On-shell expansion of the effective action: S matrix and the ambiguity-free stability criterion

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The expansion formula of the effective action in terms of the connected S-matrix elements is derived. They are expressed in terms of the on-shell quantities. The correct stability criterion of the given solution is discussed. It is free from any ambiguities which are the subjects of recent controversy.

I. INTRODUCTION

One of the powerful and systematic methods to study dynamical problems in quantum field theory is through the use of the effective action or the potential. It has become a popular tool particularly when one wants to look for the nonperturbative solution corresponding to dynamical symmetry breaking.

The stationary requirement of the effective action determines the ground-state solution, and expanding around it we get the off-shell Green's function as the expansion coefficients. The stationary requirement and the vanishing of the external source J are equivalent. Recently, one of the authors developed[1] an expansion scheme without violating the relation $J=0$ in the course of expansion. By examining the equation for small deviation and by keeping the lowest term, the correct stability criterion for the chosen solution was derived. These studies started from the complete analogy with classical analytical mechanics.

The purpose of the present paper is twofold.

(1) The previous analysis is extended to higher orders. The result is the appearance of the scattering matrix (S-matrix) elements as the expansion coefficients. It is a novel formula for the effective action and is derived in Sec. II.

(2) The lowest-order equation is used in Sec. III to provide an ambiguity-free criterion of the stability. In fact there is much debate on the ambiguities about the stability criterion in terms of the effective potential. We clarify the sources of ambiguities and keep the criterion free from ambiguities. This is possible because we are always on the mass shell. Two examples are discussed in order to illustrate the point. Section IV is devoted to several discussions.

II. ON-SHELL EXPANSION OF THE EFFECTIVE ACTION AND THE S MATRIX

In this section several hitherto unobserved important properties of the effective action are discussed. They are the statements about the formal on-shell properties of the effective action and are summarized in three formulas: (13), (29), and (33). Equation (13) has been derived in Ref. 1 but we recapitulate it for the completeness of this section. We discuss them before showing their derivation in detail.

A vacuum solution is first found by the stationary requirement of the effective action. Our three formulas are obtained by considering the motion in the vicinity of the chosen solution where the deviation is small. These analyses are analogous to the ones we perform in classical mechanics when we discuss the stability of a given solution. The effective action is replaced in the classical case simply by the classical action.

Once the ground-state solution has been fixed, the result (13) for the lowest order in the deviation gives the condition to determine the excitation level or the particle spectrum—the "on-shell" condition. The form of (13) is more general than the conventional criterion that the particle spectrum is determined by the pole of the Green's function. Our criterion includes this case and can also be applied to the situation where the conventional one cannot be used. It can be utilized, for example, in the case where the space and/or time inhomogeneous background field is present. In such a case, there does not exist at present any precise statement about how to define and determine the spectrum. Equation (13) is used as a stability criterion in the next section.

The second result, Eq. (29), includes the higher-order contributions to the deviation and we expect that they are related to the scattering amplitudes of the on-shell particles determined by (13). Indeed we find the connected S-matrix elements as coefficients of the higher-order terms.

The final result, Eq. (33), is the expression for the value of the effective action corresponding to the motion we are considering. It is also given by the connected S-matrix elements.

By these observations we can say that the effective action is really a generating functional of the on-shell quantities; it determines the ground state and the excitation spectrum above the ground state and the scattering amplitudes of these excitations. (To set up the whole Hilbert space, we have of course to prepare the field variables as arguments of the effective action which couple to all the channels with different quantum numbers.)

Let us start by defining the effective action $\Gamma[\phi]$ where we have taken for simplicity a multicomponent scalar field $\phi_i(x)$ ($i=1$–n) and the Lagrangian density is assumed to be $\mathcal{L}(\phi)$. Now $W[J]$ is first defined by the functional integral as

$$\exp(iW[J])=\int[d\phi]\exp\left[i\int_{-\infty}^{\infty}d^4x[\mathcal{L}(\phi)+J_i(x)\phi_i(x)]\right], \tag{1}$$

where the summation over the repeated index is understood. The definition of $\Gamma[\phi]$ is given by

$$\Gamma[\phi]=W[J]-\int d^4x\, J_i(x)\delta W[J]/\delta J_i(x)\ , \tag{2}$$

$$\phi_i(x)\equiv\delta W[J]/\delta J_i(x)\ . \tag{3}$$

We shall frequently use the well-known identities

$$\delta\Gamma[\phi]/\delta\phi_i(x)\equiv\Gamma_{ix}=-J_i(x)\ , \tag{4}$$

$$\Gamma^{(2)}_{ix,jy}W^{(2)}_{jy,kz}=W^{(2)}_{ix,jy}\Gamma^{(2)}_{jy,kz}=-\delta_{ik}\delta_{xz}\ , \tag{5}$$

where the integration $\int d^4y$ over the repeated space-time variables is implied and $\delta_{xz}\equiv\delta^4(x-z)$. We also define

$$\Gamma^{(n)}_{i_1x_1,i_2x_2,\ldots,i_nx_n}\equiv\delta^n\Gamma[\phi]/\delta\phi_{i_1}(x_1)\delta\phi_{i_2}(x_2)\cdots\delta\phi_{i_n}(x_n) \tag{6}$$

and similarly for $W^{(n)}$. By (4) we see that the equation of motion for $\phi_i(x)$ under the theory specified by $\mathcal{L}(\phi)$ is the stationary requirement:

$$\Gamma_{ix}=0\ . \tag{7}$$

Let one of the solutions to (7) be $\phi_i^{(0)}(x)$ and expand $\Gamma[\phi]$ around $\phi_i^{(0)}(x)$ by writing $\phi_i(x)=\phi_i^{(0)}(x)+\delta\phi_i(x)$. Then we get the *off-shell* expansion

$$\Gamma[\phi]=\Gamma[\phi^{(0)}]+\sum_{n=2}^{\infty}\frac{1}{n!}\Gamma^{(n)}_{i_1x_1,i_2x_2,\ldots,i_nx_n}\times\delta\phi_{i_1}(x_1)\delta\phi_{i_2}(x_2)\cdots\delta\phi_{i_n}(x_n), \tag{8}$$

where $\Gamma^{(n)}$ is known to be the one-particle-irreducible (1PI) Green's function of our theory evaluated at $\phi_i(x)=\phi_i^{(0)}(x)$.

