalues. For $\mu=0$ and $s < s_0$ we indeed expect $V[s]$ to become complex in higher orders in order to permit the eigenvalues associated with discrete states, other than the ground state, to develop an imaginary part, associated with the width for radiative decay. In this connection, the corresponding eigenfunctions $\phi_b[s_b]$ can be used to compute radiative decay amplitudes in the standard way, with some relativistic corrections thereby included.

To see that our assumptions are not unreasonable, we consider a soluble model involving an s-dependent potential operator V. We choose for V a separable potential of the form

$$V[s] = -\lambda(s)|v\rangle\langle v| . \quad (5.21a)$$

Here $\lambda(s)$ is an energy-dependent coupling constant and in $\mathbf{r}$ space,

$$V[s]f(\mathbf{r}) \equiv -\lambda(s)v(r)\int v^*(r)f(\mathbf{r})d\mathbf{r} \quad (5.21b)$$

with $v(r)$ square integrable. Let us assume that for some value $s = s_b < s_0$ there is a normalizable function $\phi_b(\mathbf{r})$ which is an eigenfunction of $h_{\rm op}[s_b]$ with eigenvalue $E_b = \sqrt{s_b}$. Then the Fourier transform $\tilde{\phi}_b(\mathbf{p})$ of $\phi_b(\mathbf{r})$ satisfies the equation

$$W(p)\tilde{\phi}(\mathbf{p}) - \lambda(s_b)\tilde{v}(p)\langle\tilde{v}|\tilde{\phi}\rangle = E_b\tilde{\phi}(\mathbf{p}) . \quad (5.22)$$

This implies that E_b is an eigenvalue if and only if $\lambda(s_b)$ is such that

$$1=\lambda(s_b)\int d\mathbf{p}\frac{|\tilde{v}(p)|^2}{W(p)-\sqrt{s_b}} . \tag{5.23}$$

On the other hand for this case $T(s,t)$ can be computed exactly

$$T(s,t)=\frac{\lambda(s)\langle \mathbf{p}'|v\rangle\langle v|\mathbf{p}\rangle}{1-\lambda(s)\int d\mathbf{p}\frac{|\tilde{v}(p)|^2}{W(p)-\sqrt{s}}} . \tag{5.24}$$

Since the denominator in (5.24) vanishes at $s=s_b$ if and only if Eq. (5.23) is satisfied, it follows that for a wide class of function $\lambda(s)$ and $\tilde{v}(p)$, the transition amplitude has a pole at $s=s_b$ if and only if $h_{\rm op}[s_b]$ has $\sqrt{s_b}$ as a discrete eigenvalue.

E. Effect of the delta-function potential in scalar QED

We have seen in Eq. (2.44) that in the one-photon-exchange contribution to the scattering of spin-0 charged particles there occurs a term which goes as a constant at large t, corresponding to a delta-function potential. When such a potential is iterated, it produces ultraviolet divergences. While these divergences do not contribute to the long-range potential that is of interest to us, their occurrence cannot but make us uneasy about the status of the theory that we are considering. Indeed, it is well known that the Schrödinger equation with a delta-function potential has no scattering solutions. We comment here briefly on this state of affairs; as mentioned earlier, an approach which avoids the occurrence of delta functions is described in Appendix F.

1. Composite spinless charged particles

All known spinless charged particles are regarded as composites. Examples are the ^{4}He nucleus and six-times ionized ^{16}O. For the scattering of two such composites, the scattering amplitude we obtained in Eq. (2.44) must be modified through multiplication by a factor $G(q^2)$, where q is the photon four-momentum. $G(q^2)$ is essentially the product of the charge form factors of the individual particles and satisfies

$$G(0)=1 \tag{5.25a}$$

and

$$G(q^2)\to 0 \quad \text{as } q^2\to\infty . \tag{5.25b}$$

The result of this modification is to change the potentials $U_a^{(2)}$ and $U_b^{(2)}$ of Eqs. (2.50a)–(2.50c). For $U_a^{(2)}$ the change amounts to adding an additional short-range potential which falls off exponentially with r. For $U_b^{(2)}$, the change is more significant. Instead of being a delta function this potential becomes a short-range potential which decreases exponentially with r. The form of this potential, which we will call U, is essentially the Fourier transform of $G(q^2)$. Now U can be iterated without producing ultraviolet divergences, but there is no need to do so, because its iteration will generate only short-range fourth-order potentials that we will anyway not include in $U_{\rm LR}^{(3)}$.

We note further that in the dispersion theory calculation that we use for the two-photon-exchange graphs, we only need to know the photon emission and absorption amplitudes on the photon mass shell. However, in these graphs the charged particles can be off the mass-shell ($p^2\neq m^2$). By use of the Ward identity it can be shown that, for this case,

$$G(q^2=0,p^2)=1+O(p^2-m^2) . \tag{5.26}$$

On substituting this form into the expression for $[M^{(4)}]_t$ we obtain the previous result from the first term in G. The other term generates corrections that are dependent on the charged-particle structure and which correspond to potentials that fall off at least as fast as r^{-4} for large r. These corrections and other terms due to excited states of the charged particles are similar in form to those found previously in the potential between a charge and a neutral system.[2] We conclude that the results summarized in Sec. V A apply with minor modifications to the potential acting between composite spinless charges.

2. Elementary spinless charges

It is conceivable that elementary charged particles with spin-zero exist. A possible example would occur if there is more than one doublet of Higgs mesons, in which case one charged Higgs-boson particle will survive spontaneous symmetry breaking. It has been known for a long time that the QED of spinless charges is renormalizable.[6] However, in order to carry out the renormalization, in addition to the familiar charge and mass counterterms it is necessary to introduce additional terms, such as $\lambda\phi_A^\dagger\phi_A\phi_B^\dagger\phi_B$. A term of this kind will also generate, in order λ, a constant amplitude similar in form to the second, constant term in Eq. (2.44). This appears to imply that a proper treatment of the higher-order effects of the delta-function potential $U_b^{(2)}$ will necessarily involve a study of divergent radiative corrections and the overall renormalization of the theory. This is beyond the scope of this paper.

F. Concluding remarks

We have calculated the two-photon correction to the Coulomb potential acting between charged spin-0 particles. This correction is given in Eq. (5.5). The additional potential is most conveniently expressed as a power series in r^{-1}. When substituted into Eq. (5.1), this potential, together with the Coulomb potential (2.48a), generates scattering amplitudes and bound-state energies that are correct to order e^4, and to all orders in m_A/m_B. To order zero in this ratio, the results obtained in order e^4 agree with those of the Klein-Gordon equation. The additional terms are essentially recoil corrections.

Several questions remain which we hope to address elsewhere. (i) If in Eq. (5.1) the full potential $V^{(2)}+V^{(4)}+\cdots$ is used, then the scattering amplitude obtained from the solution will, by construction, agree with the field-theory amplitude computed in the generalized ladder approximation. Suppose however that V in (5.1) is approximated by $V^{(2)}+V^{(4)}$. The amplitude obtained from the exact solution of the approximate equation will still contain terms of order e^6 and higher and of all orders in m_A/m_B. It would be of interest to know the relation between the terms beyond order e^4 generated in this way and the actual higher-order terms that would follow from quantum field theory, for example, from exchange of more than two photons. (ii) One could also study the short-distance behavior of the potential between two charges, to given order in e^2, by using the methods described. But other methods may be more convenient for this purpose. (iii) We have not included radiative corrections to the photon-exchange graphs. Because of the need to cancel infrared divergences in such corrections against similar divergences in the emission of soft photons the possibility of a description of such corrections in terms of potentials requires further study. (iv) Once we fix the lowest-order potential $V^{(2)}$, the higher-order potentials are uniquely determined in our approach. It would be interesting to explore the question of whether there is, in some well-defined sense, an optimal choice for $V^{(2)}$.

In a future paper we will extend this work to the case of the two-photon-exchange potential between charges with spin, where additional, spin-dependent terms will occur.

ACKNOWLEDGMENTS

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APPENDIX A: SCATTERING OF A SPIN-0 PARTICLE BY AN EXTERNAL FIELD

In this appendix we derive some results which, though well known, are not readily available in standard texts on relativistic quantum theory.

