Dynamics of charged bubbles in general relativity and models of particles

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The dynamics of extended models of charged particles considered as sources of the Reissner-Nordström geometry is analyzed. They consist of spherical bubbles with electric charge and surface tension. Equations of motion for each model are derived showing the existence of stable equilibrium configurations with bubble sizes of the order of the classical radius of the particle.

I. INTRODUCTION

In this paper we study the dynamics of classical models of extended charged particles in general relativity. To this end we apply the Gauss-Codazzi formalism to sources of the Reissner-Nordström geometry made of perfect fluids with electric charge and surface tension. These sources are confined to thin shells (bubbles) that are allowed to expand or contract in a way consistent with the spherical symmetry.

The theory of surface layers in general relativity was developed in 1966 by Israel[1] making use of the Gauss-Codazzi equations. He applied this formalism to the derivation of the equation of motion of an uncharged spherical shell of dust. Soon afterwards, using a similar approach, Kuchar[2] and Chase[3] derived the general equation of motion of a charged shell. In the past few years the development of the inflationary model of the universe[4] has renewed the interest in the dynamics of thin shells in general relativity, in particular the evolution of domain-wall bubbles with spherical symmetry.[5] To analyze the existence of repulsive gravitational fields in the vicinity of a domain wall Ipser and Sikivie[6] rederived the results of Israel in a simple and elegant way. They introduced a convenient notation in which all tensorial quantities are visualized immersed in space-time, thus avoiding the introduction of two types of indices. In this paper, for the sake of clarity, we adopt the same procedure for a charged shell.

In Sec. II we derive the generic equation of motion of a charged spherical bubble of ideal fluid without restriction on its equation of state. Next, in Sec. III we obtain the conservation equations obeyed by these charged shells. In Sec. IV we apply these results to the study of three classical models of extended charged particles. We consider first a charged analog of the bubble of dust discussed by Israel in Ref. 1 by introducing suitable Poincaré stresses to compensate the Coulomb repulsion. Next, we analyze the evolution of a charged spherical domain wall, which corresponds to the general-relativistic generalization of an extended model of the electron constructed by Dirac.[7] Last, we develop a model introduced by Cohen and Cohen[8] where the mass of the particle is exactly equal to the energy content of the electromagnetic field outside the bubble. In Sec. V we discuss the relevance of general relativity in the description of an elementary-particle such as the electron.

II. CHARGED SPHERICAL BUBBLES

The Gauss-Codazzi equations are ideal to study the motion of two-dimensional sources of gravitation since they reduce the problem of solving Einstein's equation to the analysis of the intrinsic and extrinsic curvatures of the timelike hypersurface S representing the history of the source. The contracted Gauss-Codazzi equations are given by[6]

$$^3R+\Pi_{ab}\Pi^{ab}-\Pi^2=-2G_{ab}\xi^a\xi^b \ , \tag{1}$$

$$h_{ab}D_c\Pi^{bc}-D_a\Pi=G_{bc}h^b{}_a\xi^c \ . \tag{2}$$

Throughout the paper all indices run from 0 to 3 and the signature of space-time is $+2$. The vector ξ_a is the unit spacelike normal of S and h_{ab} denotes the projection operator into S, i.e.,

$$h_{ab}=g_{ab}-\xi_a\xi_b \ , \tag{3}$$

where g_{ab} is the metric tensor of space-time. From this relation it follows that h_{ab} is also the three-metric tensor on S, from which the Riemann scalar curvature invariant ^3R is constructed. On the other hand, Π_{ab} is the extrinsic curvature of S. This quantity is a measure of the variation of the unit normal vector ξ_b in a direction tangent to S, namely,

$$\Pi_{ab}\equiv D_a\xi_b=\Pi_{ba} \ , \tag{4}$$

where D_a is the projection into S of the covariant derivative ∇_a of space-time

$$D_a=h_a{}^c\nabla_c \ . \tag{5}$$

In Eqs. (1) and (2) $\Pi=\Pi_a{}^a$ and G_{ab} stands for the Einstein tensor

$$G_{ab}\equiv R_{ab}-\tfrac{1}{2}g_{ab}R=8\pi T_{ab} \ , \tag{6}$$

where T_{ab} is the stress-energy tensor. Units are chosen such that the velocity of light $c=1$ and Newton's constant $G=1$. Besides, the electromagnetic quantities are expressed in nonrationalized, electrostatic units.