In order to get the *on-shell* expansion, we have to stay on the trajectory of $\phi_i(x)$ satisfying $J_i(x)=0$. Therefore, we look for another solution to (7) in the vicinity of $\phi_i^{(0)}(x)$. For that purpose we set

$$\phi_i(x)=\phi_i^{(0)}(x)+\Delta\phi_i(x) \tag{9}$$

and write $\Delta\phi_i(x)$ as

$$\Delta\phi_i(x)=\Delta\phi_i^{(1)}(x)+\Delta\phi_i^{(2)}(x)+\Delta\phi_i^{(3)}(x)+\cdots\ , \tag{10}$$

assuming $\Delta\phi_i^{(1)}(x)$ is small and $\Delta\phi_i^{(n)}(x)$ ($n\geq 2$) is of the order $[\Delta\phi_i^{(1)}(x)]^n$.

Consider the equation

$$0=\Gamma_{ix}[\phi]=\Gamma_{ix}[\phi^{(0)}+\Delta\phi]\ , \tag{11}$$

and expand it as

$$0=\Gamma^{(2)}_{ix,jy}\Delta\phi_j^{(1)}(y)+\frac{1}{2!}\Gamma^{(3)}_{ix,jy,kz}\Delta\phi_j^{(1)}(y)\Delta\phi_k^{(1)}(z)+\Gamma^{(2)}_{ix,jy}\Delta\phi_j^{(2)}(y)+\frac{1}{3!}\Gamma^{(4)}_{ix,jy,kz,lw}\Delta\phi_j^{(1)}(y)\Delta\phi_k^{(1)}(z)\Delta\phi_l^{(1)}(w)+2\times\frac{1}{2!}\Gamma^{(3)}_{ix,jy,kx}\Delta\phi_j^{(1)}(y)\Delta\phi_k^{(2)}(z)+\Gamma^{(2)}_{ix,jy}\Delta\phi_j^{(3)}(y)+\cdots\ . \tag{12}$$

Here all the coefficients are evaluated at $\phi_i(x)=\phi_i^{(0)}(x)$. We now investigate the possibility that (12) is satisfied order by order. This leads us to the on-shell expansion we are looking for.

The lowest order. The requirement that the first term of (12) vanishes, turns out to be the zero-eigenvalue equation for $\Delta\phi_i^{(1)}(y)$ (Ref. 1):

$$\Gamma^{(2)}_{ix,jy}\Delta\phi_j^{(1)}(y)=0\ . \tag{13}$$

Thus,

$$\det(\Gamma^{(2)}_{ix,jy})=0\ , \tag{14}$$

where the determinant is taken regarding ix and jy as the indices of the matrix $\Gamma^{(2)}$. We call (13) or (14) the generalized on-shell condition. In order to see the reason, let us take the space-time translational-invariant case $\phi_i^{(0)}(x)=\phi_i^{(0)}$ where $\Gamma^{(2)}_{ix,jy}$ is a function of $x-y$ and (5) and (13) take the form

$$\Gamma^{(2)}_{ij}(p^2)=-W^{-1}_{ij}(p^2)\ , \tag{15}$$

$$\Gamma^{(2)}_{ij}(p^2)\Delta\phi_j^{(1)}(p)=0\ , \tag{16}$$

in the Fourier representation

$$\Gamma^{(2)}_{ij}(p^2)=\int d^4(x-y)e^{ip(x-y)}\Gamma^{(2)}_{ix,jy}$$

with $p^2\equiv(p^\mu)^2$. Equations (15) and (16) mean that $\Delta\phi_j^{(1)}(p)$ has the support only at the pole of $W_{ij}(p^2)$, and $\Delta\phi_j^{(1)}(p)$ is the eigenvector corresponding to the pole of $W_{ij}(p^2)$. Let $U_{ij}(p^2)$ be the orthogonal matrix which diagonalizes $\Gamma^{(2)}_{ij}(p^2)$, and $W^{(2)}_{ij}(p^2)$ as

$$[U^{-1}(p^2)\Gamma^{(2)}(p^2)U(p^2)]_{ij}\equiv\hat{\Gamma}^{(2)}_{ij}(p^2)=\delta_{ij}\gamma_i(p^2)\ ,$$

and let the solution of $\gamma_i(p^2)=0$ be $p^2=m_i^2$. We first solve $\hat{\Gamma}^{(2)}_{ij}(p^2)\Delta\hat{\phi}_j(p)=0$ with the solution $\Delta\hat{\phi}_j(p)=C_j(p)\delta(p^2-m_j^2)$, where $C_j(p)$ is an arbitrary function. Then the pole part of $W^{(2)}_{ij}(p^2)$ is given by

$$W^{(2)}_{ij}(p^2)=U_{ik}(p^2)\frac{1}{\gamma'(m_k^2)(m_k^2-p^2)}U^{-1}_{kj}(p^2) \tag{17}$$

with $\gamma'(p^2)\equiv d\gamma(p^2)/dp^2$. Now the general solution of (16) is thus

$$\Delta\phi_j^{(1)}(p)=\sum_{k=1}^{n} U_{jk}(p^2)C_k(p)\delta(p^2-m_k^2) \quad (18)$$

$$\equiv \sum_{k=1}^{n} C_k(p)\Delta\phi_{j,k}^{(1)} . \quad (19)$$

We assume $m_k^2>0$. (The Goldstone particle with $m_k^2=0$ is expected if the theory has a continuous symmetry and if $\phi_i^{(0)}$ breaks the symmetry.) We also assume $\gamma'(m_k^2)>0$ because otherwise the particle of mass m_k becomes a negative-norm state by (17). Since there are two solutions $p^0=\pm(\mathbf{p}^2+m_k^2)^{1/2}\equiv\pm\omega(\mathbf{p}^2)$ for the on-shell condition $p^2=m_k^2$, we can always write

$$C_k(p)=C_k^+(\mathbf{p})\theta(p^0)+C_k^-(-\mathbf{p})\theta(-p^0)$$

$$\equiv \sum_{\epsilon=\pm} C_k^\epsilon(\epsilon\mathbf{p})\theta(\epsilon p^0) . \quad (20)$$

