Scalar field in a seven-dimensional manifold behaving as an SO(3)-covariant spinor field in space-time

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In a Kaluza theory with topology Minkowski-space × three-sphere, a metrical field equivalent to the Pauli matrices introduces a natural spin structure. Other fields occurring naturally on this manifold behave then as spinor fields when viewed from their space-time projection.

I. INTRODUCTION

In Refs. 1 and 2 we have given an eight-dimensional Riemannian space with a natural spin structure. This is achieved by considering a special metrical field enforcing a linkage between rotations of Minkowski space and rotations of an internal sphere S^3, such that a full rotation in space-time corresponds to a half rotation of the S^3. The existence of metrical fields behaving as spinor fields, when viewed from their space-time projection, has been deduced.

Here, we give a modification of the same idea with the following improvements.

(1) While in Refs. 1 and 2 the metrical field enforcing the linkage is of a statistical nature, the same is achieved in the present model by a field constant over space-time.

(2) The Pauli matrices appear in the present model, which can be viewed as components of the metrical tensor.

(3) We give here an explicit and particularly simple example of a spinor which can be verified by the reader in a few lines, while in Refs. 1 and 2 the existence of such examples was the result of a complicated and abstract theory.

(4) The internal space here is S^3 which is more reasonable than the internal space $S^3 \otimes S^1$ in Refs. 1 and 2.

II. SEVEN-DIMENSIONAL KALUZA THEORY

Consider the following eight-dimensional pseudo-Euclidean space $x^0, \ldots, x^7$ with scalar product

$$(x^0)^2 - \sum_{i=1}^{7} (x^i)^2 \ . \tag{2.1}$$

In this embedding space we consider a seven-dimensional submanifold given by

$$\sum_{i=4}^{7} (x^i)^2 = 1 \ . \tag{2.2}$$

It has the topology

$$\mathbb{R}^4 \otimes S^3 \tag{2.3}$$

and is a pseudo-Riemannian space.

We can generalize that situation by allowing an arbitrary metrical tensor

$$\gamma_{ij}(x^0, \ldots, x^7), \quad i,j = 0, \ldots, 7 \tag{2.4}$$

in the embedding space instead of (2.1). Again a Riemannian metric is introduced in the subspace (2.2). We call it EK7 (seven-dimensional Einstein-Kaluza space).

We propose the EK7, endowed with an arbitrary metric (2.4), as a model for nature, insofar as it reveals one known aspect of physical reality: namely, the occurrence of SO(3)-covariant spinor fields [although not of SO(3,1)-covariant spinor fields].

We call (2.2) the physical points, while the other points of the embedding space have been introduced for the sake of a convenient mathematical description only. The coordinates

$$(x^\mu) = (x^0, x^1, x^2, x^3) \tag{2.5}$$

will be identified with macroscopic space-time.

The situation (2.1) will be called the prevacuum. It has the isometry group

$$\mathrm{ISO}(3,1) \otimes \mathrm{SO}(4) \tag{2.6}$$

[we do not consider here discrete symmetries, therefore we write ISO(3,1) instead of IO(3,1) and SO(4) instead of O(4)] being the direct product of Poincaré transformations on space-time x^μ and of (constant) rotations of S^3.

A Poincaré transformation is a Lorentz transformation $\Lambda^\mu{}_\nu \in \mathrm{SO}(3,1)$ about a special point P (origin) followed by a translation a^μ:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \ . \tag{2.7}$$

(Obviously, the Lorentz-transformation $\Lambda^\mu{}_\nu$ contained in a Poincaré transformation is unique, i.e., independent of the choice of the origin P.)

III. THE HIGGS FIELD

On the prevacuum of the EK7 we assume a vector field with the following nonvanishing components:

$$H^\mu = \sum_{a,b=4}^{7} \sigma^\mu_{ab} x^a x^b, \quad \mu = 0,1,2,3 \ , \tag{3.1}$$

where σ^μ are the following constant, real matrices:

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$$\sigma^0=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\quad \sigma^1=\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix},$$
$$\sigma^2=\begin{bmatrix}0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0\end{bmatrix},\quad \sigma^3=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}. \tag{3.2}$$

For matrices with an even number of rows and columns, we use a partition into 2×2 submatrices. For reasons of computational convenience we can introduce a complex notation for real 2×2 matrices:

$$a=a'+ia''=\begin{bmatrix}a'&-a''\\a''&a'\end{bmatrix},\quad a\in\mathbb{C},\quad a',a''\in\mathbb{R}, \tag{3.3}$$

(3.3) is the usual representation of complex numbers by real 2×2 matrices. Both sides of (3.3) have the same algebraic (computational) properties.