1. External scalar field

The interaction of a spin-0 particle of mass m with an external scalar field $A(x)$ can be described by a Klein-Gordon- (KG-) type of equation of the form

$$[\partial_\mu \partial^\mu + m^2 + A(x)]\phi(x) = 0 \ . \tag{A1}$$

This can be regarded as the Euler-Lagrange equation arising from addition of a term $\phi^\dagger(x)\phi(x)A(x)$ to the usual Lagrangian for a complex spin-0 field. For a static external field we write $A(x) = 2mU(\mathbf{x})$ and consider a scattering-state wave function of the form $\phi(x) = \phi(\mathbf{x})\exp(-iEt)$. Then (A1) takes the form

$$(E^2 - E_{\text{op}}^2)\phi(\mathbf{x}) = 2mU\phi(\mathbf{x}) \ . \tag{A2}$$

We wish to compute the scattering amplitude to second order in U. To this end it is convenient to introduce a new wave function χ defined by

$$\chi(\mathbf{x}) = [(E + E_{\text{op}})/2E]^{1/2}\phi(\mathbf{x}) \tag{A3}$$

and to rewrite (A2) in the Schrödinger-type form

$$E\chi = (E_{\text{op}} + V_E)\chi \ , \tag{A4}$$

where

$$V_E = [2m/(E + E_{\text{op}})]^{1/2} U [2m/(E + E_{\text{op}})]^{1/2} \ . \tag{A5}$$

The amplitude T for a transition from a plane wave state $|\,\mathbf{p}\rangle$ to a state $|\,\mathbf{p}'\rangle$ of the same energy E is then given by

$$T = \langle \mathbf{p}' \,|\, V_E + V_E(E - E_{\text{op}})^{-1}V_E + \cdots \,|\, \mathbf{p}\rangle \ . \tag{A6}$$

On use of the form (A5) for V_E and the relations

$$E_{\text{op}}\,|\,\mathbf{p}\rangle = E(\mathbf{p})\,|\,\mathbf{p}\rangle, \quad E_{\text{op}}\,|\,\mathbf{p}'\rangle = E(\mathbf{p}')\,|\,\mathbf{p}'\rangle \tag{A7}$$

we get, with $E = E(\mathbf{p}) = E(\mathbf{p}')$,

$$T = (m/E)\langle \mathbf{p}' \,|\, U + U[2m/(E - E_{\text{op}}) \times (E + E_{\text{op}})]U \,|\, \mathbf{p}\rangle + \cdots \ . \tag{A8}$$

The term of order U^2 contains contributions from states which in quantum field theory correspond to the presence of virtual particle-antiparticle pairs. These may be isolated by introducing a lowest-order effective potential $V_{\text{eff}}^{(1)}$, defined by

$$V_{\text{eff}}^{(1)} = (m/E_{\text{op}})^{1/2} U (m/E_{\text{op}})^{1/2} \tag{A9}$$

and noting that

$$(E - E_{\text{op}})^{-1}(E + E_{\text{op}})^{-1} = [(E - E_{\text{op}})^{-1} - (E + E_{\text{op}})^{-1}]/2E_{\text{op}} \ . \tag{A10}$$

It follows that T may be written in the form

$$T = \langle \mathbf{p}' \,|\, V_{\text{eff}}^{(1)} + V_{\text{eff}}^{(1)}(E - E_{\text{op}})^{-1}V_{\text{eff}}^{(1)} \,|\, \mathbf{p}\rangle + \langle \mathbf{p}' \,|\, V_{\text{eff}}^{(2)} \,|\, \mathbf{p}\rangle + \cdots \ , \tag{A11}$$

where

$$V_{\text{eff}}^{(2)} = -(m/E_{\text{op}})^{1/2} U [m/E_{\text{op}}(E + E_{\text{op}})] U (m/E_{\text{op}})^{1/2} = -V_{\text{eff}}^{(1)}(E + E_{\text{op}})^{-1}V_{\text{eff}}^{(1)} \ . \tag{A12}$$

The last term in (A11) contains the pair-effects. Note that the minus sign in (A12) corresponds to the fact that in a field-theory calculation the energy denominator associated with, say, the production by $V_{\text{eff}}^{(1)}$ of the final particle of momentum $\mathbf{p}'$ and an antiparticle of momentum $\boldsymbol{l}$, in the presence of the initial particle of momentum $\mathbf{p}$, is just $E - [E(\mathbf{p}) + E(\mathbf{p}') + E(\boldsymbol{l})] = -[E + E(\boldsymbol{l})]$. Thus in field theory we get

$$T_{\text{pair}} = -\int \frac{d\boldsymbol{l}}{(2\pi)^3} \frac{\langle \mathbf{p}' \,|\, V_{\text{eff}}^{(1)} \,|\, \boldsymbol{l}\rangle\langle \boldsymbol{l} \,|\, V_{\text{eff}}^{(1)} \,|\, \mathbf{p}\rangle}{E + E(\boldsymbol{l})} \tag{A13}$$

which coincides with $\langle \mathbf{p}' \,|\, V_{\text{eff}}^{(2)} \,|\, \mathbf{p}\rangle$.

In the nonrelativistic regime we may replace E and E_{op} by m in (A12), yielding

$$V_{\text{eff}}^{(2)} \approx -\frac{U^2}{2m} \ . \tag{A14}$$

This coincides with the leading term for $V^{(4)}$ in the scalar Yukawa theory, in the limit $m_B \to \infty$, provided we identify m with m_A, in Eq. (3.85) of the text, and U with the

Yukawa potential $U^{(2)}$, defined by (2.41c). Thus the potential $V^{(4)}$, in the $m_B \to \infty$ limit of this theory, represents, to order g^4, the difference between using (5.1) and (A1).

2. External electromagnetic field

For a spin-0 particle of mass m and charge e moving in an external electromagnetic field with four-potential $A_\mu(x)$, the KG equation is

$$[(\partial_\mu - ieA_\mu)(\partial^\mu - ieA^\mu) + m^2]\phi(x) = 0 . \qquad (A15)$$

For an electrostatic field, we may take $A^i = 0$ and $A^0 = A^0(\mathbf{x})$ only. With $U(\mathbf{x}) = eA^0(\mathbf{x})$, Eq. (A15) becomes, for $\phi(x) = \phi(\mathbf{x})e^{-iEt}$,

$$((E-U)^2 - E_{\rm op}^2)\phi(\mathbf{x}) = 0$$

or

$$(E^2 - E_{\rm op}^2)\phi(\mathbf{x}) = (2EU - U^2)\phi(\mathbf{x}) . \qquad (A16)$$

On defining $\mathcal{X}(\mathbf{x})$ as in (A3) this takes a form analogous to (A4):

$$E\mathcal{X} = (E_{\rm op} + \tilde{V}_E)\mathcal{X} , \qquad (A17)$$

where

$$\tilde{V}_E = (E + E_{\rm op})^{-1/2}(2EU - U^2)(E + E_{\rm op})^{-1/2} . \qquad (A18)$$

The transition amplitude is now given by

$$T = \langle \mathbf{p}' \,|\, \tilde{V}_E + \tilde{V}_E(E - E_{\rm op})^{-1}\tilde{V}_E + \cdots \,|\, \mathbf{p} \rangle . \qquad (A19)$$

Keeping only terms through order U^2 we get

$$T = \frac{1}{\sqrt{2E}} \left\langle \mathbf{p}' \middle| (2EU - U^2) + (2EU) \times \frac{1}{E - E_{\rm op}} \frac{1}{E + E_{\rm op}} (2EU) \middle| \mathbf{p} \right\rangle \frac{1}{\sqrt{2E}} + \cdots$$

or

$$T = \left\langle \mathbf{p}' \middle| U + U \frac{2E}{(E - E_{\rm op})(E + E_{\rm op})} U \middle| \mathbf{p} \right\rangle - \left\langle \mathbf{p}' \middle| \frac{U^2}{2E} \middle| \mathbf{p} \right\rangle + \cdots .$$

Using the fact that

$$\frac{2E}{(E - E_{\rm op})(E + E_{\rm op})} = \frac{1}{E - E_{\rm op}} + \frac{1}{E + E_{\rm op}}$$

we may rewrite T in the form

$$T = \langle \mathbf{p}' \,|\, \tilde{V}_{\rm eff}^{(1)} + \tilde{V}_{\rm eff}^{(1)}(E - E_{\rm op})^{-1}\tilde{V}_{\rm eff}^{(1)} \,|\, \mathbf{p} \rangle + \langle \mathbf{p}' \,|\, \tilde{V}_{\rm eff}^{(2)} \,|\, \mathbf{p} \rangle + \cdots , \qquad (A20a)$$

where

$$\tilde{V}_{\rm eff}^{(1)} = U \qquad (A20b)$$

and

$$\tilde{V}_{\rm eff}^{(2)} = U \left[\frac{1}{E + E_{\rm op}} - \frac{1}{2E} \right] U = U \frac{E - E_{\rm op}}{(2E)(E + E_{\rm op})} U . \qquad (A20c)$$

We now identify U with the Coulomb interaction U_C by setting $e = e_A$ and $A^0 = e_B / 4\pi r$, the electrostatic potential of a point source "B." We may then compare the result (A20) with the $m_B = \infty$ limit of scalar QED.