We are going to apply the Gauss-Codazzi equations to determine the motion of infinitely thin, massive, spherically symmetrical shells (bubbles) with electric charge and surface tension. Birkhoff's theorem guarantees that the geometry outside the bubble is given by the Reissner-Nordström solution and that the interior geometry is flat. Introducing spherical coordinates, the corresponding exterior (+) and interior (−) metrics are

$$(ds^2)_+ = -\left[1-\frac{2M}{r}+\frac{e^2}{r^2}\right]dt^2 + \left[1-\frac{2M}{r}+\frac{e^2}{r^2}\right]^{-1} dr^2+r^2 d\omega^2 , \quad r>R(t) , \tag{7}$$

$$(ds^2)_- = -dt^2+dr^2+r^2 d\omega^2, \quad r<R(t) , \tag{8}$$

where $d\omega^2 = d\theta^2+\sin^2\theta\, d\phi^2$. In these coordinates the equation governing the history of the bubble is

$$r=R(t) . \tag{9}$$

It can be shown that the discontinuity of the extrinsic curvature $[\Pi_{ab}]$ is related to the stress-energy tensor S_{ab} of the hypersurface S by the Lanczos equation[9]

$$-8\pi S_{ab}=\gamma_{ab}-h_{ab}\gamma , \tag{10}$$

where

$$\gamma_{ab}=[\Pi_{ab}], \quad \gamma=\gamma_c{}^c \tag{11}$$

and

$$[\Pi_{ab}]=\Pi_{+ab}-\Pi_{-ab} . \tag{12}$$

We are interested in material sources S_{ab} consisting of a perfect fluid with a surface energy density σ and a surface tension τ, namely,

$$S_{ab}=(\sigma-\tau)u_a u_b-\tau h_{ab} , \tag{13}$$

where the four-velocity u^a is a timelike unit vector orthogonal to the spacelike unit normal ξ_a, i.e.,

$$u_a u^a=-1, \quad \xi_a u^a=0, \quad \xi_a \xi^a=1 . \tag{14}$$

With the help of these relations and Eqs. (7)–(9) we find the components of the four-vectors u^a and ξ_a at both sides of the bubble:

$$u_+^a=(-\beta g_{00}^{-1},\dot{R},0,0), \quad u_-^a=(\alpha,\dot{R},0,0) , \tag{15}$$

$$\xi_a^+=(-\dot{R},-\beta g_{00}^{-1},0,0), \quad \xi_a^-=(-\dot{R},\alpha,0,0) , \tag{16}$$

where an overdot denotes the derivative with respect to the proper time of an observer lying on the surface S which does not measure any flux in his local frame. In addition,

$$\alpha=(1+\dot{R}^2)^{1/2}, \quad \beta=\left[1+\dot{R}^2-\frac{2M}{R}+\frac{e^2}{R^2}\right]^{1/2} . \tag{17}$$

Referring to Maxwell's tensor $T_a{}^b$, its only nonvanishing components outside the bubble are

$$T_0{}^0=T_1{}^1=-T_2{}^2=-T_3{}^3=-\frac{e^2}{8\pi r^4} . \tag{18}$$

We now proceed to derive the equation of motion of the bubble in terms of the quantities σ and τ. To begin with, following Israel,[1] we evaluate the normal component of the four-acceleration a^b on opposite sides of the bubble

$$(\xi_b a^b)_+ \equiv \xi_b u^a \nabla_a u^b|_+ = -u^a u^b \Pi_{ab}|_+ = \frac{1}{\beta}\left[\ddot{R}+\frac{1}{R^2}\left[M-\frac{e^2}{R}\right]\right] , \tag{19}$$

$$(\xi_b a^b)_- \equiv \xi_b u^a \nabla_a u^b|_- = -u^a u^b \Pi_{ab}|_- = \frac{1}{\alpha}\ddot{R} . \tag{20}$$