These coefficients $C_k^\pm(\pm\mathbf{p})$ are to be determined by the boundary conditions on $\Delta\phi_i^{(1)}(x)$. We have to specify, for example, $\Delta\phi^{(1)}(t_0,x)$ and $d\Delta\phi^{(1)}(t_0,x)/dt_0$ at some time $t=t_0$. For each k, $\Delta\phi_{j,k}(p)$ in (19) is an eigenvector corresponding to the particle with mass m_k. Since $W_{ix,jy}^{(2)}$ is the causal two-point Green's function, our Eq. (13) coincides with the usual definition of the particle spectrum.

Higher-order terms. By (12), $\Delta\phi_i^{(2)}(x)$ is given by

$$\Delta\phi_i^{(2)}(x)=\frac{1}{2!}W_{ix,jy}^{(2)}\Gamma_{jy,kz,lw}^{(3)}\Delta\phi_k^{(1)}(z)\Delta\phi_l^{(1)}(w)$$

$$=\frac{1}{2!}W_{ix,jy,kz}^{(3)}W_{iy,j'y'}^{(2)-1}W_{kz,k'z'}^{(2)-1}\Delta\phi_{j'}^{(1)}(y')$$

$$\times\Delta\phi_{k'}^{(1)}(z') , \quad (21)$$

where we have used the following identity derivable from (5):

$$\Gamma_{ix,jy,kz}^{(3)}=W_{i'x',j'y',k'z'}^{(3)}W_{i'x',ix}^{(2)-1}W_{j'y',jy}^{(2)-1}\; W_{k'z',kz}^{(2)-1} . \quad (22)$$

In the solution (21), we do not have to add the solution of the homogeneous equation $\Gamma_{ix,jy}^{(2)}\Delta\phi_j^{(2)}(y)=0$ since in the sum (10) it can be absorbed by redefining $\Delta\phi_i^{(1)}(x)$. For $\Delta\phi_i^{(3)}(x)$, a new feature arises; we recover the one-particle-reducible graphs and get $W^{(4)}$. We differentiate (22) with respect to $\phi_l(w)$ and obtain the relation between $\Gamma^{(4)}$ and $W^{(4)},W^{(3)},W^{(2)}$. Using this formula, $\Delta\phi^{(3)}$ in (12) can be solved as

$$\Delta\phi_i^{(3)}(x)=\frac{1}{3!}W_{ix,jy,kz,lw}^{(4)}W_{iy,j'y'}^{(2)-1}W_{kz,k'z'}^{(2)-1}$$

$$\times W_{lw,l'w'}^{(2)-1}\Delta\phi_{j'}^{(1)}(y')\Delta\phi_{k'}^{(1)}(z')\,\Delta\phi_{l'}^{(l)}(w') . \quad (23)$$

By a straightforward mathematical induction we can easily show that the recovery of the one-particle-reducible graphs persists for general $\Delta\phi_i^{(n)}(x)$ and it is given by

$$\Delta\phi_i^{(n)}(x)=\frac{1}{n!}W_{ix,i_1x_1,i_2x_2,\ldots,i_nx_n}^{(n+1)}W_{i_1x_1,i_1'x_1'}^{(2)-1}$$

$$\times W_{i_2x_2,i_2'x_2'}^{(2)-1}\cdots W_{i_nx_n,i_n'x_n'}^{(2)-1}$$

$$\times\Delta\phi_{i_1'}^{(1)}(x_1')\Delta\phi_{i_2'}^{(1)}(x_2')\cdots\Delta\phi_{i_n'}^{(1)}(x_n') . \quad (24)$$

Equation (24) can also be derived by a similar procedure as in the case of the proof the tree theorem for any field theory. Insert (9) in (7) and expand (7) in the power series of $\Delta\phi_i(x)$:

$$0=\Gamma_{ix}[\phi]$$

$$=\Gamma_{ix,jy}^{(2)}\Delta\phi_j(y)+\frac{1}{2!}\Gamma_{ix,jy,kz}^{(3)}\Delta\phi_j(y)\Delta\phi_k(z)+\cdots .$$

The solution is given as

$$\Delta\phi_i(x)=\Delta\phi_i^0(x)$$

$$+W_{ix,jy}^{(2)}\left[\frac{1}{2!}\Gamma_{jy,kz,lw}^{(3)}\Delta\phi_k(z)\Delta\phi_l(w)+\cdots\right] , \quad (25)$$

where $\Delta\phi_i^0(x)$ is the solution of the homogeneous equation (13) so that we can set $\Delta\phi_i^0(x)=\Delta\phi_i^{(1)}(x)$. The iterative solution of (25) is known to produce all the "tree diagrams," but in our case the bare vertex is replaced by the full 1PI vertices $\Gamma^{(3)},\Gamma^{(4)},\ldots$, and the bare propagator by the full propagator $W^{(2)}$. Since all the "tree diagrams" exhaust all the one-particle-reducible diagrams, we have proved (24).

