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Deuteron as a toroidal Skyrmion

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The minimal-energy baryon-number-two solution of the Skyrme model is a static soliton which is approximately toroidal in shape. The symmetries of this solution imply that the ground state and the first excited state of this soliton have the quantum numbers of the deuteron d and its unbound isovector $^1S_0$ state. These identifications are tested by calculating the static electromagnetic properties of the deuteron and the transition moment for photodisintegration of the deuteron via the excitation, $\gamma d \rightarrow {}^1S_0$. The results are consistent with the interpretation of the deuteron as a quantum state of a toroidal Skyrmion.

I. INTRODUCTION

Since the original work of Skyrme,[1] physicists have been intrigued by the idea that in the case of $N_f=2$ flavors the low-lying baryons may be constructed as solitons, or Skyrmions, in an $\mathrm{SU}(N_f)\otimes\mathrm{SU}(N_f)$ chiral-invariant field theory of the pseudoscalar mesons. The fundamental assumption underlying this construction is that the conserved topological charge B of the meson field theory be identified with baryon number, so that individual baryons are solitons of unit topological charge. This identification remained pure conjecture, however, until 1983, when Witten[2] proved that this idea could be extended to $N_f \geq 3$, and that solitons of odd (even) B are fermions (bosons) if the number of colors N_c in the underlying gauge theory of strong interactions is odd. As $N_c=3$ in quantum chromodynamics, the $B=1$ Skyrmions are fermions. Moreover, upon semiclassical quantization, they exhibit the pattern of quantum numbers of the observed baryons, and possess static properties (charge radii, magnetic moments, etc.) in reasonable agreement with experiment.[3,4] The model has also been successful in describing pion-nucleon scattering phase shifts[5] and nucleon electromagnetic form factors.[6]

The SU(2) Skyrme model is also expected to have stable soliton solutions $U_B(\mathbf{r})$ with topological charge $B>1$. In particular, it is known[7,8] that a static $B=2$ solution $U_2(\mathbf{r})$ exists which is classically stable against decay to two widely separated $B=1$ Skyrmions. Given the success of the model in the $B=1$ sector, it is natural to ask whether these solitons with $B>1$ provide realistic models for nuclei.[9] The purpose of our investigations here is to test this idea for the configuration $U_2(\mathbf{r})$ and to ascertain whether the deuteron can be modeled by a quantum state of this $B=2$ Skyrmion.

Let us first give a brief synopsis of our current understanding of minimal-energy configurations in the $B=2$ sector of the SU(2) Skyrme model. The initial investigations[7] focused on the extraction of an internucleon potential. These efforts were based on Skyrme's original observation[1] that the $B=2$ solution describing well-separated Skyrmions may be approximated by a product *Ansatz.* If we place the centers of the two Skyrmions symmetrically about the origin along the x axis, then this product *Ansatz* takes the form

$$U(\mathbf{r};s,A)=U_1(\mathbf{r}+s\hat{\mathbf{x}})AU_1(\mathbf{r}-s\hat{\mathbf{x}})A^\dagger , \tag{1}$$

where

$$U_1(\mathbf{r})=\exp[iF(r)\hat{\mathbf{r}}\cdot\boldsymbol{\tau}] \tag{2}$$

is the minimal-energy $B=1$ Skyrmion (also known as the Skyrme hedgehog). Here A is a constant SU(2) matrix describing the relative isospin orientation of the two Skyrmions and $F(r)$ is the chiral angle whose profile has been numerically evaluated elsewhere.[3,4] Let H be the static Skyrme Hamiltonian and $M_1=H[U_1(\mathbf{r})]$ be the classical mass of a single Skyrmion. Then the difference

$$V(s,A)=H[U(\mathbf{r};s,A)]-2M_1 \tag{3}$$

can, for sufficiently large s, be interpreted as the static interaction energy of two Skyrmions. It is a function of the half-separation s and the SU(2) matrix A. An internucleon potential can then be obtained by projecting this function of A onto states of definite spin and isospin.