In this notation (3.2) reads

$$\sigma^0=\begin{bmatrix}i&0\\0&i\end{bmatrix},\quad \sigma^1=\begin{bmatrix}0&1\\1&0\end{bmatrix},$$
$$\sigma^2=\begin{bmatrix}0&-i\\i&0\end{bmatrix},\quad \sigma^3=\begin{bmatrix}1&0\\0&-1\end{bmatrix}, \tag{3.4}$$

i.e., $\sigma^1,\sigma^2,\sigma^3$ are the Pauli matrices and σ^0 is the unit matrix.

The prevacuum, given by its metrical tensor (2.1), together with the vector field (3.1), will be proposed as a model for the physical vacuum. We will show that the vector field has endowed the prevacuum with a spin structure. Furthermore, the vector field breaks the symmetry group (2.6) of the prevacuum to a smaller one:

$$\mathbb{R}\otimes \mathrm{ISO}(3)\times \mathrm{U}(1)\,. \tag{3.5}$$

Here, $\mathbb{R}$ is the additive group of translations of the origin of time x^0. ISO(3) is the Poincaré group of three-space x^1,x^2,x^3 (equal to the Euclidean group consisting of translations and rotations). $\mathrm{U}(1)=\mathrm{SO}(2)\subset\mathrm{SO}(4)$ are the residual rotations of the S^3 leaving the vector field H^μ invariant and will tentatively be identified with the electromagnetic U(1).

It would be more desirable instead of (3.5) to have

$$\mathrm{ISO}(3,1)\otimes \mathrm{U}(1)\,, \tag{3.5$'$}$$

i.e., Poincaré invariance in space-time and a residual U(1) symmetry in the internal space S^3. This difficulty will be discussed in Sec. VI.

As will be shown in Sec. IV, the symmetry breaking to the group (3.5) enforced by the vector field H^μ will come out because in the direct product

$$\mathrm{SO}(4)\simeq \mathrm{SU}(2)_1\otimes \mathrm{SU}(2)_2\,, \tag{3.6}$$

one factor, e.g., $\mathrm{SU}(2)_1$, will be linked with the spatial rotations contained in ISO(3), while the second factor $\mathrm{SU}(2)_2$ will be broken to U(1). ($\simeq$ means local group isomorphism. For more details see Ref. 1, Sec. II.) A symmetry breaking of an internal SU(2) to an U(1) occurs also in Glashow-Salam-Weinberg theory.[3-10] Thus, tentatively we identify $\mathrm{SU}(2)_2$ with the SU(2) of weak interaction and H^μ with the Higgs field.

A spatial rotation contained in ISO(3) can be given as

$$\mathbf{x}'=(1-\cos\theta)(\mathbf{x}\cdot\mathbf{n})\mathbf{n}+\cos\theta\mathbf{x}-\sin\theta(\mathbf{n}\times\mathbf{x})\,. \tag{3.7}$$

Here, the notation

$$\mathbf{x}=(x^1,x^2,x^3)\,, \tag{3.8}$$

and the vector product $\times$ have been used. Equation (3.7) gives the transformation of coordinates (3.8) when the primed frame is obtained from the unprimed one by a rotation with angle θ about an axis given by the unit vector $\mathbf{n}=(n_x,n_y,n_z)$. The above-mentioned linkage between rotations $(\mathbf{n},\theta)\in\mathrm{ISO}(3)$ and elements of $\mathrm{SU}(2)_1\subset\mathrm{SO}(4)$ is given by

$$x'=Dx\,, \tag{3.9}$$

where x is the row of coordinates of the embedding space,

$$x^T=(x^4,x^5,x^6,x^7)\,, \tag{3.10}$$

and D is given by

$$D=\begin{bmatrix}\cos\frac{\theta}{2}&-n_z\sin\frac{\theta}{2}&n_y\sin\frac{\theta}{2}&-n_x\sin\frac{\theta}{2}\\ n_z\sin\frac{\theta}{2}&\cos\frac{\theta}{2}&n_x\sin\frac{\theta}{2}&n_y\sin\frac{\theta}{2}\\ -n_y\sin\frac{\theta}{2}&-n_x\sin\frac{\theta}{2}&\cos\frac{\theta}{2}&n_z\sin\frac{\theta}{2}\\ n_x\sin\frac{\theta}{2}&-n_y\sin\frac{\theta}{2}&-n_z\sin\frac{\theta}{2}&\cos\frac{\theta}{2}\end{bmatrix}. \tag{3.11}$$