Now Eq. (24) is well defined since $W^{(2)-1}\Delta\phi$ cancels the pole of $W^{(n+1)}$ and we are just taking the residue of the pole of $W^{(n+1)}$. Therefore, $\Delta\phi_i^{(n)}$ is expected to be related to the S-matrix elements. In order to see this, let us suppose again that $\phi_i^{(0)}(x)$ is a constant. By (17) and (18) we have

$$W_{ij}^{(2)-1}(p^2)\Delta\phi_j^{(1)}(p)=-U_{ij}(p^2)\gamma_j'(m_j^2)C_j(p)$$

$$\times(p^2-m_j^2)\delta(p^2-m_j^2) , \quad (26)$$

which is inserted into (24). Then with (20) we get

$$\Delta\phi_i^{(n)}(p)=\frac{1}{n!}\int\cdots\int\sum_{\epsilon_1,\epsilon_2,\ldots,\epsilon_n=\pm}\;\sum_{j_1,j_2,\ldots,j_n=1}^{n}W_{ip;i_1,-\epsilon_1p_1,\ldots,i_n,-\epsilon_np_n}^{(n+1)}$$

$$\times\prod_{\alpha=1}^{n}U_{i_\alpha j_\alpha}(p_\alpha^2)\gamma'(m_{j_\alpha}^2)\delta(p_\alpha^2-m_{j_\alpha}^2)\theta(\epsilon_\alpha p_\alpha^0)C_{j_\alpha}^{\epsilon_\alpha}(\epsilon_\alpha\mathbf{p}_\alpha)\frac{d^4p_\alpha}{(2\pi)^4} . \quad (27)$$

By defining the wave-function renormalization factor of the kth channel through $Z_k=\gamma'(m_k^2)^{-1}$, we define the connected S matrix $\hat{S}(1,n)$ by

$$\hat{S}^{(1,n)}_{ip;j_1p_1,j_2p_2,\ldots,j_np_n}=\prod_{\alpha=1}^{n}\lim_{p_\alpha^2\to m_{j_\alpha}^2}\frac{i}{\sqrt{Z_{j_\alpha}}}U_{i_\alpha j_\alpha}(p_\alpha^2)\theta(\epsilon_\alpha p_\alpha^0)(m_{j_\alpha}^2-p_\alpha^2)W^{(n+1)}_{ip,i_1-\epsilon_1p_1,i_2-\epsilon_2p_2,\ldots,i_n-\epsilon_np_n}\ . \tag{28}$$

This represents the scattering of $n+1$ particles where the first particle is off the mass shell and the remaining n particles are on the mass shell. Each particle has the corresponding internal quantum number and if ϵ_α is $+$ $(-)$ then the αth particle is incoming (outgoing) with the four-momentum p_α. Thus, our final formula for $\Delta\phi_i(p)$ is

$$\Delta\phi_i(p)=\Delta\phi^{(1)}(p)+\sum_{n=2}^{\infty}\frac{1}{n!i^n}\sum_{j_\alpha,\epsilon_\alpha}\prod_{\alpha=1}^{n}\int C^{r\epsilon_\alpha}_{j_\alpha}(\mathbf{p}_\alpha)\frac{d^3\mathbf{p}_\alpha}{(2\pi)^4 2\omega_{j_\alpha}(\mathbf{p}_\alpha^2)}\hat{S}^{(1,n)}_{ip;j_1p_1,j_2p_2,\ldots,j_np_n}\ , \tag{29}$$

where $C_j^{r\epsilon}\equiv C_j^{\epsilon}/\sqrt{Z_j}$. If we take the residue of the pole corresponding to the first particle, the usual connected S-matrix elements $S^{(n)}$ emerge:

$$\Delta K[C]\equiv\int\frac{d^4p}{(2\pi)^4}\Delta\phi_i(p)U_{ij}(p^2)(m_j^2-p^2)\delta(p^2-m_j^2)\sum_{\epsilon=\pm}C_j^{r\epsilon}(\epsilon\mathbf{p})\theta(\epsilon p^0)$$
$$=\sum_{n=3}^{\infty}\frac{1}{(n-1)!}\frac{1}{i^n}\prod_{\alpha=1}^{n}\int\frac{d^3\mathbf{p}_\alpha}{(2\pi)^4 2\omega_{j_\alpha}(\mathbf{p}_\alpha^2)}C^{r\epsilon_\alpha}_{j_\alpha}(\mathbf{p}_\alpha)S^{(n)}_{j_1p_1,j_2p_2,\ldots,j_np_n}\ . \tag{30}$$

Equation (29), with (18) and (20) for $\Delta\phi_i^{(1)}(p)$, is the formula for $\Delta\phi_i(p)$ in terms of $C_j^{r\pm}(\mathbf{p})$ and the connected S-matrix elements. Since $C_j^{r\pm}(\mathbf{p})$ is fixed by the initial data for $\Delta\phi_i(p)$, we see that *the connected S-matrix elements completely determine* $\Delta\phi_i(p)$ *or* $\Delta\phi_i(x)$.

Conversely, if we consider $\Delta\phi_i(p)$ as a functional of the initial data $C_j^{r\pm}(\mathbf{p})$, then the connected S-matrix element is obtained by the formula

$$\delta^n\Delta K[C]/\delta C^{r\epsilon_1}_{j_1}(\mathbf{p}_1)\delta C^{r\epsilon_2}_{j_2}(\mathbf{p}_2)\cdots\delta C^{r\epsilon_n}_{j_n}(\mathbf{p}_n)=\frac{1}{(n-1)!}\frac{1}{i^n}\prod_{\alpha=1}^{n}\frac{1}{(2\pi)^4 2\omega_{j_\alpha}(\mathbf{p}_\alpha^2)}S^{(n)}_{j_1p_1,j_2p_2,\ldots,j_np_n}\ . \tag{31}$$