A numerical study of the product *Ansatz* (1) was first performed by Jackson, Jackson, and Pasquier,[7] who evaluated $V(s,A)$ as a function of s for several values of the matrix A, and asymptotically (i.e., $s\rightarrow\infty$) for all values of A. Their results suggested that the interaction energy is minimized if the isospin axes of one Skyrmion are rotated 180° about any axis perpendicular to the line of separation. In terms of the product *Ansatz* (1) this corresponds to a relative isospin orientation of $A=e^{i\pi\hat{\mathbf{n}}\cdot\boldsymbol{\tau}/2}$, with $\hat{\mathbf{n}}$ any unit vector perpendicular to $\hat{\mathbf{x}}$. For definiteness, we take $\hat{\mathbf{n}}=\hat{\mathbf{z}}$, or $A=i\tau_3$, arriving at the configuration

$$U_s(x,y,z)=U_1(x+s,y,z)\tau_3U_1(x-s,y,z)\tau_3 . \tag{4}$$

In previous work[8] we confirmed this conjecture with a numerical evaluation $V(s, A)$ for all values of s and A. In particular, taking the parameters of the Skyrme model to be $f_\pi = 108$ MeV, $e = 4.84$, and $m_\pi / e f_\pi = 0.263$ as in Ref. 4, we found that $V(s, i\tau_3)$ achieves its minimum at $s = s_0 = 2.8/ef_\pi = 1.1$ fm, corresponding to a classical binding energy of $-V(s_0, i\tau_3) = 1.06 f_\pi / e = 24$ MeV.

Thus the minimal-energy product-*Ansatz* configuration $U_{s_0}(\mathbf{r})$ describes two rather loosely bound Skyrmions with an internucleon separation $2s_0$ greater than 2 fm. Cognizant that the physical deuteron displays similar properties we suggested in Ref. 8 that $U_{s_0}(\mathbf{r})$ could be a sufficiently accurate variational estimate of the true $B=2$ minimal-energy configuration $U_2(\mathbf{r})$. Quantizing the collective modes of $U_{s_0}(\mathbf{r})$ in a manner directly analogous to the treatment of the $B=1$ Skyrmion by Adkins, Nappi, and Witten,[3] we found the lowest-energy state d indeed had the correct spin ($j=1$), isospin ($i=0$), and parity ($P=+$) quantum numbers of the deuteron. The identification of the state d with the physical deuteron was further strengthened by a calculation of its mean charge radius, and its magnetic and quadrupole moments. The values obtained for these observables agreed with experiment to within 30% (Ref. 8).

A problem with our analysis, however, was that the spectrum contained a state d' nearly degenerate in energy with the identified deuteron state d, possessing identical quantum numbers and similar static properties, a prediction clearly in conflict with experiment. This led us to suggest that the near degeneracy was an artifact of our variational approximation, and that the energy of the state d' would be much larger for the true solution. This suggestion was dramatically confirmed by Verbaarschot, Walhout, Wambach, and Wyld (VWWW), who numerically evaluated U_2 by minimizing the full Hamiltonian in the $B=2$ sector.[10] They discovered that U_2 possesses additional symmetries which in fact completely eliminate the state d' from the spectrum. The solution U_2 was also found independently by Kopeliovich and Shtern.[11].

To explain this result and to motivate the succeeding analysis, let us pause for a moment to consider the exact and approximate symmetries of the product *Ansatz* (4). Under planar reflections, $U_s(\mathbf{r})$ obeys

$$U_s(-x,y,z) = \tau_2 U_s(x,y,z)^\dagger \tau_2 \ , \tag{5a}$$

$$U_s(x,-y,z) = \tau_1 U_{-s}(x,y,z)^\dagger \tau_1 \ , \tag{5b}$$

$$U_s(x,y,-z) = U_{-s}(x,y,z)^\dagger \ . \tag{5c}$$

It also has a simple transformation law under parity, which takes an arbitrary configuration $U(\mathbf{r})$ into $U(-\mathbf{r})^\dagger$:

$$PU_s(\mathbf{r}) = U_s(-\mathbf{r})^\dagger = \tau_3 U_s(\mathbf{r}) \tau_3 \ . \tag{6}$$

Note that relation (6) is not independent of Eqs. (5a)–(5c), but represents the product of all three reflections.