On the other hand, by application of Eqs. (10) and (13), the sum and difference of Eqs. (19) and (20) is found to be

$$2\{\xi_b a^b\} = \left\{-\frac{2\tau}{\sigma}(h^{ab}+u^a u^b)\Pi_{ab}-\frac{2}{\sigma}S^{ab}\Pi_{ab}\right\} , \tag{21}$$

$$[\xi_b a^b] = -u^a u^b \gamma_{ab} = 4\pi(\sigma-2\tau) . \tag{22}$$

Here the curly and square brackets of a quantity A stand for the following expressions:

$$\{A\}=\tfrac{1}{2}(A_+ + A_-) , \tag{23}$$

$$[A]=A_+ - A_- . \tag{24}$$

The first term on the right-hand side of Eq. (21) is equal to

$$\left\{-\frac{2\tau}{\sigma}(h^{ab}+u^a u^b)\Pi_{ab}\right\} = -\frac{2\tau}{\sigma}(\alpha+\beta)\frac{1}{R} . \tag{25}$$

The second term is easily evaluated by taking the difference of Eq. (1) on opposite sides of S:

$$-\frac{2}{\sigma}\{S^{ab}\Pi_{ab}\} = \frac{e^2}{4\pi R^4 \sigma} . \tag{26}$$

Consequently,

$$2\{\xi_b a^b\} = -\frac{2\tau}{\sigma}(\alpha+\beta)\frac{1}{R}+\frac{e^2}{4\pi R^4\sigma} . \tag{27}$$

This result, combined with Eqs. (19), (20), and (22), gives us

$$(\alpha+\beta)\ddot{R} = -\frac{\alpha}{R^2}\left[M-\frac{e^2}{R}\right]-\frac{2\tau\alpha\beta(\alpha+\beta)}{\sigma R}+\frac{e^2\alpha\beta}{4\pi R^4\sigma} , \tag{28}$$

$$(\alpha-\beta)\ddot{R} = -\frac{\alpha}{R^2}\left[M-\frac{e^2}{R}\right]+4\pi\alpha\beta(\sigma-2\tau) . \tag{29}$$

On eliminating $\ddot{R}$ from these two equations one obtains the result

$$\left[\sigma-2\tau+\frac{e^2}{4\pi(\alpha+\beta)R^3}\right]\left[\sigma-\frac{M-\dfrac{e^2}{2R}}{2\pi(\alpha+\beta)R^2}\right]=0 . \tag{30}$$

Although this is an algebraic equation of second degree in σ, only one of the two roots holds: namely,

$$\sigma=\frac{M-\dfrac{e^2}{2R}}{2\pi(\alpha+\beta)R^2}\ , \tag{31}$$

which, on account of Eq. (17) can be put in the form

$$\alpha-\beta=4\pi\sigma R\ . \tag{32}$$

To show that this equation always holds we proceed as follows: from Eqs. (10) and (13) we obtain the relation

$$-8\pi\sigma=\gamma_{ab}u^a u^b+\gamma\ . \tag{33}$$

By Eq. (11), the right-hand side of this equation can be written in terms of the extrinsic curvature tensor Π_{ab}:

$$-8\pi\sigma=[\Pi_{ab}u^a u^b]+[\Pi]\ . \tag{34}$$

Furthermore, it follows from Eqs. (3) and (4) that

$$\Pi=\nabla_a\xi^a\ . \tag{35}$$

The value of this quantity on both sides of the bubble may be obtained from Eqs. (15) and (16). A straightforward calculation gives

$$\Pi_+=\frac{1}{\beta}\left[\frac{2}{R}\beta^2+\frac{1}{R^2}\left[M-\frac{e^2}{R}\right]+\ddot{R}\right]\ , \tag{36}$$

$$\Pi_-=\frac{1}{\alpha}\left[\frac{2}{R}\alpha^2+\ddot{R}\right]\ . \tag{37}$$

When these values are inserted in Eq. (34), and use is made of Eqs. (19) and (20), Eq. (32) is proven.