Note first that with the convention

$$V^{(n)} = z_{\rm op} V^{(n)} z_{\rm op} , \qquad (A21)$$

adopted in Sec. II [Eq. (2.55)] and used throughout Sec. IV, and the fact that, as follows from (2.51a),

$$\lim_{m_B \to \infty} z_{\rm op} = 1 , \qquad (A22)$$

we have, on use of (2.57a), the relation

$$\lim_{m_B \to \infty} V_{\rm LR}^{(2)} = U_{\rm LR}^{(2)} = U_C . \qquad (A23)$$

Thus $V_{\rm LR}^{(2)}$ agrees with $\tilde{V}_{\rm eff}^{(1)}$, given by (A20b), in this limit. With regard to $\tilde{V}_{\rm eff}^{(2)}$, we restrict ourselves to a comparison of (A20c) with $V_{\rm LR}^{(4)}$ in the limit of large m or equivalently, in the domain $\mathbf{p}^2 = \mathbf{p}'^2 \ll m^2$. In this region we may replace E and $E_{\rm op}$ by $m + p^2/2m$ and $m + \mathbf{p}_{\rm op}^2/2m$, respectively, so that

$$\tilde{V}_{\rm eff}^{(2)} \approx \frac{U(\mathbf{p}^2 - \mathbf{p}_{\rm op}^2)U}{8m^3} . \qquad (A24)$$

Acting between plane-wave states $|\mathbf{p}\rangle$ and $|\mathbf{p}'\rangle$ of the same energy, the right-hand side of (A24) is equivalent to

$$\frac{1}{8m^3}\frac{1}{2}([\mathbf{p}_{\rm op}^2, U]U + U[U, \mathbf{p}_{\rm op}^2]) = \frac{1}{16m^3}[[\mathbf{p}_{\rm op}^2, U], U] = \frac{-1}{8m^3}(\nabla U)^2 . \qquad (A25)$$

With $U = e_A e_B / 4\pi r$ and m identified with m_A, this becomes

$$-\frac{(e^2/4\pi)^2}{8m_A^3 r^4} , \qquad (A26)$$

in agreement with Eq. (4.68) for the $p^2 = 0$ limit of the leading term at large r of $U_A^{(4)}(r, p^2)$. So in this case as well, $V^{(4)}$ represents, to order e^4, the additional term that is required because we use (5.1) instead of (A15).

APPENDIX B: THE RELATIVISTIC TWO-BODY COULOMB EQUATION

To examine the qualitative changes in the Coulomb wave function that are introduced by the relativistic kinematics in Eq. (5.1) let us first recall some facts about the nonrelativistic two-body Schrödinger equation for a pure Coulomb potential, viz.,

$$\left[\frac{\mathbf{p}_{\rm op}^2}{2m_{AB}}+U_C(r)\right]\phi=\frac{k^2}{2m_{AB}}\phi \ , \tag{B1}$$

where m_{AB} is the reduced mass.

The asymptotic form of ϕ, with an incident plane wave of momentum $\mathbf{k}=k\hat{\mathbf{z}}$ is given by[15]

$$\phi=e^{i(kz+N)}+\frac{f_C}{r}e^{i(kr-N)}+O\left[\frac{1}{r^2}\right] , \tag{B2a}$$

where

$$N=a\ln k(r-z)+b \ , \tag{B2b}$$

with

$$a=\frac{e_A e_B}{4\pi}\frac{m_{AB}}{k}, \quad b=\arg\Gamma(1+ia) \ , \tag{B2c}$$

and f_C, the scattering amplitude, is identical with that given by the Born approximation, viz.,

$$f_C=-a/k(1-\cos\theta) \ . \tag{B2d}$$

If a short-range potential $\delta U_{\rm SR}$ is added to U_C, f_C must be replaced by $f_C+\delta f$.

The simplest relativistic analog of (B1) for scalar QED is obtained by including only the lowest-order long-range part of V, viz., $z_{\rm op}U_C z_{\rm op}$ where $z_{\rm op}=z(\mathbf{p}_{\rm op})$ is defined by (2.47). Thus we study the equation

$$W_{\rm op}\phi+z_{\rm op}U_C z_{\rm op}\phi=W(\mathbf{k})\phi \ , \tag{B3}$$

where

$$\begin{aligned} W(\mathbf{k})&=E_A(\mathbf{k})+E_B(\mathbf{k}) \ , \\ W_{\rm op}&\equiv E_A(\mathbf{p}_{\rm op})+E_B(\mathbf{p}_{\rm op}) \ . \end{aligned} \tag{B4}$$

To facilitate comparison between (B3) and (B1) we note first that, as some algebra shows,

$$W(\mathbf{p})-W(\mathbf{k})=\frac{\mathbf{p}^2-\mathbf{k}^2}{2m_{AB}}g(\mathbf{p},\mathbf{k}) \ , \tag{B5}$$

with

$$g(\mathbf{p},\mathbf{k})=\frac{(2m_{AB})[W^2(\mathbf{p})+W^2(\mathbf{k})]}{[E_A(\mathbf{p})E_B(\mathbf{p})+E_A(\mathbf{k})E_B(\mathbf{k})][W(\mathbf{p})+W(\mathbf{k})]} \ . \tag{B6}$$

For $\mathbf{p}=\mathbf{k}$, (B6) simplifies to

$$g(k)\equiv g(\mathbf{k},\mathbf{k})=\frac{m_{AB}}{m_{AB}(k)} \ , \tag{B7a}$$

where $m_{AB}(k)$ is a relativistic "reduced energy,"

$$m_{AB}(k)=\frac{E_A(k)E_B(k)}{E_A(k)+E_B(k)} \ . \tag{B7b}$$

It follows from (B5) that (B3) may be written in the form

$$\frac{\mathbf{p}_{\rm op}^2}{2m_{AB}}\phi+g_{\rm op}^{-1}z_{\rm op}U_C z_{\rm op}\phi=\frac{k^2}{2m_{AB}}\phi \ , \tag{B8}$$

where $g_{\rm op}=g(\mathbf{p}_{\rm op},\mathbf{k})$. Since for large r, ϕ consists of a plane wave $e^{i\mathbf{k}\cdot\mathbf{r}}$ plus terms of order $1/r$, the operator factors $g_{\rm op}$ and $z_{\rm op}$ may, to leading order in $1/r$, be replaced by the values they have when acting directly on $e^{i\mathbf{k}\cdot\mathbf{r}}$, viz., by $g(\mathbf{k},\mathbf{k})=g(k)$ and by $z(k)=[1+k^2/E_A(k)E_B(k)]^{1/2}$, respectively. Thus for large r, (B8) takes the form

$$\left[\frac{p_{\rm op}^2}{2m_{AB}}+\eta(k)U_C+\delta U\right]\phi=\frac{k^2}{2m_{AB}}\phi \ , \tag{B9}$$

where

$$\eta(k)=\frac{z^2(k)}{g(k)} \tag{B10}$$

and δU is a "short-range" operator, arising from this replacement. The asymptotic form of ϕ can therefore be obtained by substituting for a everywhere in (B2b) the quantity

$$a_{\rm rel}=\eta(k)a \ , \tag{B11}$$

which leads to the replacement

$$f_C\to f_C^{\rm rel}\equiv\eta(k)f_C \ , \tag{B12}$$

and by adding to $f_C^{\rm rel}$ a term δf arising from δU. Thus f_C is replaced in (B2a) by

$$f^{\rm rel}=f_C^{\rm rel}+\delta f \ .$$

Inclusion of a term such as $V^{(4)}$ in (5.1), will change δf but will leave $f_C^{\rm rel}$ unaffected.

APPENDIX C: ANALYSIS OF THE INFRARED DIVERGENCE PROBLEM

We consider in this appendix the fourth-order Feynman graphs for the scalar Yukawa theory, in order to clarify the occurrence and cancellation of infrared divergences.