Our final formula is obtained by inserting (24) into (10) and then into $\Gamma[\phi]$:

$$\Gamma[\phi]=\Gamma[\phi^{(0)+}\Delta\phi]=\Gamma[\phi^{(0)}]+\sum_{n=3}^{\infty}\frac{1}{n!}\tilde{W}^{(n)}_{i_1x_1,i_2x_2,\ldots,i_nx_n}\Delta\phi^{(1)}_{i_1}(x_1)\Delta\phi^{(1)}_{i_2}(x_2)\cdots\Delta\phi^{(1)}_{i_n}(x_n)\ , \tag{32}$$

where $\tilde{W}^{(n)}$ is $W^{(n)}$ with the external legs amputated by $W^{(2)-1}$. It is evaluated at $\phi_i(x)=\phi_i^{(0)}(x)$. The above formula is easily obtained by a simple mathematical induction. Compare (32) with the off-shell expansion (8); in (32) the term $n=2$ is missing and $\Delta\phi_i^{(1)}(x)$ specifies the on-shell condition. In terms of the connected S-matrix element $S^{(n)}$ we get the on-shell expansion for the value of the effective action:

$$\Gamma[\phi]=\Gamma[\phi^{(0)}]+\sum_{n=3}^{\infty}\frac{1}{n!}\frac{1}{i^n}\sum_{j_\alpha\epsilon_\alpha}\prod_{\alpha=1}^{n}\int\frac{d^3\mathbf{p}_\alpha}{(2\pi)^4 2\omega_{j_\alpha}(\mathbf{p}_\alpha^2)}C^{r\epsilon_\alpha}_{j_\alpha}(\mathbf{p}_\alpha)S^{(n)}_{j_1p_1,\ldots,j_np_n}\ . \tag{33}$$

This has the structure that once $\phi^{(0)}$ and $C_j^{r\pm}(\mathbf{p})$ are given as initial data, $\Gamma[\phi]$ is fixed by the connected S-matrix elements. $\Gamma[\phi]$ can also be regarded as a generating functional of $S^{(n)}$.

Since we always stay on the configuration satisfying (11), it looks at first sight strange that we have $\Gamma[\phi]\neq\Gamma[\phi^{(0)}]$ (Ref. 2). The answer to this question lies in the fact that the stationary requirement (11) does not determine $\phi_i(x)$ completely and we have $C^{\pm}(\mathbf{p})$ left undetermined. These freedoms are utilized in order to arrive at the configuration which has a different value of the action. Note also that in the classical analytical mechanics, the variational principle leads to the equation of motion without taking the variation at the boundary. The function $C^{\pm}(\mathbf{p})$ corresponds to this freedom of the boundary value.

There are ample applications of our formula in various fields. We list several examples below.

(1) Although our discussions have been given in terms of the relativistic quantum field theory, the results given above are applicable to any quantum-mechanical system where we can calculate the effective action in a straightforward way (even easier than the field-theoretical case). To determine the spectra and the scattering among the modes corresponding to these spectra is the central problem of any quantum theory.

(2) Restricted to the quantum field theory, it appears to be an interesting application to introduce a source $J(x,y)$ coupled to a bilocal field $\phi(x)\phi(y)$. In this case, (13) becomes an exact bound-state (BS) equation[1,3] and we can obtain an exact expression for the S-matrix elements of the scattering of these bound states. These can be done

as a straightforward extension of the present method

(3) Our method is in contrast with the approach of several authors[4-6] who make use of the c-number field, the background field, satisfying the classical equation of motion. In particular, Jevicki and Lee[6] derived a full order relation which expresses the S matrix implicitly in terms of the effective action evaluated at the stationary solution. Our scheme, however, does not rely on the classical field and the particle spectrum itself is determined within the formalism and the S-matrix elements among these calculated eigenstates naturally emerge. Extension to the bound-state problem will be impossible for the background-field method.

(4) The effective action for the system of the finite temperature has been discussed in several papers.[7-9] By our formalism, we can study the spectrum and the scattering matrix for the finite-temperature system in a systematic way.

III. AMBIGUITY-FREE CRITERION FOR THE STABILITY

It became recognized and has been discussed by many people[10-14] that the effective potential possesses ambiguity especially when we choose a composite operator as an argument of the potential. In fact the second derivative of the effective potential evaluated at the stationary solution is not unique and it depends on how one calculates the effective potential.[10-16] Moreover, it has been found[10-12] that in some cases the effective potential is not bounded from below. There is also a case where the chosen solution corresponds to the state of the lowest energy yet the second derivative of the effective potential at the solution is not positive definite. The stability of the solution seems to depend on the procedure one has employed in evaluating the effective potential.

The purpose of this section is to answer the above controversial problems using formula (13). Based on this equation the general criterion for the stability has been given in Ref. 1. The crucial observations which constitute our starting point of this section are summarized in (i)–(iii) below. For this purpose we need the effective potential $V[\phi]$ which is obtained by the first term of the local expansion of $\Gamma[\phi]$:

$$\Gamma[\phi]=\int d^4x\,[-V(\phi(x))+\tfrac{1}{2}Z(\phi(x))\partial_\mu\phi_i(x)\partial^\mu\phi_i(x) + \cdots] \,. \tag{34}$$

Now we state our observations.

(i) The stability problem is essentially a time-dependent phenomenon and it *cannot* be discussed by the effective potential. In order to make clear this point, let us take a classical mechanical system with the particle coordinate $q(t)$ and the Lagrangian

$$\mathcal{L}(q)=\tfrac{1}{2}m\dot{q}^{\,2}-V(q)\quad(\dot{q}\equiv dq/dt)\,.$$

Let one of the static solutions to the equation of motion $\delta I[q]/\delta q(t)=0$ be $q^{(0)}$ where I is the action defined by $I[q]=\int\mathcal{L}(q)dt$. The stability of $q^{(0)}$ is determined by inserting $q(t)=q^{(0)}+\Delta q(t)$ into $\delta I[q]/\delta q(t)=0$ and retaining the linear term in $\Delta q(t)$:

$$m\,\Delta\ddot{q}(t)=-V''(q^{(0)})\Delta q(t)\quad(V''\equiv d^2V/dq^2)\,.$$

The requirement that $\Delta q(t)$ does not contain a blowing up solution is equivalent to $V''(q^{(0)})>0$, which is the well-known stability condition of the solution $q^{(0)}$. Note that the condition $V''(q^{(0)})>0$ follows because the kinetic energy term is known to be $\frac{1}{2}m\dot{q}^{\,2}$. In any quantum system, however, we have to *calculate* $Z(\phi(x))$ and also all the terms with higher derivatives for the discussion of the (space-)time dependence of the solution. The condition $V''>0$ is not necessarily the stability criterion.