The expression (32) leaves undetermined the value of the surface tension τ. Therefore, it leads to a different equation of motion for every arbitrary choice of τ. Three cases of special interest will be worked out in Sec. IV.

III. CONSERVATION EQUATIONS

Taking the difference of the Codazzi equation (2) on opposite sides of S, and using Eqs. (3), (14), and (18), one finds

$$h_{ac}D_bS^{cb}=0\ , \tag{38}$$

whence, introducing expression (13) for S^{cb}, there results

$$(\sigma-\tau)h_{ac}u^bD_bu^c+u_aD_b[(\sigma-\tau)u^b]-h_a{}^bD_b\tau=0\ . \tag{39}$$

On contracting this equation with u^a one obtains

$$D_b(\sigma u^b)-\tau D_bu^b=0\ . \tag{40}$$

After substituting this result into Eq. (39) and rearranging terms, we arrive at

$$(\sigma-\tau)h_{ab}u^bD_bu^c-(h_a{}^b+u_au^b)D_b\tau=0\ . \tag{41}$$

We verify that the conservation equations (40) and (41) are just the same as in the case of a chargeless shell.[6]

For a spherical bubble, a straightforward calculation gives

$$D_au^a=2\frac{\dot{R}}{R}\ , \tag{42}$$

so that Eq. (40) reduces simply to

$$\dot{\sigma}+(\sigma-\tau)2\frac{\dot{R}}{R}=0\ , \tag{43}$$

where

$$\dot{\sigma}=(D_b\sigma)u^b\ . \tag{44}$$

Equation (42) entails also the conservation of the charge carried by the spherical bubble. The charge density q is given by

$$q=\frac{e}{4\pi R^2}\ . \tag{45}$$

Therefore, taking the total derivative,

$$-\frac{\dot{q}}{q}=2\frac{\dot{R}}{R}\ . \tag{46}$$

Comparison with Eq. (42) shows that the right-hand side of this equation is equal to D_au^a, whence we obtain the conservation law for the electric charge

$$D_aj^a=D_a(qu^a)=0\ .$$

Referring now to Eq. (41), for a spherical bubble the second term vanishes by virtue of Eqs. (3), (15), and (16). When $\sigma\neq\tau$ this yields

$$h_{ac}a^c\equiv h_{ac}u^bD_bu^c=0\ . \tag{47}$$

That is, the projection into the hypersurface S of the four-acceleration vanishes. This equation may also be expressed in the form

$$a^m+\xi^m u^bu^c\Pi_{bc}=0\ . \tag{48}$$

When this equation is contracted with ξ_m, it gives

$$\xi_ma^m=-u^bu^c\Pi_{bc}\ , \tag{49}$$

a result that we already used in deriving Eqs. (19) and (20).

IV. EQUATIONS OF MOTION

It is the purpose of this section to discuss three examples of charged spherical bubbles with surface tension when the energy density is given by Eq. (32). Each one gives rise to a model of an extended charged particle with interesting properties.

A. Poincaré's model

Following the ideas of Poincaré[10] we introduce a surface tension τ to compensate the Coulomb repulsion. This amounts to choosing τ in such a way that the last two terms on the right-hand side of Eq. (28) cancel each other. We thus obtain the expression for the Poincaré stress,

$$\tau=\frac{e^2}{8\pi(\alpha+\beta)R^3}\ , \tag{50}$$

leading to the equation of motion

$$\ddot{R}=-\left[\frac{\alpha}{\alpha+\beta}\right]\frac{1}{R^2}\left[M-\frac{e^2}{R}\right]\ . \tag{51}$$

In this case the radius of the bubble performs stable oscillations around the classical radius of the particle:

$$R_0=\frac{e^2}{M} . \tag{52}$$

In the limiting case when the charge e is made to tend to zero Eq. (51) becomes the equation for a bubble of dust obtained by Israel[1] in 1966. Despite the fact that $\tau\neq 0$, the material of the charged shell behaves as if it were made of dust. To verify this property we introduce, following Israel,[1] a fictitious background space $\tilde{V}$ such that the hypersurface S is regular in $\tilde{V}$ with extrinsic curvature $\tilde{\Pi}_{ab}$. Combining Eqs. (19), (20), (27), and (50) one obtains

$$u^a u^b \tilde{\Pi}_{ab}=0 . \tag{53}$$

On account of Eq. (48) this is equivalent to

$$\tilde{a}^m=0 . \tag{54}$$

This means that the points of the shell follow timelike geodesics in $\tilde{V}$. This is the characteristic property of a shell of dust established in Ref. 1 for the uncharged bubble.