The fact that the amplitudes M_L and M_X, associated with the box and crossed-box Feynman diagrams shown in Figs. 3(a) and 3(b), each have an infrared divergence

(IRD) if one sets $\mu=0$ in Eqs. (3.2a) and (3.2b) is obvious from power counting. For example, with $t=Q^2\neq 0$, for small k the integrands of M_L and M_X are proportional, apart from a factor t^{-1}, to $(-k^2 p_A\cdot k p_B\cdot k)^{-1}$ and $(k^2 p_A\cdot k p_B'\cdot k)^{-1}$, respectively. If we work in the Euclidean region, with $d^4k\to K^3 dK\, d\Omega_3$, we see that each integrand behaves as K^{-1} for small K, corresponding to a logarithmic divergence in both M_L and M_X. Moreover, for $Q=-(p_B-p_B')\neq 0$, these divergences do not cancel. As shown in the text by use of dispersion theory techniques, the difference $M^{(4)}-M_I=M_L+M_X-M_I$, where M_I is the iteration amplitude obtained by using the one-meson-exchange potential $V^{(2)}$ in second-order perturbation theory, is IRD free in the $\mu\to 0$ limit. This fact may appear somewhat mysterious, because although it is plausible that M_L contains iteration effects from $V^{(2)}$, which are responsible at least in part for its IRD, this is not true for M_X. To be specific, while M_L, if regarded as a sum of contributions from time-ordered diagrams, contains pieces which involve only two-particle intermediate states and which can therefore in part be associated with iteration of $V^{(2)}$, the corresponding time-ordered diagrams for M_X always involve states of three or more particles, which are beyond the ken of $V^{(2)}$. The reader may therefore be puzzled, as we originally were, as to just what the mechanism of the IRD cancellation is from the viewpoint of ordinary perturbation theory. The purpose of this appendix is to describe this mechanism.

The answer, in brief, is as follows. Of the 24 time-ordered diagrams associated with M_L at most six, those involving no virtual particle-antiparticle pairs, can have enough small energy denominators to lead to an IR divergence problem. These diagrams are shown in Fig. 5. Of these only the first four, each of which involves some pure two-body intermediate states, actually have IRD's in the $\mu=0$ limit. The contributions of these four "dangerous" diagrams are, however, not purely of "potential type." They become so only if one neglects the recoil energy of one of the initial or final particles occurring in a three-body intermediate state, relative to the energy of the exchanged meson. If these recoil energies are at first neglected, one finds that the resulting contributions are precisely canceled by those of M_I (whether or not $\mu\to 0$). However, the remainder is still IR divergent. This is fortunate, because as analysis shows, the left-over divergence in M_L-M_I serves to precisely cancel the IRD in the crossed-diagram contribution M_X.

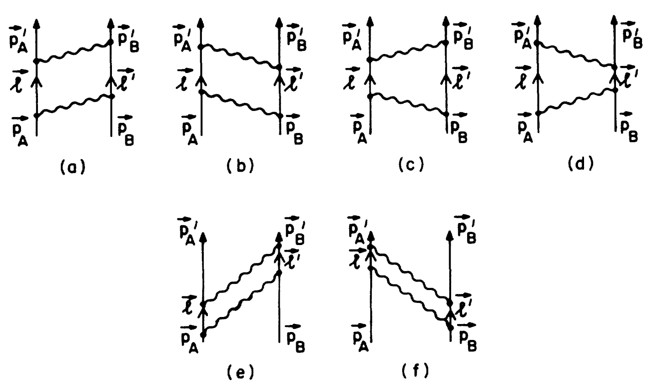

FIG. 5. Six of the 12 time-ordered graphs describing two-meson exchange, without creation of virtual pairs or crossing of meson lines. (a)–(d) Graphs involving some intermediate states with no mesons; (e), (f) graphs involving some intermediate states with two mesons.

We now give the details. If the propagators in (3.2) and (3.4) are separated into positive- and negative-frequency parts via

$$D_A^{-1}(l)=\frac{1}{2E_A(l)}\left[\frac{1}{l^0-E_A(l)+i\epsilon}-\frac{1}{l^0+E_A(l)-i\epsilon}\right] \tag{C1}$$

the integration over l^0 can be carried out by closing the Feynman contour, say, in the upper half-plane. The resulting sum of terms may, after some algebraic reorganization, be written as another sum of terms, each of which has a simple interpretation from the viewpoint of old-fashioned perturbation theory and can be associated with a suitable time-ordered diagram. In Figs. 5 and 6 we exhibit 6 of the 24 such diagrams associated with M_L and M_X, respectively. The remaining $2\times 18=36$ diagrams all correspond to processes involving some intermediate states with more than two "heavy" particles. Thus, they contain at least one large energy denominator and need not concern us at present. It should be noted that the amplitude associated with the "no-pair" diagrams shown in Figs. 5 and 6 can be most simply obtained by studying

$$T_{fi}^{(4)}=\langle f\,|\,H_I G_0 H_I G_0 H_I G_0 H_I\,|\,i\rangle\ , \tag{C2}$$

where H_I is the interaction piece of the total Hamiltonian $H=H_0+H_I$ with $G_0=(E-H_0+i\epsilon)^{-1}$, and retaining only those terms which do not involve more than two heavy particles in intermediate states, the so-called "no-pair terms."

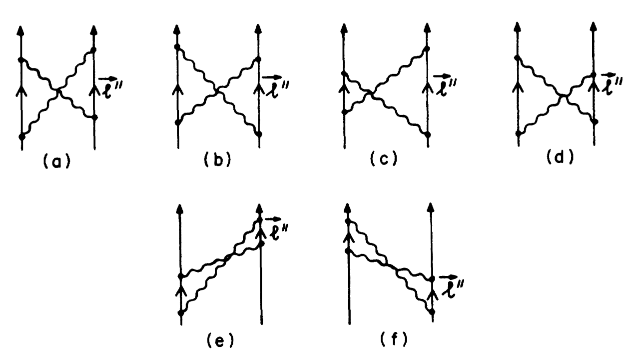

FIG. 6. The remaining six no-pair two-meson-exchange graphs, obtained from those of Fig. 5 by crossing the meson lines.

1. IR analysis of the no-pair part of M_L

Whichever way one proceeds, one finds that M_L^{np}, the sum of the "no-pair diagrams" shown in Fig. 5, is given by

$$M_L^{np}=\sum_{\alpha=a}^{f} M_{L;\alpha}\ , \tag{C3}$$

where, with $\mathbf{k}=\mathbf{p}_A-\mathbf{l}$, $\mathbf{k}'=\mathbf{l}-\mathbf{p}'_A$,

$$\mathcal{E}_A=E_A(\mathbf{l}),\quad \mathcal{E}_B=E_B(\mathbf{l}')\ ,\qquad \omega=(\mathbf{k}^2+\mu^2)^{1/2},\quad \omega'=(\mathbf{k}'^2+\mu^2)^{1/2}\ , \tag{C4}$$

we have

$$M_{L;\alpha}=G^4\int\frac{d\mathbf{l}}{(2\pi)^3}(16\mathcal{E}_A\mathcal{E}_B\omega\omega')^{-1}I_\alpha\ . \tag{C5}$$

Here

$$I_a=(E'_B-\mathcal{E}_B-\omega'+i\epsilon)^{-1}(d+i\epsilon)^{-1}(E_A-\mathcal{E}_A-\omega+i\epsilon)^{-1}\ , \tag{C6a}$$

$$I_b=(E'_A-\mathcal{E}_A-\omega'+i\epsilon)^{-1}(d+i\epsilon)^{-1}(E_B-\mathcal{E}_B-\omega+i\epsilon)^{-1}\ , \tag{C6b}$$

$$I_c=(E'_B-\mathcal{E}_B-\omega'+i\epsilon)^{-1}(d+i\epsilon)^{-1}(E_B-\mathcal{E}_B-\omega+i\epsilon)^{-1}\ , \tag{C6c}$$

$$I_d=(E'_A-\mathcal{E}_A-\omega'+i\epsilon)^{-1}(d+i\epsilon)^{-1}(E_A-\mathcal{E}_A-\omega+i\epsilon)^{-1}\ , \tag{C6d}$$

and

$$I_e=(E'_B-\mathcal{E}_B-\omega'+i\epsilon)^{-1}(E_A-E'_A-\omega-\omega'+i\epsilon)^{-1}(E_A-\mathcal{E}_A-\omega+i\epsilon)^{-1}\ , \tag{C6e}$$

$$I_f=(E'_A-\mathcal{E}_A-\omega'+i\epsilon)^{-1}(E_B-E'_B-\omega-\omega'+i\epsilon)^{-1}(E_B-\epsilon_B-\omega+i\epsilon)^{-1}\ , \tag{C6f}$$

with

$$d\equiv E_A+E_B-\omega_A-\omega_B\ . \tag{C7}$$

The denominators in each case are just equal to the energy E_A+E_B of the initial state minus the energy of an intermediate state.