IV. PROOF OF (3.5) AND (3.11)

Invariance of our Higgs field (3.1) under (constant) translations of space-time is obvious. Invariance under the combined (linked) transformation (2.7) and (3.9), where (2.7) is given by (3.7), means

$$H'^\mu(x')=\sum_{a,b=4}^{7}\sigma^\mu_{ab}x'^a x'^b=\sum_{a,b,c,d=4}^{7}\sigma^\mu_{ab}D^a_c D^b_d x^c x^d=\sum_{\nu=0}^{3}\Lambda^\mu{}_\nu H^\nu(x)=\sum_{\nu=0}^{3}\sum_{c,d=4}^{7}\Lambda^\mu{}_\nu\sigma^\nu_{cd}x^c x^d\,, \tag{4.1}$$

being equivalent to

$$D^T\sigma^\mu D=\Lambda^\mu{}_\nu\sigma^\nu\,. \tag{4.2}$$

Going over the complex notation (3.3),

$$B^{\dagger}\sigma^{\mu}B=\Lambda^{\mu}{}_{\nu}\sigma^{\nu} \tag{4.3}$$

with

$$B=\cos\frac{\theta}{2}+i\mathbf{n}\sigma\sin\frac{\theta}{2}\ , \tag{4.4}$$

which is satisfied, as is well known from two-component spinor calculus (see, e.g., Ref. 11, Sec. 17). The rotations (3.11) form the subgroup $SU(2)_1$ in (3.6) of SO(4). The other group $SU(2)_2$ is given by

$$D_2=\begin{bmatrix} \cos\frac{\theta_2}{2} & -n_{2z}\sin\frac{\theta_2}{2} & n_{2y}\sin\frac{\theta_2}{2} & n_{2x}\sin\frac{\theta_2}{2} \\ n_{2z}\sin\frac{\theta_2}{2} & \cos\frac{\theta_2}{2} & n_{2x}\sin\frac{\theta_2}{2} & -n_{2y}\sin\frac{\theta_2}{2} \\ -n_{2y}\sin\frac{\theta_2}{2} & -n_{2x}\sin\frac{\theta_2}{2} & \cos\frac{\theta_2}{2} & -n_{2z}\sin\frac{\theta_2}{2} \\ -n_{2x}\sin\frac{\theta_2}{2} & n_{2y}\sin\frac{\theta_2}{2} & n_{2z}\sin\frac{\theta_2}{2} & \cos\frac{\theta_2}{2} \end{bmatrix} . \tag{4.5}$$

The D of (3.11) will be denoted by D_1. D_1 and D_2 commute and a general element D of SO(4) can be written as

$$D=D_1D_2=D_2D_1=(-D_1)(-D_2)\ , \tag{4.6}$$

corresponding to the decomposition (3.6). The decomposition of a general D is unique up to a sign as indicated in the last part of (4.6). Each subgroup (3.11) or (4.5) is isomorphic (not only locally isomorphic) to an SU(2). For more details, see Ref. 1, Sec. 2.

The complex notation (3.3) fails here because the submatrices in (4.5) do not have the form (3.3). All statements given in the following can be formulated and proved by using in each step the corresponding real representations, such as (4.5) only. However, it is much more convenient to introduce the special real 2×2 matrix

$$\sigma=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} . \tag{4.7}$$

A general real 2×2 matrix can then be written as

$$a=a^{(1)}+a^{(2)}\sigma\ , \tag{4.8}$$

where $a^{(1)}$ and $a^{(2)}$ are complex numbers in the sense of (3.3). The quantity σ, which from a mathematical point of view is a quaternion, has the properties

$$\sigma^2=1\ , \tag{4.9}$$

$$\sigma z=z^{*}\sigma,\quad z\in\mathbb{C}\ , \tag{4.10}$$

where an asterisk denotes complex conjugation. Equation (4.5) can then be written as

$$D_2=\tilde{\varepsilon}B_2\tilde{\varepsilon} \tag{4.11}$$

with

$$\tilde{\varepsilon}=\begin{bmatrix}1 & 0\\ 0 & \sigma\end{bmatrix},\quad \tilde{\varepsilon}^2=1\ , \tag{4.12}$$

$$B_2=\cos\frac{\theta_2}{2}+i\mathbf{n}_2\sigma\sin\frac{\theta_2}{2} \tag{4.13}$$

which is analogous to (4.4).