A nice property of this model is that the bubble in equilibrium does not exert any gravitational attraction or repulsion on test particles lying arbitrarily close to its surface. To prove this assertion we combine Eqs. (22), (27), (32), and (50) to get

$$(\xi_b a^b)_+=-(\xi_b a^b)_-=2\pi(\sigma-2\tau)=\frac{M-\dfrac{e^2}{R}}{(\alpha+\beta)R^2} , \tag{55}$$

showing that the normal component of the acceleration vanishes at the classical radius (52) and that $\sigma=2\tau$ there. A similar result was found in Ref. 6 in the case of a plane wall with reflection symmetry.

To obtain a first integral of the equation of motion (51) we insert into the conservation equation (43) the value of τ/σ derived from Eqs. (32) and (50), namely,

$$\frac{\dot{\sigma}}{\sigma}+2\frac{\dot{R}}{R}=\frac{2\dot{R}}{R}\frac{\tau}{\sigma}=\frac{e^2\dot{R}}{R(2MR-e^2)} , \tag{56}$$

which is easily integrated to give

$$\sigma=\frac{k}{R^2}\left[M-\frac{e^2}{2R}\right] , \tag{57}$$

where k is the integration constant. This result, combined with Eq. (31), yields

$$\alpha+\beta=\text{const} . \tag{58}$$

This first integral was found by Israel[1] in the limiting case of a chargeless bubble of dust. It reduces the solution of the equation of motion (51) to a single quadrature.

B. Charged domain wall

The inflationary universe model[4] has motivated a great interest in the study of the evolution of domain walls.[5] In this case the stress-energy surface tensor corresponds to a perfect fluid with the equation of state $\sigma=\tau$. The conservation equation (40) shows that σ is an integral of the motion

$$\sigma=\tau=\text{const} . \tag{59}$$

For a spherical, charged, domain wall this result when introduced in Eq. (32) gives, after differentiation, the equation of motion

$$\frac{1}{\alpha}\ddot{R}=\frac{e^2}{8\pi\sigma R^4}-\frac{2\alpha}{R}+6\pi\sigma . \tag{60}$$

Of course, one arrives at the same equation by applying either Eq. (28) or (29).

From Eq. (32), we can obtain also an explicit equation for the mass of the shell:

$$M=\frac{e^2}{2R}+4\pi R^2\sigma(1+\dot{R}^2)^{1/2}-8\pi^2R^3\sigma^2 . \tag{61}$$

Let us examine the nature of the terms occurring in Eq. (61). The first one is the electrostatic energy lying in three-space outside the bubble, the second one is the relativistic surface energy, and the last one is the surface-surface binding energy. In fact Eq. (61) is a first integral of the equation of motion (60). Solving this equation for $\dot{R}$, the solution can be reduced immediately to a quadrature.

The equation of motion (60) shows the existence of a stable equilibrium configuration at a radius R_0 lying, according to Eqs. (29) and (32), within the range

$$\frac{e^2}{2M}<R_0<\frac{e^2}{M} . \tag{62}$$

In the Newtonian approximation of Eq. (61), i.e., in the limit

$$\frac{2M}{R}-\frac{e^2}{R^2}\ll 1,\quad \dot{R}^2\ll 1 , \tag{63}$$

it follows that

$$M\approx\frac{e^2}{2R}+4\pi R^2\sigma(1+\tfrac{1}{2}\dot{R}^2) . \tag{64}$$

This is just the total energy of a nonspinning classical extended electron model introduced by Dirac[7] in 1962. In this approximation there is a configuration of stable equilibrium at the radius

$$R_0=\frac{3e^2}{4M} , \tag{65}$$

which is in agreement with condition (62).