(ii) The original theory is recovered only at $J_i=0$ which is equivalent to the equation of motion (7) of $\phi_i(x)$. We have always to satisfy (7) in order to discuss our theory governed by $\mathcal{L}(\phi)$. This is quite different from the variational approach where the theory itself is not modified. The second derivative of the effective potential is, however, an off-shell quantity in general (except for the case where it has the vanishing second derivative) so that it involves the information of the theory with $J_i(x)\neq 0$.

(iii) The stability of a given solution is a physical statement and it is determined once the theory is fixed. It follows that the stability criterion should not depend on (1) the gauge we have chosen in the case of the gauge theory or on (2) the renormalization scheme or on (3) the operator we have chosen to study the stability. The last statement needs clarification. We choose the operator O to study the ground state which might have the finite expectation value of O. But we can equally utilize another O' and should get the same physical results (such as the energy of the condensed state and the stability) as long as O and O' have the same quantum number and therefore mix. The fact that we get different answers to the stability problem for different operators is precisely the ambiguity problem stated above.

Now we state our stability criterion[1] which clarifies the points (i)–(iii) above: any solution to (13) does not blow up for large t. If $\phi_i^{(0)}(x)$ is space-time independent, $\phi_i^{(0)}(x)=\phi_i^{(0)}$, then we have the solution (18). In x space, we see from (18) that the stability condition of our solution $\phi_i^{(0)}$ is that there do not exist any poles of $W^{(2)}$ (evaluated at $\phi_i=\phi_i^{(0)}$) in the spacelike region ($m_i^2<0$), the absence of the tachyonic pole. Otherwise $\Delta\phi_i^{(1)}(t,\mathbf{p})$ for $\mathbf{p}^2<-m_j^2$ blows up for large t, as $\exp[\pm i(\mathbf{p}^2+m_j^2)^{1/2}t]$.

Our derivation naturally answers questions (i)–(iii) since it is the position of the pole of the Green's function which determines the stability. Several examples are given below for composite operators since it is for this case that the ambiguity has been pointed out.

A. The local field

We take as an example the Gross-Neveu (GN) model,[17] the O(N)-symmetric two-dimensional fermionic model, with the Lagrangian

$$\mathcal{L}^{\rm GN}=\bar{\psi}(i\partial\!\!\!/)\psi+\tfrac{1}{2}g^2(\bar{\psi}\psi)^2\,.$$

The source $J(x)$ is introduced in two ways.

(I) $J(x)$ couples to the auxiliary field of $\sigma(x)$. In this case we add the term $-\frac{1}{2}(\sigma-g\bar{\psi}\psi)^2+J\sigma$ to $\mathcal{L}^{\rm GN}$ and calculate $W_I[J]$ by the Lagrangian

$$\tilde{\mathcal{L}}^{\rm GN} \equiv \bar\psi i\partial\psi - \tfrac{1}{2}\sigma^2 + g\sigma\bar\psi\psi + J\sigma \ .$$

(II) $J(x)$ couples to $g\bar\psi(x)\psi(x)$. We consider here the Lagrangian

$$\mathcal{L}^{\rm GN} - \tfrac{1}{2}(\sigma - g\bar\psi\psi - J)^2 + gJ\bar\psi\psi = \tilde{\mathcal{L}}^{\rm GN} - \tfrac{1}{2}J^2 \ .$$

The only difference between (I) and (II) is the term $-\frac{1}{2}J^2$. We have to calculate the effective action Γ for two cases in order to discuss (13). We use σ in what follows as an argument of Γ in conformity with the usual notation.

For case (I), $\Gamma_{\rm I}$ is known to be obtained, for large N, by the stationary-phase contribution of σ satisfying

$$\begin{aligned} -\sigma(x) - iNg\,{\rm Tr}[A^{-1}(x,x)] &= -J(x) \\ &= \delta\Gamma_{\rm I}[\sigma]/\delta\sigma(x) \ , \\ A(x,y) &= [i\partial\!\!\!/ + g\sigma(x)]\delta^2(x-y) \ . \end{aligned} \tag{35}$$

We use in the following this $\sigma(x)$ for $\sigma_{\rm I}(x) = \delta W_{\rm I}[j]/\delta J(x)$. For the space-time-independent solution $\sigma^{(0)}(x) = \sigma^{(0)}$ of $\delta\Gamma_{\rm I}/\delta\sigma = 0$, after renormalization we get, in Fourier space,

$$\begin{aligned} \delta\Gamma_{\rm I}/\delta\sigma_r &= \sigma_r^{(0)}\{-1 + (\lambda_r/2\pi)[\ln(\sigma_0^2/\sigma_r^{(0)2}) + 2]\} \\ &= 0 \ , \end{aligned} \tag{36}$$

$$\begin{aligned} \Gamma^{(2)}(p^2) &\equiv \delta^2\Gamma_{\rm I}/\delta\sigma_r^2(p^2) \\ &= -1 + (\lambda_r/2\pi)[\ln(\sigma_0^2/\sigma_r^{(0)2}) + 2 - B] \ , \\ B &\equiv \sqrt{(1 - M^2/p^2)}\ln\left[\frac{\sqrt{-p^2+M^2} + \sqrt{-p^2}}{\sqrt{-p^2+M^2} - \sqrt{-p^2}}\right] \ , \end{aligned} \tag{37}$$

where $\lambda_r = g_r^2 N$, $M^2 = 4g_r^2\sigma_r^{(0)2}$, and r denotes the renormalized quantity. We have employed the renormalization condition[15] $\partial^2 V/\partial\sigma_r^2|_{\sigma_r=\sigma_0} = 1$ with V representing the corresponding effective potential. From (36) and (37), we find that Eq. (16) becomes [we delete the superscript (1) for $\Delta\sigma_{\rm I}^{(1)}$ in the following]

$$\Gamma^{(2)}(p^2)\Delta\sigma_{\rm I}(p) = 0$$

and has the following nontrivial solution $\Delta\sigma_{\rm I}(p)$ for (i) $p^2 = M^2 > 0$ (stable) corresponding to the symmetry-breaking solution $|\sigma_r| = \sigma_0\exp(1 - \pi/\lambda_r)$ and (ii) $p^2 = -g_r^2\sigma_0^2\exp(2 - 2\pi/\lambda_r) < 0$ (unstable) corresponding to the symmetric solution $\sigma^{(0)} = 0$. These agree with the well-known results.