We now observe that the product *Ansatz* (4), for finite s, has only one nontrivial symmetry, obtained by composing (5a) and (6), or equivalently (5b) and (5c):

$$U_s(x,-y,-z) = \tau_1 U_s(x,y,z) \tau_1 \ . \tag{7}$$

Equation (7) states that a spatial rotation of U_s by π radians about the axis of separation is equivalent to an isospin rotation by π radians about the same axis. On the other hand, U_s has no simple properties under spatial rotations about the y or z axes.

As shown in Ref. 8, symmetry (7) is crucial in obtaining the correct spectrum of quantum numbers. As we discuss in more detail below, any symmetry such as (7) is associated in the quantum theory with a Finkelstein-Rubinstein[12,13] constraint on the Hilbert space of physically allowed states. These constraints implement the requirements of the Pauli exclusion principle and, in particular, provide that individual $B=1$ Skyrmions are quantized as fermions. In the case of symmetry (7), the associated constraint disallows the wave function with the quantum numbers $i=j=0$, and thereby ensures that the lowest-energy state has the quantum numbers of the deuteron ($i=0$, $j=1$). Any additional symmetries of the static configuration would lead to further Finkelstein-Rubinstein constraints on the physical Hilbert space.

As we have already noted, the product *Ansatz* U_s has no nontrivial symmetries outside of relation (7). Consider, however, the situation as the individual Skyrmions are separated to infinite distance, $s \to \infty$. Using the approximate relation

$$U_{-s}(x,y,z) \simeq \tau_3 U_s(x,y,z) \tau_3 \quad \text{as } s \to \infty \ , \tag{8}$$

in conjunction with Eqs. (5b) and (5c), one may show that the product *Ansatz* enjoys approximate symmetries analogous to (7) which become exact in the infinite s limit. It was the suggestion of VWWW that these additional symmetries are in fact exact for the minimal-energy $B=2$ configuration U_2. Specifically,

$$U_2(-x,y,z) = \tau_2 U_2(x,y,z)^\dagger \tau_2 \ , \tag{9a}$$

$$U_2(x,-y,z) = \tau_2 U_2(x,y,z)^\dagger \tau_2 \ , \tag{9b}$$

$$U_2(x,y,-z) = \tau_3 U_2(x,y,z)^\dagger \tau_3 \ . \tag{9c}$$

Composing these two at a time yields the symmetries

$$U_2(x,-y,-z) = \tau_1 U_2(x,y,z) \tau_1 \ , \tag{10a}$$

$$U_2(-x,y,-z) = \tau_1 U_2(x,y,z) \tau_1 \ , \tag{10b}$$

$$U_2(-x,-y,z) = U_2(x,y,z) \ . \tag{10c}$$

Besides symmetry (10a), which is identical to relation (7) obeyed by the product *Ansatz,* there are two additional symmetries, Eqs. (10b) and (10c), associated with rotations by π radians about the y and z axes. The associated Finkelstein-Rubinstein constraints eliminate the spurious state d', leaving a unique state d with the quantum numbers of the deuteron.[10]

The discrete symmetries (10) do not in fact exhaust all the symmetries of the solution U_2. As originally conjectured by Manton,[14] and later confirmed by direct examination of the numerical solution,[15] U_2 possesses a continuous cylindrical symmetry. If we choose the z axis as the axis of symmetry, then the cylindrical symmetry of

U_2 takes the form

$$U_2(\rho,\phi+\alpha,z)=e^{-i\alpha\tau_3}U_2(\rho,\phi,z)e^{i\alpha\tau_3}, \quad (11)$$

where α is any real constant. Symmetry (10c) is just a special case of (11) with $\alpha=\pi$.

One consequence of (11) is that both the baryon-number and energy densities of the configuration U_2 are independent of the azimuth ϕ. Closer examination of the numerical solution reveals that the densities are peaked in the (ρ,z) plane about $\rho_0\simeq 1.4/ef_\pi\simeq 0.5$ fm and $z_0=0$. Therefore, qualitatively speaking, U_2 has the structure of a toroid and looks roughly like a donut.