C. Charged bubble with vanishing energy density

From Eq. (32) it follows that the gravitational mass of a charged bubble is equal to

$$M=\frac{e^2}{2R}+2\pi(\alpha+\beta)R^2\sigma . \tag{66}$$

This equation means that when the radius of the shell is one-half the classical electron radius $R_0=e^2/2M$ the surface energy density σ vanishes and hence the mass is given by the energy content of the electric field. It would thus be expected that for a bubble having a configuration of stable equilibrium with the radius R_0 its whole mass should be of electromagnetic origin. A model of a classical extended charged particle having this property was constructed in 1969 by Cohen and Cohen.[11]

To guarantee the existence of a bubble performing stable oscillations around $R_0=e^2/2M$ we add to the Poincaré stress (50) another term such that Eq. (28) becomes

$$(\alpha+\beta)\ddot{R}=-\frac{\alpha}{R^2}\left[M-\frac{e^2}{2R}\right] . \tag{67}$$

The associated surface tension is found to be

$$\tau=\frac{e^2}{16\pi\beta R^3} . \tag{68}$$

When the bubble is in equilibrium at R_0 its surface tension reduces to

$$\tau_0=\frac{e^2}{16\pi R_0{}^3}=\frac{M_0}{8\pi R_0{}^2} . \tag{69}$$

This formula agrees with the value obtained in Ref. 13 in the limit $a=0$. On the other hand, the value given in Ref. 8 is not correct.

We must look now into the question of whether the total mass is of electromagnetic origin. The gravitational mass of the bubble at equilibrium is determined by Tolman's formula[14]

$$M=\int(T_0{}^0-T_1{}^1-T_2{}^2-T_3{}^3)\sqrt{-g}\,dV . \tag{70}$$

In the problem under consideration this integral should be evaluated upon the shell and over all three-space outside the bubble. The first one is the contribution to the mass of the surface tension τ_0 and is given by

$$M_{\tau_0}=-4\pi R_0{}^2(S_0{}^0-S_2{}^2-S_3{}^3)=-8\pi R_0{}^2\tau_0 , \tag{71}$$

and, using Eqs. (13) and (69), one obtains

$$M_{\tau_0}=-\frac{e^2}{2R_0} . \tag{72}$$

That is, the surface tension makes a negative contribution to the gravitational mass. On the other hand, Maxwell's tensor (18) gives the value

$$M_{\mathrm{em}}=\frac{e^2}{R_0} . \tag{73}$$

That is, the energy content of the gravitational field created by Maxwell's tensor is just equal to the electromagnetic energy. This result is a direct consequence of Eq. (70), since the electromagnetic stress-energy tensor is traceless.

Adding up the two contributions (72) and (73), the total mass amounts to

$$M=\frac{e^2}{2R_0} . \tag{74}$$

We conclude therefore that the contribution of the surface tension exactly cancels the gravitational energy produced by the electromagnetic field. This remarkable cancellation explains why the total mass of the particle is equal to the energy contained in the electromagnetic field in spite of the presence of the Poincaré stresses.

V. CONCLUSIONS

To analyze the dynamics of charged-particle models we have considered some exact solutions of the coupled Einstein-Maxwell equations associated with extended sources of the Reissner-Nordström field. At first sight it might appear that general-relativity effects are negligible when dealing with an elementary particle such as the electron. This looks almost evident when one verifies that the gravitational radius is about 40 orders of magnitude smaller than the classical radius. However, as was shown in this paper, general relativity is of great help for several reasons. First, the Gauss-Codazzi formalism gives the coupling between the material source and the gravitational and electromagnetic fields. Second, all equations are relativistically covariant. Third, gravitational effects are not negligible, as was evident in the evaluation of the gravitational mass in the third example discussed in the preceding section. As pointed out by Tolman in Ref. 14, the Newtonian potential gives a better approximation to the exact relativity expression for the gravitational mass than going at once to flat space-time. Last, the evolution and junction conditions of charged bubbles are easily obtained by means of the Gauss-Codazzi formalism in the form developed by Israel in Ref. 1.

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