The internal symmetry group U(1) in (3.5) of our physical vacuum is given explicitly by

$$\mathbf{n}_2=(0,0,1),\quad 0\le\theta_2<4\pi \tag{4.14}$$

in (4.13) or (4.5). Invariance of the Higgs field under this group means [compare (4.1)]

$$D_2^T\sigma^{\mu}D_2=\sigma^{\mu} \tag{4.15}$$

or in quaternionic notation

$$\tilde{\varepsilon}B_2^{\dagger}\tilde{\varepsilon}\sigma^{\mu}\tilde{\varepsilon}B_2\tilde{\varepsilon}=\sigma^{\mu}\ . \tag{4.16}$$

Transposition T of real matrices corresponds in quaternionic notation to Hermitian conjugation $\dagger$, when Hermitian conjugation $A^{\dagger}$ of an arbitrary quaternionic matrix A, i.e., a matrix consisting of quaternionic entries, such as (4.12), is defined as taking the transposition of the matrix and taking the Hermitian conjugate of each entry. We have the following rules:

$$\sigma^{\dagger}=\sigma\ , \tag{4.17}$$

$$z^{\dagger}=z^{*},\quad z\in\mathbb{C}\ , \tag{4.18}$$

$$(A_1A_2)^{\dagger}=A_2^{\dagger}A_1^{\dagger}\ , \tag{4.19}$$

$$\tilde{\varepsilon}^{\dagger}=\tilde{\varepsilon}\ . \tag{4.20}$$

For

$$B_2=\mathrm{diag}(\exp(i\theta_2/2),\exp(-i\theta_2/2))\ , \tag{4.21}$$

(4.16) can easily be verified. Thus we have proved that the symmetry group of our physical vacuum is at least (3.5).

To prove that there are no further symmetries, we consider the general symmetry of the prevacuum, i.e., a Poincaré transformation combined with a rotation of the S^3, which leave the Higgs-field invariant; i.e., we have to consider the general solution of (4.2), where D is an arbitrary orthogonal matrix and $\Lambda^\mu{}_\nu$ is an arbitrary Lorentz transformation. The case $\mu=0$ in (4.2), $\sigma^0=1$ and $D^T=D^{-1}$, implies that $\Lambda^\mu{}_\nu$ is a spatial rotation. Let us denote by D_Λ the D given by (3.11), which corresponds to the rotation $\Lambda^\mu{}_\nu$ and satisfies (4.2), i.e.,

$$D_\Lambda^T \sigma^\mu D_\Lambda = \Lambda^\mu{}_\nu \sigma^\nu . \tag{4.22}$$

By (4.6) the general orthogonal matrix D can be written as

$$D = D_2 D_1^{-1} D_\Lambda , \tag{4.23}$$

whence it follows from (4.2) that

$$D_2^T \sigma^\mu D_2 = D_1^T \sigma^\mu D_1 . \tag{4.24}$$

In quaternionic form, we find [compare (4.4) and (4.13)]

$$\tilde{\epsilon} B_2^+ \tilde{\epsilon} \sigma^\mu \tilde{\epsilon} B_2 \tilde{\epsilon} = B_1^\dagger \sigma^\mu B_1 . \tag{4.25}$$

We have to show that the only solution of that equation is $B_1=\pm 1$ and B_2 given by (4.21). [$B=-1$ can be absorbed in B_2 without changing the D in (4.23), compare (4.6)]. Writing

$$B_1 = \begin{bmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{bmatrix}, \quad |\alpha|^2+|\beta|^2=1 , \tag{4.26}$$

and similarly for B_2, we can work out the matrix equations (4.25), decompose it into complex and quaternionic part to obtain

$$\alpha_2^* \beta_2 = 0 , \tag{4.27}$$

$$\alpha_2^*\alpha_2 - \beta_2\beta_2^* = \alpha^*\alpha^* + \beta\beta = \alpha^*\alpha^* - \beta\beta = \alpha^*\alpha - \beta\beta^* . \tag{4.28}$$

There follows $\beta=0$, and $\alpha=\pm 1$. The case $\alpha_2=0$ in (4.27) leads to a contradiction. Thus, we have $\beta_2=0$ and $|\alpha_2|=1$.