Consider the next case (II). Since $W_{\rm II} = W_{\rm I} - \frac{1}{2}J^2$, $\sigma_{\rm II}(x) \equiv \delta W_{\rm II}/\delta J(x)$ is given by $\sigma_{\rm I}(x) - J(x) = \sigma(x) - J(x)$ where $\sigma(x)$ satisfies (35). We see therefore that $W^{(2)}_{{\rm II}x,y} \equiv \delta^2 W_{\rm II}/\delta J(x)\delta J(y)$ equals $\delta\sigma(x)/\delta J(y) - \delta^2(x-y)$. Using this relation and the equation obtained by taking $\delta/\delta J$ of (35), we get, for a space-time-invariant solution $\sigma^{(0)}$,

$$W_{\rm II}^{(2)}(p^2) = -\Gamma^{(2)}(p^2)^{-1} - Z_\sigma^{-1} \ ,$$

where Z_σ is the renormalization constant of the σ propagator. It is given, by using the cutoff Λ, as

$$Z_\sigma = -1 + (\lambda_r/2\pi)[\ln(\Lambda^2/g_r^2\sigma_0^2) - 2] \ .$$

Note that the solution $\sigma^{(0)}$ is the same for (I) and (II) since at $J=0$, $\sigma_{\rm II}(x) = \sigma_{\rm I}(x)$. Therefore, Eq. (16) takes the form

$$\begin{aligned} C\Gamma^{(2)}(p^2)\Delta\sigma_{\rm II}(p) &= 0 \ , \\ C &\equiv Z_\sigma[\Gamma^{(2)}(p^2) + Z_\sigma]^{-1} \ . \end{aligned}$$

Since the factor C does not vanish for finite p^2, we get the same stability condition as case (I). The second derivatives of the effective potential are different for the two cases and are given by

$$\begin{aligned} V_{\rm I}''(\sigma_r^{(0)}) &= 1 - (\lambda_r/\pi)\ln(\sigma_0/\sigma_r^{(0)}) \ , \\ V_{\rm II}''(\sigma_r^{(0)}) &= \{Z_\sigma/[Z_\sigma - 1 + (\lambda_r/\pi)\ln(\sigma_0/\sigma_r^{(0)})]\} \\ &\quad \times [1 - (\lambda_r/\pi)\ln(\sigma_0/\sigma_r^{(0)})] \ . \end{aligned}$$

They are renormalization-scheme dependent and for $V_{\rm II}''$ it depends even on the cutoff.

The result is easily understood in terms of $W[J]$: It is clear that $-\frac{1}{2}J^2$ does affect the second derivative of the effective potential but it is also clear that it does not have any effect on the position of the pole of $W^{(2)}$. It can also be shown that the ambiguity of adding an arbitrary polynomial[14] of J does not affect the stability.

B. The bilocal field.

We next discuss the following action of the N-component boson field:

$$S = \tfrac{1}{2}i\phi_i^a G_{0ij}^{-1}\phi_j^a + (1/4N)\phi_i^a\phi_j^a V_{ij,kl}\phi_k^b\phi_l^b \ , \tag{38}$$

where $a,b = 1$–N and G_0 represents the free propagator. The subscripts $i,j,\ldots,$ represent the space-time coordinates as well as other degrees of freedom. The action (38) includes most of the models discussed in the literature if we take $V_{ij,kl}$ as the nonlocal potential. We now introduce a bilocal auxiliary field σ_{ij} by adding the term

$$-(N/4)[\sigma_{ij} - (1/N)\phi_i^a\phi_j^a]V_{ij,kl}[\sigma_{kl} - (1/N)\phi_k^b\phi_l^b]$$

to (38). We thus consider

$$S = \tfrac{1}{2}\phi_i^a(iG_{0ij}^{-1} + V_{ij,kl}\sigma_{kl})\phi_j^a - (N/4)\sigma_{ij}V_{ij,kl}\sigma_{kl} \ . \tag{39}$$

(I) We add the source term $NJ_{ij}\sigma_{ij}$ to (39) and define $W_{\rm I}[J]$. For large N, it is easy to obtain

$$\begin{aligned} W_{\rm I}[J]/N &= \tfrac{1}{2}i\,{\rm Tr}\ln(G_0^{-1} - iV\sigma) \\ &\quad - \tfrac{1}{4}\sigma_{ij}V_{ij,kl}\sigma_{kl} + J_{ij}\sigma_{ij} \ , \end{aligned} \tag{40}$$

where σ satisfies the stationary-phase conditions

$$\tfrac{1}{2}V_{ij,kl}(G_0^{-1} - iV\sigma)^{-1}_{kl} - \tfrac{1}{2}V_{ij,kl}\sigma_{kl} + J_{ij} = 0 \ , \tag{41}$$

where $V\sigma$ represents $V_{ij,kl}\sigma_{kl}$. We use this σ_{ij} for $\sigma_{{\rm I}ij} = (1/N)\delta W_{\rm I}[J]/\delta J_{ij}$. The corresponding effective action is given by