The first four diagrams are "dangerous" (D) in the sense that even for $\mathbf{Q}^2\neq 0$, they become IR divergent if we let $\mu\to 0$. For future reference we note that I_D, the sum of the dangerous I_α,

$$I_D=\sum_{\alpha=a}^{d} I_\alpha \tag{C8a}$$

may be written in the compact form

$$I_D=F'(d+i\epsilon)^{-1}F\ , \tag{C8b}$$

where

$$F=(E_B-\mathcal{E}_B-\omega+i\epsilon)^{-1}+(E_A-\mathcal{E}_A-\omega+i\epsilon)^{-1}\ ,\qquad F'=(E'_B-\mathcal{E}_B-\omega'+i\epsilon)^{-1}+(E'_A-\mathcal{E}_A-\omega'+i\epsilon)^{-1}\ . \tag{C8c}$$

2. The no-recoil part of M_L^{np}

Suppose now that we "neglect recoil," that is, neglect $E'_B-\mathcal{E}_B$ and $E'_A-\mathcal{E}_A$ relative to ω' and $E_B-\mathcal{E}_B$ and $E_A-\mathcal{E}_A$ relative to ω. Then

$$I_D\to I_D^{\text{no rec}}=\frac{4}{\omega\omega'}\frac{1}{d+i\epsilon}\ . \tag{C9}$$

Correspondingly, if we define

$$M_{L;D}\equiv\sum_{\alpha=a}^{d}M_{L;\alpha}=G^4\int\frac{d\mathbf{l}}{(2\pi)^3}\frac{1}{16\mathcal{E}_A\mathcal{E}_B\omega\omega'}I_D\ , \tag{C10}$$

then

$$M_{L;D}\to M_{L;D}^{\text{no rec}}=\frac{G^4}{4}\int\frac{d\mathbf{l}}{(2\pi)^3}\frac{1}{\mathcal{E}_A\mathcal{E}_B}\frac{1}{\omega^2\omega'^2}\frac{1}{d+i\epsilon}\ . \tag{C11}$$

In the c.m. system we have, therefore,

$$M_{L;D}^{\text{no rec}}=\frac{G^4}{4g^4}\int\frac{d\mathbf{l}}{(2\pi)^3\mathcal{E}_A\mathcal{E}_B}\frac{\langle\mathbf{p}'\,|\,U^{(2)}\,|\,\mathbf{l}\rangle\langle\mathbf{l}\,|\,U^{(2)}\,|\,\mathbf{p}\rangle}{E_A+E_B-\mathcal{E}_A-\mathcal{E}_B+i\epsilon}\ , \tag{C12a}$$

where now $\mathcal{E}_B=E_B(-\mathbf{l})=E_B(\mathbf{l})$. Since $G^2=4m_Am_Bg^2$ we get, in terms of $V^{(2)}$ rather than $U^{(2)}$,

$$M_{L;D}^{\text{no rec}}=N_f^{-1}N_i^{-1}\left\langle\mathbf{p}'\left|V^{(2)}\frac{1}{E-h_0+i\epsilon}V^{(2)}\right|\mathbf{p}\right\rangle\ , \tag{C12b}$$

where $E=E_A+E_B$. Thus

$$T_{L;D}^{\text{no rec}}=\left\langle\mathbf{p}'\left|V^{(2)}\frac{1}{E-h_0+i\epsilon}V^{(2)}\right|\mathbf{p}\right\rangle\ , \tag{C13}$$

which is just the first term in (2.15b). In view of this result we identify $M_{L;D}^{\text{no rec}}$ as $M_I^{(4)}$, the "iteration part" of $M^{(4)}$:

$$M_I^{(4)}\equiv M_{L;D}^{\text{no rec}}\ . \tag{C14}$$

Thus the definition (2.40b) of $V^{(2)}$ has the virtue that, if used in second order, it reproduces *exactly* the no-recoil part of $M_L^{(4)}$, as defined in the c.m. system.

3. IR analysis of the no-pair part of the crossed ladder diagram

Turning to M_X, we have

$$M_X^{np}=\sum_{\alpha=a}^{f} M_{X;\alpha} \tag{C15}$$

with

$$M_{X;\alpha}=G^4\int \frac{dl}{(2\pi)^3}\frac{1}{16\mathcal{E}_A\mathcal{E}_B''\omega\omega'}J_\alpha , \tag{C16a}$$

where

$$\mathcal{E}_B''=E_B(l''),\quad l''=\mathbf{p}_B+\mathbf{k}' . \tag{C16b}$$

With

$$d''=E_A+E_B-\mathcal{E}_A-\mathcal{E}_B'' , \tag{C17}$$

we have

$$J_a=(E_B'-\mathcal{E}_B''-\omega+i\epsilon)^{-1}(d''-\omega-\omega'+i\epsilon)^{-1}(E_A-\mathcal{E}_a-\omega+i\epsilon)^{-1} , \tag{C18a}$$

$$J_b=(E_A'-\mathcal{E}_A-\omega'+i\epsilon)^{-1}(d''-\omega-\omega'+i\epsilon)^{-1}(E_B-\mathcal{E}_B''-\omega'+i\epsilon)^{-1} , \tag{C18b}$$

$$J_c=(E_B-\mathcal{E}_B''-\omega+i\epsilon)^{-1}(d''-\omega-\omega'+i\epsilon)^{-1}(E_B-\mathcal{E}_B''-\omega'+i\epsilon)^{-1} , \tag{C18c}$$

$$J_d=(E_A'-\mathcal{E}_A-\omega'+i\epsilon)^{-1}(d''-\omega-\omega'+i\epsilon)^{-1}(E_A-\mathcal{E}_A-\omega+i\epsilon)^{-1} , \tag{C18d}$$

and, with $d_A'=E_A-E_A'$, $d_B'=E_B-E_B'$,

$$J_e=(E_B'-\mathcal{E}_B''-\omega+i\epsilon)^{-1}(d_A'-\omega-\omega'+i\epsilon)^{-1}(E_A-\mathcal{E}_A-\omega+i\epsilon)^{-1} , \tag{C18e}$$

$$J_f=(E_A'-\mathcal{E}_A-\omega'+i\epsilon)^{-1}(d_B'-\omega-\omega'+i\epsilon)^{-1}(E_B'-\mathcal{E}_B''-\omega'+i\epsilon)^{-1} . \tag{C18f}$$

We have labeled the diagrams in such a way that diagram (α) of Fig. 4 is obtained from the corresponding (α) of Fig. 3 by interchanging the photon-line end points on the world line of particle B. Inspection shows that the dangerous terms are now (a),(b), and (e),(f) rather than (a),(b) and (c),(d).

Thus we define

$$M_{X;D}\equiv\sum_{\alpha\neq c,d} M_{X;\alpha}=G^4\int \frac{dl}{(2\pi)^3}\frac{1}{16\mathcal{E}_A\mathcal{E}_B''\omega\omega'}J_D , \tag{C19}$$

where

$$J_D=J_a+J_b+J_e+J_f . \tag{C20}$$

4. Cancellation of IR divergence

We now show that the IR divergence of $M_{X;D}$ for $\mu\to 0$ just cancels the IR divergence contained in the recoil correction $M_{L;D}^{\text{rec}}$ to $M_{L;D}$:

$$M_{L;D}^{\text{rec}}\equiv M_{L;D}-M_{L;D}^{\text{no rec}} . \tag{C21}$$

To see this we keep μ finite and study the integrand of $M_{L;D}^{\text{rec}}$ and the dangerous part $M_{X;D}$ of M_X^{np} at the points $l=\mathbf{p}$ and $l=\mathbf{p}'$, corresponding to $\mathbf{k}=0$ and $\mathbf{k}'=0$, respectively. It is at these points of the integration that divergences occur as $\mu\to 0$. For $l\to\mathbf{p}$ we have

$$\begin{aligned}&\omega\to\mu,\quad \omega'\to\nu\equiv(\mathbf{Q}^2+\mu^2)^{1/2}\\ &\mathcal{E}_A\to E_A,\quad \mathcal{E}_B''\to E_B' .\end{aligned} \tag{C22}$$

Since $E_A'=E_A$ and $E_B'=E_B$ in the c.m. system, we get

$$J_a\to\frac{1}{-\omega}\frac{1}{-\omega-\omega'}\frac{1}{-\omega},\quad J_b\to\frac{1}{-\omega'}\frac{1}{-\omega-\omega'}\frac{1}{-\omega'} , \tag{C23a}$$

$$J_e\to\frac{1}{-\omega}\frac{1}{-\omega-\omega'}\frac{1}{-\omega},\quad J_f\to\frac{1}{-\omega'}\frac{1}{-\omega-\omega'}\frac{1}{-\omega'} , \tag{C23b}$$

so that

$$J_D\to\frac{-1}{\mu+\nu}\left[\frac{2}{\mu^2}+\frac{2}{\nu^2}\right] \tag{C24}$$

and the integrand of (C19) reduces, apart from a factor $(2\pi)^{-3}$, to

$$\frac{1}{16E_AE_B}\left[\frac{-1}{\mu+\nu}\right]\left[\frac{2}{\mu^3\nu}+\frac{2}{\mu\nu^3}\right] \tag{C25}$$

at $l=\mathbf{p}$ (as well as at $l''=-\mathbf{p}'$). Because of the l^2dl factor in the volume element dl, it is the μ^{-3} factor which signals the presence of an IR divergence, i.e., a nonintegrable $|l-\mathbf{p}|^3$ behavior if we let $\mu\to 0$.