V. A SCALAR FIELD BEHAVING AS A SPINOR FIELD

Consider the following scalar field

$$s(x,x^\mu) = \sum_{a=4}^{7} \psi_a(x^\mu) x^a \tag{5.1}$$

given by the four real macroscopic fields

$$\psi_a(x^\mu), \quad a=4,5,6,7 . \tag{5.2}$$

Under inhomogeneous rotations (2.7), where $\Lambda^\mu{}_\nu$ is a rotation (3.7) which is linked with an internal rotation (3.11), the scalar field (5.1) transforms as

$$s(x,x^\mu) = s(x',x'^\mu) \tag{5.3}$$

leading to

$$\psi_b(x^\mu) = \sum_{a=4}^{7} \psi'_a(x'^\mu) D_b^a, \quad b=4,\ldots,7 . \tag{5.4}$$

Using the complex abbreviations

$$\psi(x^\mu) = \begin{bmatrix} \psi_1(x^\mu) \\ \psi_2(x^\mu) \end{bmatrix} = \begin{bmatrix} \psi_4(x^\mu)+i\psi_5(x^\mu) \\ \psi_6(x^\mu)+i\psi_7(x^\mu) \end{bmatrix} , \tag{5.5}$$

we find

$$\psi'(x'^\mu) = B\psi(x^\mu) \tag{5.6}$$

with B given by (4.4). That is the well-known transformation law of a complex two-component spinor field.

Thus far, we have taken the formal view of considering passive, i.e., coordinate transformations. Any formula of a passive transformation can be read actively by considering a mapping of the manifold into itself and by changing the fields at any definite point to new ones. By postulating field equations (not specified in this paper), e.g., Einstein field equations, which are form-invariant under general coordinate transformation, the new fields again satisfy the field equations. The physical vacuum, specified by certain fields, should satisfy the field equations (note verified in this paper). Since the field equations are nonlinear, an arbitrary "superposition" upon the physical vacuum, e.g., the scalar field (5.1) will not satisfy the field equations. Suppose, however, that a special field (5.5) will satisfy the field equations. Since the physical vacuum is invariant under a symmetry of the vacuum, the transformed field (5.6) will again satisfy the field equations.

For reasons of simplicity, we have chosen a scalar field as an example for a field behaving as a spinor field when viewed macroscopically, i.e., when projected upon spacetime. Other examples, e.g., metrical fields, can be formed to behave as spinor fields. This is achieved by considering fields with an appropriate functional dependence on the inner space variable x. A complete classification of all possibilities is given in Ref. 12.

Finally, we note that the complex notation introduced in (5.5) can be used for a compact formulation of the vector field (3.1), i.e.,

$$H^\mu = \phi^\dagger \sigma^\mu \phi \tag{5.7}$$

with

$$\phi = \begin{bmatrix} x^4+ix^5 \\ x^6+ix^7 \end{bmatrix} . \tag{5.8}$$

The complex coordinates (5.8) for the S^3 can also be used for an economical formulation of the rotations of the S^3, i.e.,

$$\phi' = B\phi \tag{5.9}$$

with B given by (4.4) corresponding to D given by (3.11). This is discussed more completely in Ref. 1, Sec. 2.

VI. DISCUSSION

We have provided a model consisting of a manifold together with fields occurring naturally on manifolds, namely, tensor fields. Some of these fields behave as

SO(3)-covariant spinor fields. Thus, we have constructed a manifold with a natural spin structure. This is in contrast to most other models where spinors are introduced *ad hoc,* i.e., with no motivation why double-valued representations of the symmetry group of the vacuum should occur.

It seems worthwhile to recapitulate the essential results of this paper in a group-theoretical language. We have considered the Cartesian product of Minkowski space and a three-sphere, endowed with its natural product metric, together with a vector field H. The vector field is chosen so that there is a unique lift of the three-dimensional rotation group on Minkowski space to a faithful action of SU(2) as a group of isometries of the seven-dimensional geometry that leaves H fixed. As a result, if there is a vacuum state for the metric together with a vector field of the form (5.7), scalar fields on the seven-dimensional manifold will behave as SO(3)-covariant (although not Lorentz covariant) spinor fields on spacetime.

It is an inessential detail of the present formulation that in (3.1) we used a vector field enforcing the linkage between SO(3) rotation of space-time and internal rotational of the S^3 and thus providing the manifold with a natural spin structure. By replacing (3.1) by

$$H_{\mu m}=\sum_{a=4}^{7}\sigma_{\mu m p}x^{p},\quad \mu=0,1,2,3,\quad m=4,\ldots,7 \tag{6.1}$$

we obtained a metrical field satisfying the same purpose. Similarly, we could construct second-rank tensor fields behaving as spinor fields. Thus, a purely metrical theory can be obtained.

The lack of Lorentz invariance of the physical vacuum does not mean that particles cannot exist (electrons, photons, etc.) having a larger symmetry. This is analogous to the situation that fields exists (e.g., the free Maxwell field) which have a symmetry (conformal invariance) larger than Poincaré invariance of the vacuum. However, other perhaps less well-known particles should exist, which spoil the Lorentz invariance,

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