$$\Gamma_{\rm I}/N = \tfrac{1}{2}i\,{\rm Tr}\ln(G_0^{-1} - iV\sigma) - \tfrac{1}{4}\sigma_{ij}V_{ij,kl}\sigma_{kl}$$

so that we get the Schwinger-Dyson (SD) equation as a stationary equation; since we have

$$\delta\Gamma_{\rm I}/\delta\sigma_{ij}=\tfrac{1}{2}NV_{ij,kl}[(G_0^{-1}-iV\sigma)_{kl}^{-1}-\sigma_{kl}]\ ,$$

by introducing V^{-1} through $V_{ij,kl}^{-1}V_{kl,mn}=\delta_{im}\delta_{jn}$, we find

$$\sigma_{ij}^{-1}=G_{0ij}^{-1}-iV_{ij,kl}\sigma_{kl}\ .$$

Denoting one of the solutions by $\sigma^{(0)}$, Eq. (16) takes the form

$$\tfrac{1}{2}NV_{ij,mn}\sigma_{mp}^{(0)}\sigma_{qn}^{(0)}(-\sigma_{pk}^{(0)-1}\sigma_{1q}^{(0)-1}+iV_{pq,kl})\Delta\sigma_{{\rm I}kl}=0\ . \tag{42}$$

This is the BS equation in large-N limit.[16,1]

(II) We adopt the source term $J_{ij}\phi_i^a\phi_j^a$ and calculate $W_{\rm II}[J]$. The large-N limit is obtained by the similar procedure as in (I). We find, after some calculations,

$$W_{\rm II}[J]/N=W_{\rm I}[J]/N-J_{ij}V_{ij,kl}^{-1}J_{kl}\ ,$$

where $W_{\rm I}[J]$ is given by (40) and (41). The term $-JV^{-1}J$ is the source of the ambiguity of the effective potential. Now, by defining $\sigma_{{\rm II}ij}$ through $\delta W_{\rm II}[J]/\delta J_{ij}$, we calculate $\Gamma_{\rm II}[\sigma_{\rm II}]$. The result is

$$\Gamma_{\rm II}[\sigma]/N=\tfrac{1}{2}i{\rm Tr}\ln\sigma^{-1}+\tfrac{1}{2}i\,{\rm Tr}G_0^{-1}\sigma+\tfrac{1}{4}\sigma_{ij}V_{ij,kl}\sigma_{kl}\ .$$

This agrees with the one calculated from the diagrammatic rule.[18,19] The stationary equation $0=\delta\Gamma_{\rm II}/\delta\sigma_{\rm II}$ is the same as for case (I) since at $J=0$, $\sigma_{{\rm I}ij}=\sigma_{{\rm II}ij}$. The stability equation is

$$-(iN/2)(-\sigma_{ik}^{(0)-1}\sigma_{lj}^{(0)-1}+iV_{ij,kl})\Delta\sigma_{{\rm II}kl}=0\ . \tag{43}$$

We conclude that the stability criteria (42) and (43) are the same.[20]

We can generalize the above formulation easily to the fermion field and can discuss QED. It has been found[12,13] in QED-type theory that the effective potential $V_{\rm AF}$ (auxiliary field method) predicts the stability of the chiral-symmetry-breaking solution while $V_{\rm CJT}$ (Cornwall-Jackiw-Tomboulis method) leads to the conclusion that the solution is at the saddle point. Our criterion (16) leads to the *identical* condition for these two methods. In order to determine whether or not the solution which breaks chiral symmetry is stable, we have to solve (43). This amounts to solving the BS equation for the fermion-antifermion system, where the coefficients are evaluated at the solution of the SD equation.

C. Free field case

For $V_{ij,kl}=0$, the effective action for $\sigma_{ij}=(1/N)\phi_i^a\phi_j^a$ is given by

$$\Gamma[\sigma]/N=(i/2){\rm Tr}\ln\sigma^{-1}+(i/2){\rm Tr}G_0^{-1}\sigma$$

so the effective potential is not bounded from below.[8,9] But it does not cause any trouble. Since $\delta\Gamma/\delta\sigma=0$ means $\sigma=G_0$, Eq. (16) takes the form

$$A_+(P,q)A_-(P,q)\Delta\sigma(P,q)=0\ ,$$

$$A_\pm=(P/2\pm q)^2-\mu^2\ ,$$

where $P(q)$ or μ is the total (relative) momentum or the mass of the two-particle system. The nontrivial solution of $\Delta\sigma$ exists when (1) $A_+=0, A_-\neq 0$, or $A_-=0, A_+\neq 0$ and (2) $A_+=A_-=0$. The condition (1) is actually related to the stability in the single-particle channel. Since $P^2>4\mu^2$, the condition (2) predicts the stability of the solution $\sigma=G_0$ in the two-particle channel if the one-particle channel is stable, i.e., $\mu^2>0$.

IV. DISCUSSIONS

From the discussions of the previous sections, we can say that the effective action is really a generating functional of the *physical* quantities. Expansion coefficients appearing in (33) are all physical: $\Gamma[\phi^{(0)}]$ is the value of the action of the ground state. If $\phi^{(0)}$ is a time-independent solution, then $\Gamma[\phi^{(0)}]$ is the negative of the total energy of the ground state times the whole time interval. The higher-order terms are on-shell scattering amplitudes. They are (i) gauge invariant (remember that $C^{r\pm}$ is gauge invariant) in the case of the gauge theory, (ii) independent of the renormalization scheme, and (iii) independent of the choice of the operators as the arguments of the effective action as long as these operators have the same quantum numbers.

The first property may open the possibility of the gauge-invariant approximation scheme. The third is equivalent to the well-known statement that we can use any operator as an interpolating field provided that it couples to the channel we are interested in. The applications of the formula (33) are reserved for future study.

We want to emphasize again that our procedure is a general one and it can be applied to a system as long as it is described by a Lagrangian or a Hamiltonian. We can also discuss the static case and the time-dependent case (stationary or nonstationary) by the same formalism.

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