Let us study the integrand of $M_{L;D}^{\text{rec}}$ at the same point, $l=\mathbf{p}$. The relevant quantity is

$$\Delta I_D\equiv I_D-I_D^{\text{no rec}}=\frac{1}{d+i\epsilon}\left[F'F-\frac{4}{\omega\omega'}\right] . \tag{C26}$$

We may write, suppressing the $i\epsilon$, and using $E_B'=E_B$, $E_A'=E_A$,

$$F=\frac{-2}{\omega}+\frac{E_B-\mathscr{E}_B}{(E_B-\mathscr{E}_B-\omega)\omega}+\frac{E_A-\mathscr{E}_A}{(E_A-\mathscr{E}_A-\omega)\omega}\ , \tag{C27}$$

$$F'=\frac{-2}{\omega'}+\frac{E_B-\mathscr{E}_B}{(E_B-\mathscr{E}_B-\omega')\omega'}+\frac{E_A-\mathscr{E}_A}{(E_A-\mathscr{E}_A-\omega')\omega'}\ . \tag{C28}$$

For $l\to\mathbf{p}$, $\mathscr{E}_A\to E_A$, $\mathscr{E}_B\to E_B$ and, as a short calculation shows

$$\eta_A\equiv\lim_{l\to\mathbf{p}}\frac{E_A-\mathscr{E}_A}{d+i\epsilon}=\frac{E_A}{E_A+E_B}\ ,$$

$$\eta_B\equiv\lim_{l\to\mathbf{p}}\frac{E_B-\mathscr{E}_B}{d+i\epsilon}=\frac{E_B}{E_A+E_B}\ .$$

Since $\eta_A+\eta_B=1$, one finds that

$$\lim_{l\to\mathbf{p}}\Delta I_D=\frac{2}{\omega\omega'}\left[\frac{1}{\omega}+\frac{1}{\omega'}\right]\Bigg|_{l=\mathbf{p}}\ . \tag{C29}$$

It follows that the integrand of $M^{\rm rec}_{L;D}$ is given, at $l=\mathbf{p}$, by

$$\frac{1}{16E_AE_B}\frac{1}{\mu\nu}\frac{2}{\mu\nu}\left[\frac{1}{\nu}+\frac{1}{\mu}\right]\ . \tag{C30}$$

For $\mu\to 0$, (C25) is proportional to $-2/\mu^3\nu^2$ whereas (C30) is proportional to $+2/\mu^3\nu^2$, so the μ^{-3} singularity cancels. More precisely, the sum of (C30) and (C25) is given by

$$(16E_AE_B)^{-1}4\mu^{-2}\nu^{-2}(\mu+\nu)^{-1}\ , \tag{C31}$$

which only has a μ^{-2} singularity. This completes the proof of the absence of IR divergences in $M^{\rm rec}_{L;D}+M_{X;D}$ and hence in $M^{(4)}-M_I$.

APPENDIX D: INFINITE-MASS LIMIT OF THE GENERALIZED LADDER DIAGRAMS

The purpose of this appendix is to study the $m_B=\infty$ limit of an approximation to the Feynman amplitude $M(s,t)$, called the generalized ladder approximation (GLA). For simplicity we confine our attention to the scalar Yukawa theory studied in Sec. II. In this approximation one considers only those Feynman graphs which are generalized ladders, i.e., graphs in which n mesons are exchanged by particles A and B ($n=1,2,\ldots$), with arbitrary crossings of the meson lines allowed. Thus $M_{\rm GLA}(s,t)$, the GLA to $M(s,t)$ is given by

$$M_{\rm GLA}(s,t)=\sum_{n=1}^{\infty}M^{(2n)}(s,t)\ , \tag{D1}$$

where $M^{(n)}(s,t)$ is the sum of those graphs of order G^{2n}, $n!$ in number, which describes the exchange of n mesons.

The associated c.m. system transition amplitude $T_{\rm GLA}$ is related to $M_{\rm GLA}$ via Eq. (2.10b), viz.,

$$T_{\rm GLA}=M_{\rm GLA}/4E_A(\mathbf{p})E_B(\mathbf{p})\ . \tag{D2}$$

We now define T_A as the $m_B\to\infty$ limit of $T_{\rm GLA}$, with the understanding that $g_B=G_B/2m_B$, p and t are fixed. Thus

$$T_A\equiv\lim_{m_B\to\infty}T_{\rm GLA}=\sum_{n=1}^{\infty}T_A^{(2n)} \tag{D3}$$

with

$$T_A^{(2n)}=\lim_{m_B\to\infty}M^{(2n)}/4E_A(p)m_B\ . \tag{D4}$$

Note that in this limit the c.m. frame and the laboratory frame, with B as the target particle, coincide. Moreover, with fixed three-momenta

$$\lim_{m_B\to\infty}p_B/m_B=\lim_{m_B\to\infty}p'_B/m_B=(1,0,0,0) \tag{D5a}$$

and since $Q_0=-(E_B-E'_B)$,

$$\lim_{m_B\to\infty}Q_0=0\ . \tag{D5b}$$

From Eqs. (2.36) and (2.38b) we see that

$$T_A^{(2)}=-(m_A/E_A)g^2/(Q^2+\mu^2)\ . \tag{D6}$$

On comparison with the first term in (A8) we see that $T_A^{(2)}$ coincides with the lowest-order term obtained from the external KG equation, with U the Yukawa potential, given by (2.40).

To compute $T_A^{(4)}$ we use the symmetrized form (3.4) of $M^{(4)}$. Consider first the factor B_+ in (3.4),

$$B_+=D_B^{-1}+D_B'^{-1} \tag{D7a}$$

with D_B and D'_B, defined by (3.3a) and (3.3b), given by

$$D_B=2p_B\cdot k+k^2+i\epsilon,\quad D'_B=2p_B\cdot k'+k'^2+i\epsilon \tag{D7b}$$

on the B-particle mass shell. It follows from (D5a) that

$$\lim_{m_B\to\infty}m_BB_+=\frac{1}{2k_0+i\epsilon}+\frac{1}{2k'_0+i\epsilon} \tag{D8a}$$

and from (D5b), since $Q_0=k_0+k'_0$, that

$$k_0+k'_0=0\ . \tag{D8b}$$

Hence (D8a) reduces to

$$\lim_{m_B\to\infty}m_BB_+=-\pi i\delta(k_0)\ . \tag{D9}$$

Note further that for $k_0=k'_0=0$, the factor A_+ in (3.4) reduces to

$$(2\mathbf{p}_A\cdot\mathbf{k}-\mathbf{k}^2)^{-1}+(2\mathbf{p}_A\cdot\mathbf{k}'-\mathbf{k}'^2)^{-1}\ . \tag{D10}$$

These two terms give equal contributions to the integral, because of the symmetry between $\mathbf{k}$ and $\mathbf{k}'$ and the first term may be rewritten in the form

$$[E_A^2(\mathbf{p})-E_A^2(\boldsymbol{l})]^{-1}\ ; \tag{D11a}$$

here $\boldsymbol{l}$ is the three-momentum of A in intermediate states,

$$l = \mathbf{p} - \mathbf{k} \tag{D11b}$$

and we have dropped the subscript A on $\mathbf{p}_A$. On denoting $E_A(\mathbf{p})$ and $E_A(l)$ by E_A and E'_A, respectively, we see that in the limit $m_B \to \infty$, we may make the replacement

$$A_+ \to 2(E_A^2 - E_A'^2)^{-1} \ . \tag{D12}$$

It then follows from (D4), (3.4), (D9), and (D12), together with the relation $G^4 = 16g^4 m_A^2 m_B^2$, that

$$T_A^{(4)} = \left[\frac{m_A^2}{E_A}\right] g^4 \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1}{\mathbf{k}^2+\mu^2} \frac{1}{\mathbf{k}'^2+\mu^2} \frac{2}{E_A^2 - E_A'^2} \ . \tag{D13}$$