In this paper the implications of the discrete symmetries (10) and the cylindrical symmetry (11) of U_2 are examined in more detail. In addition to identifying the ground state of this soliton with the deuteron, it is shown that some of its excited states can be interpreted as unbound states of the two-nucleon system. These identifications are tested with a calculation of the static electromagnetic properties of the deuteron and the transition moment for its photodisintegration into the unbound isovector $^1S_0$ state. A calculation of the electromagnetic form factors of the deuteron in the Skyrme model will be presented in a forthcoming paper.[16]

The outline of the paper is as follows. In Sec. II we begin with a general discussion of cylindrically symmetric static configurations in the SU(2) Skyrme model. Imposing relations (9) retrieves the specific form of the minimal-energy configuration U_2, described by two functions of ρ and z, namely, $F(\rho,z)$ and $\Theta(\rho,z)$. Solving the static equations of motion for F and Θ, and computing the energy and baryon-number densities, we obtain results in agreement with the previous computations of VWWW (Refs. 10 and 15) and Kopeliovich and Shtern.[11] In Sec. III the semiclassical quantization of U_2 is performed. Here we shall find that the symmetries of U_2 have, via the Finkelstein-Rubinstein constraints, profound implications for the spectrum of the quantized theory. In particular, these constraints ensure that the lowest-lying state has the quantum numbers of the deuteron. Next, in Sec. IV we give our results for the static electromagnetic properties of the deuteron state as well as its photodisintegration moment. We also offer some comments on the comparison of our results with experiment, emphasizing quantities independent of the values of the Skyrme-model parameters f_π and e. Finally, in Sec. V we contrast our approach, in which nuclei are interpreted as quantum states of solitons in the Skyrme model, with the "potential approach" advocated by others.[7] While our approach may at first seem counterintuitive since it does not build up nuclei out of individual nucleons, it can be tested by calculating the physical properties of the solitons. The calculations presented in this paper do in fact support our interpretation of the deuteron as a quantum state of a toroidal Skyrmion.

II. THE CLASSICAL CONFIGURATION $U_2(\mathbf{r})$

The dynamics of the SU(2)⊗SU(2) Skyrme model is described by the Lagrangian density

$$\begin{aligned}L=&\frac{f_\pi^2}{16}\mathrm{Tr}(\partial_\mu U\,\partial^\mu U^\dagger)\\&+\frac{1}{32e^2}\mathrm{Tr}[U^\dagger\partial_\mu U,U^\dagger\partial_\nu U]^2\\&+\frac{1}{8}m_\pi^2 f_\pi^2\mathrm{Tr}(U-1)\,.\end{aligned} \quad (12)$$

U is an SU(2) matrix describing the three pseudoscalar pion fields $\pi^a(x)$, $a=1,2,3$, viz., $U(x)=\exp[2i\pi^a(x)\tau^a/f_\pi]$. For the parameters f_π, e, and m_π, we employ the values listed below, as obtained from a semiclassical fit to the masses of the nucleon and delta:[4]

$$f_\pi=108\ \mathrm{MeV},\quad e=4.84,\quad m_\pi/ef_\pi=0.263\,. \quad (13)$$

As is well known, configurations $U(\mathbf{r})$ obeying the boundary condition

$$U(\mathbf{r})|_{|\mathbf{r}|\to\infty}=1 \quad (14)$$

fall into topological sectors labeled by the baryon number

$$B=\int d^3r\,B_0(\mathbf{r})\,, \quad (15)$$

where

$$B^\mu(x)=\frac{\epsilon^{\mu\nu\rho\tau}}{24\pi^2}\mathrm{Tr}(U^\dagger\partial_\nu U)(U^\dagger\partial_\rho U)(U^\dagger\partial_\tau U) \quad (16)$$

($\epsilon^{0123}=-1$) is the conserved-baryon-number current. We are interested in that subclass of configurations, $U_B(\mathbf{r})$, $B=1,2,\ldots$, which minimize the Hamiltonian functional,

$$\begin{aligned}H[U(\mathbf{r})]=\int d^3r\,\mathrm{Tr}\Bigg[&\frac{f_\pi^2}{16}\nabla U^\dagger\cdot\nabla U\\&-\frac{1}{32e^2}[U^\dagger\partial_i U,U^\dagger\partial_j U]^2\\&-\frac{1}{8}m_\pi^2 f_\pi^2(U-1)\Bigg]\,,\end{aligned} \quad (17)$$

subject to the condition that, in each case, the total baryon number B is fixed. For $B=1$ it has been established[17] that the minimal energy configuration U_1 is precisely the Skyrme solution given by Eq. (2). As mentioned in the Introduction it has recently been ascertained[10,14,15] that U_2 respects the discrete symmetries (10) and the cylindrical symmetry (11). The form and/or symmetries of the remaining minimal-energy configurations $U_B(\mathbf{r})$, $B>2$, are as yet unknown.