This agrees with the sum of the second and third terms in (A11) obtained from the KG equation. The agreement may not be immediately obvious because (A11) was written for a different purpose. However, we may use the relations

$$\frac{2}{E_A^2 - E_A'^2} = \left[\frac{1}{E_A - E'_A} - \frac{1}{E_A + E'_A}\right] \frac{1}{E'_A} \tag{D14}$$

and [see (2.39)]

$$\langle \mathbf{p}' \mid U \mid l \rangle = -g^2/(\mathbf{k}'^2 + \mu^2) \ ,$$

$$\langle l \mid U \mid \mathbf{p} \rangle = -g^2/(\mathbf{k}^2 + \mu^2) \ ,$$

to rewrite (D13) in the form

$$T_A^{(4)} = \int \frac{dl}{(2\pi)^3} \langle p' \mid V_{\text{eff}}^{(1)} \mid l \rangle \langle l \mid V_{\text{eff}}^{(1)} \mid p \rangle \left[\frac{1}{E_A(\mathbf{p}) - E_A(l)} - \frac{1}{E_A(\mathbf{p}) + E_A(l)}\right] , \tag{D15}$$

where $V_{\text{eff}}^{(1)}$ is given by (A9). The agreement with (A11) is now manifest.

To extend this approach to any order n, we write $M^{(2n)}$ in a form used long ago in an analysis of the eikonal approximation in quantum field theory,[16] viz.,

$$M^{(2n)} = \frac{i^{n-1} G^{2n}}{(2\pi)^{4(n-1)}} \int d^4k_1 d^4k_2 \cdots d^4k_n \delta^{(4)}\left[Q - \sum_{i=1}^{n} k_i\right] D^{(n)} A^{(n)} B^{(n)} \ , \tag{D16}$$

where

$$D^{(n)} = (k_1^2 - \mu^2)^{-1} (k_2^2 - \mu^2)^{-1} \cdots (k_n^2 - \mu^2)^{-1} \ , \tag{D17a}$$

$$A^{(n)} = [(p_A - k_1)^2 - m_A^2]^{-1} [(p_A - k_1 - k_2)^2 - m_A^2]^{-1} \cdots [(p_A - k_1 - k_2 - \cdots - k_{n-1})^2 - m_A^2]^{-1} \ , \tag{D17b}$$

and

$$B^{(n)} = \sum_{\pi} [(p_B + k_{\pi(1)})^2 - m_B^2]^{-1} [(p_B + k_{\pi(1)} + k_{\pi(2)})^2 - m_B^2]^{-1} \cdots [(p_B + k_{\pi(1)} + \cdots + k_{\pi(n-1)})^2 - m_B^2]^{-1} \ , \tag{D17c}$$

with $\pi(i)$ denoting a permutation of the indices $1,2,\ldots,n$. If we replace $A^{(n)}$ by a symmetrized form analogous to $B^{(n)}$ and, correspondingly, divide the right-hand side of (D16) by $n!$, then for $n=2$, (D16) reduces to (3.2). For the present purpose it is more convenient to use the form (D16) as written.

The counterpart of the limit (D8a) is

$$\lim_{m_B \to \infty} m_B^{n-1} B^{(n)} = \frac{1}{2^n} S_n(k_1^0 + i\epsilon_1, k_2^0 + i\epsilon_2, \ldots, k_n + i\epsilon_n) \ , \tag{D18}$$

where

$$S_n(a_1, a_2, \ldots, a_n) \equiv \sum_{\pi} \frac{1}{a_{\pi(1)}} \frac{1}{a_{\pi(1)} + a_{\pi(2)}} \cdots \frac{1}{a_{\pi(1)} + a_{\pi(2)} + \cdots + a_{\pi(n-1)}} \ . \tag{D19}$$

We have extracted a factor of 2 from each denominator and given labels to the ϵ's to emphasize that they tend to zero independently. Since also $Q^0 \to 0$ for $m_B \to \infty$, the factor $\delta(Q^0 - k_1^0 - \cdots - k_n^0)$ reduces to $\delta(k_1^0 + k_2^0 + \cdots + k_n^0)$ and we are led to consider the quantity

$$Y_n = S_n(k_1^0 + i\epsilon_1, \ldots, k_n^0 + i\epsilon_n) \delta(k_1^0 + k_2^0 + \cdots + k_n^0) \ . \tag{D20}$$

We now make use of two theorems. The first is an algebraic identity which played a key role in the analysis mentioned above,[16] viz., the following.

Theorem 1:

$$S_n(a_1, a_2, \ldots, a_n) = \frac{a_1 + a_2 + \cdots + a_n}{a_1 a_2 \cdots a_n} \ . \tag{D21}$$

The second theorem is of a distribution-theoretic nature; it gives a formula for $\overline{Y}_n$, the limit of Y_n as $\epsilon_i \to 0$,

$$\bar{Y}_n \equiv \lim_{\epsilon_i \to 0} Y_n \ . \quad \text{(D22)}$$

We simply state it here and give a proof based on (D21), in Appendix E.

Theorem 2:

$$\bar{Y}_n = (-2\pi i)^{n-1}\delta(k_1^0)\delta(k_2^0)\cdots\delta(k_n^0) \ . \quad \text{(D23)}$$

Accepting (D23) we see, on using the definitions (D4) and (D16), recalling the relation $G^2=4g^2m_A m_B$, and eliminating the integrations over the k_i^0 via the delta functions, that

$$T_A^{(n)} = \frac{(2g^2 m_A)^n}{(2\pi)^{3(n-1)}} \int d\mathbf{k}_1 \cdots d\mathbf{k}_n \delta^{(3)}\left[\mathbf{Q} - \sum_{i=1}^{n} \mathbf{k}_i\right] D_0^{(n)} A_0^{(n)} \ , \quad \text{(D24a)}$$

where $D_0^{(n)}$ and $A_0^{(n)}$ denote the values of $D^{(n)}$ and $A^{(n)}$ for $k_1^0=k_2^0=\cdots=k_n^0=0$. On introducing new variables $\mathbf{l}_1,\mathbf{l}_2,\ldots,\mathbf{l}_n$ via

$$\mathbf{k}_1=\mathbf{p}_A-\mathbf{l}_1,\ \ \mathbf{k}_2=\mathbf{l}_1-\mathbf{l}_2,\ldots,\ \ \mathbf{k}_{n-1}=\mathbf{l}_{n-2}-\mathbf{l}_{n-1},\ \ \mathbf{k}_n=\mathbf{l}_{n-1}-\mathbf{l}_n \ ,$$

we may write $D_0^{(n)}$ and $A_0^{(n)}$ in the forms

$$D_n^{(0)} = (-1)^n \frac{1}{(\mathbf{l}_{n-1}-\mathbf{l}_n)^2+\mu^2} \cdots \frac{1}{(\mathbf{l}_1-\mathbf{l}_2)^2+\mu^2}\frac{1}{(\mathbf{p}_A-\mathbf{l}_1)^2+\mu^2} \ ,$$

and

$$A_n^{(0)} = \frac{1}{E_A^2(\mathbf{p}_A)-E_A^2(\mathbf{l}_{n-1})+i\epsilon} \cdots \frac{1}{E_A^2(\mathbf{p}_A)-E_A^2(\mathbf{l}_1)+i\epsilon} \ .$$

Since $-g^2/[(\mathbf{l}_i-\mathbf{l}_{i+1})^2+\mu^2] = \langle \mathbf{l}_{i+1}|U|\mathbf{l}_i\rangle$ when U is the Yukawa potential, we see that $T_A^{(n)}$ is given by

$$T_A^{(2n)} = \frac{1}{(2\pi)^{3(n-1)}} \int d\mathbf{l}_1 \cdots d\mathbf{l}_{n-1} \left[\langle \mathbf{p}'_A|2m_A U|\mathbf{l}_{n-1}\rangle\langle \mathbf{l}_{n-1}|2m_A U|\mathbf{l}_{n-2}\rangle \cdots \langle \mathbf{l}_1|2m_A U|\mathbf{p}_A\rangle \times \frac{1}{E_A^2(\mathbf{p}_A)-E_A^2(\mathbf{l}_{n-1})+i\epsilon} \cdots \frac{1}{E_A^2(\mathbf{p}_A)-E_A^2(\mathbf{l}_2)+i\epsilon} \right] \quad \text{(D24b)}$$

$$T_A^{(2n)} = \left\langle \mathbf{p}'_A \left| (2m_A U)\frac{1}{E_A^2-(E_A^{\text{op}})^2+i\epsilon}(2m_A U)\cdots\frac{1}{E_A^2-(E_A^{\text{op}})^2+i\epsilon}(2m_A U) \right| \mathbf{p}_A \right\rangle \ . \quad \text{(D25)}$$

We may compare this with the amplitude obtained from the external-field Klein-Gordon equation, viz.,

$$[E^2-(E_A^{\text{op}})^2-2m_A U]\phi=0 \quad \text{(D26a)}$$

for a static external field U. For a scattering problem (D26a) may be written in the integral form

$$\phi=\phi_0+\frac{1}{E^2-(E_A^{\text{op}})^2+i\epsilon}(2m_A U)\phi \ , \quad \text{(D26b)}$$

where ϕ_0 is a plane wave of momentum $\mathbf{p}_A$. From (D26b) we see at once that, with $E=E_A(\mathbf{p}_A)$, the nth-order amplitude generated from the KG equation coincides with (D25) and hence with the $m_B=\infty$ limit of the sum of the generalized ladder graphs of order $2n$.

With the use of the same methods, analogous results can be proved for the case of interacting spin-$\frac{1}{2}$ and spin-0 particles, or for two spin-$\frac{1}{2}$ particles, with the KG equation replaced by the external-field Dirac equation.

An immediate consequence of the theorem proved here is that the two-body Bethe-Salpeter equation,[17] with the kernel defined by the sum of all two-body irreducible generalized ladder diagrams, but radiative corrections neglected, is equivalent in the $m_B \to \infty$ limit to the external-field KG or Dirac equation. This fact about the BS equation has been known for a long time.[18]

APPENDIX E: PROOF OF A DISTRIBUTION THEORETIC EQUALITY

The purpose of this appendix is to provide a proof of Theorem 2, used in Appendix D (Ref. 19). This theorem states that with $x_1,\ldots,x_n$ real variables ($n \geq 2$) and

$$S_n(x_1,\ldots,x_n) \equiv \sum_{\pi} \frac{1}{x_{\pi(1)}}\frac{1}{x_{\pi(1)}+x_{\pi(2)}} \cdots \frac{1}{x_{\pi(1)}+\cdots+x_{\pi(n-1)}} \ , \quad \text{(E1)}$$

where π is any permutation of $1,2,\ldots,n$, that the quantity Y_n, defined by

$$Y_n \equiv S_n(x_1+i\epsilon_1,\ldots,x_n+i\epsilon_n)\delta(x_1+\cdots+x_n)\,, \qquad (E2)$$

has, in the limit $\epsilon_i \to 0$ ($i=1,2,\ldots,n$), the value

$$\bar{Y}_n = (-2\pi i)^{n-1}\delta(x_1)\cdots\delta(x_n)\,. \qquad (E3)$$

To prove (E3) we first use the identity (D21) to write $S_n(x_1,x_2,\ldots,x_n)$ in the form

$$S_n(x_1,x_2,\ldots,x_n) = S_{n-1}(x_1,x_2,\ldots,x_{n-1})\frac{1}{x_n}+\frac{1}{x_1x_2\cdots x_{n-1}}$$
$$= S_{n-1}(x_1,x_2,\ldots,x_{n-1})\left[\frac{1}{x_n}+\frac{1}{\sum_{n=1}}\right]\,, \qquad (E4)$$

where $\sum_{n-1}=x_1+x_2+\cdots+x_{n-1}$. Because of the factor $\delta(x_1+x_2+\cdots+x_n)$ in (E2) we may replace x_n in (E4) by $-\sum_{n-1}$ so that Y_n may be written in the form

$$Y_n = S_{n-1}(x_1+i\epsilon_1,\ldots,x_{n-1}+i\epsilon_{n-1})\left[\frac{1}{-\sum_{n-1}+i\epsilon_n}+\frac{1}{\sum_{n-1}+i\epsilon_n'}\right]\delta(x_1+x_2+\cdots+x_n)\,, \qquad (E5)$$

where $\epsilon_n'=\epsilon_1+\epsilon_2+\cdots\epsilon_{n-1}$. The factor in large parentheses may be replaced, for infinitesimal ϵ_i's, by $(-2\pi i)\delta(x_1+x_2+\cdots+x_{n-1})$ and we infer that

$$\bar{Y}_n = (-2\pi i)\bar{Y}_{n-1}\delta(x_1+x_2+\cdots+x_n)\,. \qquad (E6)$$

Using the same formula for $\bar{Y}_{n-1}$ we can "roll back" this equation to get

$$\bar{Y}_n = (-2\pi i)^{n-1}\delta(x_1)\delta(x_1+x_2) \times\cdots\delta(x_1+x_2+\cdots+x_n)\,, \qquad (E7)$$

which is equivalent to (E3).

It may be worthwhile noting that a number of other identities may be obtained as corollaries of (E3), by replacing $(x_i+i\epsilon_i)^{-1}$ by $P(1/x_i)-i\pi\delta(x_i)$ and equating the resulting sum of products to the form given by (E3). For example, for $n=3$, we get

$$\left[P\frac{1}{x_1}P\frac{1}{x_2}+\text{c.p.}\right]\delta(x_1+x_2+x_3) = -\pi^2\delta(x_1)\delta(x_2)\delta(x_3) \qquad (E8)$$

and

$$\left[P\frac{1}{x_1}\delta(x_2)+\text{c.p.}\right]\delta(x_1+x_2+x_3)=0\,. \qquad (E9)$$

Some of these identities may be difficult to prove directly.

APPENDIX F: ITERABLE SECOND-ORDER POTENTIAL FOR SCALAR QED

As mentioned in Sec. II D 2, it is possible to define an alternative second-order potential with the property that its iteration does not lead to ultraviolet divergences. An example of such a potential, call it V', is given by

$$V' = V_a' + V_b'\,, \qquad (F1)$$

where

$$V_a' = z_{\text{op}}' U_C z_{\text{op}}'\,. \qquad (F2)$$

Here $z_{\text{op}}'=z'(\mathbf{p}_{\text{op}})$ with

$$z'(\mathbf{p}) \equiv [1+\mathbf{p}^2/2E_A(f)E_B(f)]^{1/2}\,,$$

and

$$V_b' = y_{\text{op}} U_b' y_{\text{op}} \qquad (F3a)$$

with

$$U_b' = \mathbf{p}_{\text{op}}\cdot U_C \mathbf{p}_{\text{op}}/2m_A m_B\,. \qquad (F3b)$$

It is easy to verify that V' satisfies the constraint (2.16a), i.e., reproduces the lowest-order field theory amplitude. The iteration amplitude $T'^{(4)}$ associated with V' is given, in an obvious notation by

$$T'^{(4)} = T'^{(4)}_{aa}+T'^{(4)}_{ab}+T'^{(4)}_{ba}+T'^{(4)}_{bb}\,. \qquad (F4)$$

The analog of the previously divergent term arising from the iteration of the delta-function term (2.57b) is the term $T'^{(4)}_{bb}$, defined by

$$T'^{(4)}_{bb} = \int [d\mathbf{k}/(2\pi)^3]\langle \mathbf{p}'|V_b'|\mathbf{k}\rangle\langle \mathbf{k}|V_b'|\mathbf{p}\rangle/[W(\mathbf{p})-W(\mathbf{k})]\,, \qquad (F5)$$

where $W(\mathbf{p})=E_A(\mathbf{p})+E_B(\mathbf{p})$. For large $\mathbf{k}$ the product of the matrix elements in the numerator of the integrand of (F5) is proportional to $\mathbf{p}'\cdot\mathbf{k}\mathbf{p}\cdot\mathbf{k}/k^6$ so that the integrand behaves as $1/k^5$. Since $d\mathbf{k} \propto k^2 d\mathbf{k}$, the integral is convergent. It can similarly be shown that the other terms in (F4) are ultraviolet finite.

We have also verified that the associated sixth-order iteration amplitude $T^{(6)}$ is ultraviolet finite. Because the high-momentum behavior of V' is no worse than that of the Coulomb interaction it seems likely that the Schrödinger equation (2.6) with V' as the potential has well-behaved scattering solutions, unlike the corresponding equation with V given by (2.55) and (2.57).

Note that unlike U_C, U_b' involves derivative operators in coordinate space. The matrix elements of the operator V' differ from those of $V^{(2)}$ off the energy shell and this difference will be reflected in a change in the higher-order potentials $V^{(4)}$, $V^{(6)}$, etc. Since the difference between $V^{(2)}$ and V' is a long-range "potential," the long-range parts of $V^{(4)}$, etc., will also change.

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[19]The particular method described here was developed after a discussion with A. H. Mueller.