The cylindrical symmetry of a chiral field $U(\rho,\phi,z)$ does not imply that U is independent of the azimuth ϕ, but rather that a rotation $\delta\phi=\alpha$ about the axis of symmetry may be compensated by an isospin rotation:

$$U(\rho,\phi+\alpha,z)=A(\alpha)U(\rho,\phi,z)A(\alpha)^\dagger\,. \quad (18)$$

The matrices $A(\alpha)$ must form an U(1) subgroup of SU(2), which without loss of generality can be chosen to have the form

$$A(\alpha)=e^{i\alpha m\tau^3/2}\,. \quad (19)$$

Single valuedness under $\phi \to \phi + 2\pi$ requires that m be an integer. The most general configuration satisfying Eqs. (18) and (19) is

$$U(\rho,\phi,z) = e^{-im\phi\tau_3/2} \exp[iF_a(\rho,z)\tau_a] e^{im\phi\tau_3/2} , \tag{20}$$

where the quantities F_a, $a=1,2,3$, are functions of ρ and z. The spherically symmetric Skyrme *Ansatz* (2) must clearly be a special case of (20). The correspondence is achieved by taking $m=1$ and

$$F_1 = F(r)\sin\theta, \quad F_2 = 0, \quad F_3 = F(r)\cos\theta . \tag{21}$$

From Eq. (11) we see that U_2 describes a cylindrically symmetric configuration (20) with a winding number m equal to 2. The discrete symmetries (9), however, impose additional restrictions on the functions $F_a(\rho,z)$. The planar reflections (9a) and (9b), which in cylindrical coordinates correspond to transformations of azimuth $\phi \to \pi - \phi$ and $\phi \to -\phi$, respectively, prescribe that $F_2 = 0$. Symmetry (9c) then requires that

$$\begin{aligned} F_1(\rho,-z) &= F_1(\rho,z) , \\ F_3(\rho,-z) &= -F_3(\rho,z) . \end{aligned} \tag{22}$$

Comparing with the Skyrme hedgehog (21) this suggests a polar coordinate representation for U_2, defining $F(\rho,z)$ and $\Theta(\rho,z)$ via

$$\begin{aligned} F_1(\rho,z) &= F(\rho,z)\sin\Theta(\rho,z) , \\ F_3(\rho,z) &= F(\rho,z)\cos\Theta(\rho,z) . \end{aligned} \tag{23}$$

Conditions (22) are then replaced by

$$\begin{aligned} F(\rho,-z) &= F(\rho,z) , \\ \Theta(\rho,-z) &= -\Theta(\rho,z) + l\pi , \end{aligned} \tag{24}$$

where l is an odd integer which we specify below. With definitions (23) we may represent the minimal-energy configurations U_1 and U_2 with the unified form

$$\begin{aligned} U_m(\rho,\phi,z) = {} & e^{-im\phi\tau_3/2} e^{-i\Theta(\rho,z)\tau_2/2} e^{iF(\rho,z)\tau_3} \\ & \times e^{i\Theta(\rho,z)\tau_2/2} e^{im\phi\tau_3/2}, \quad m=1,2 . \end{aligned} \tag{25}$$

By reproducing the known results of the Skyrme hedgehog (corresponding to $m=1$), this unified form proved quite useful in checking the accuracy of our numerical computations.

The boundary condition (14) implies that

$$F(\rho,z) \to 0 \quad \text{as } \rho^2 + z^2 \to \infty . \tag{26}$$

Along the axis of symmetry ($\rho = 0$) we also demand that $U^\dagger \nabla U$ be continuous so that the energy density there is finite. This latter condition implies that $F(\rho,z)$ is continuous and that

$$\sin F(0,z) \sin\Theta(0,z) = 0 . \tag{27}$$

Hence $\Theta(0,z)$ must be an integer multiple of π but can have jump discontinuities wherever $\sin F(0,z)$ is zero. The simplest possibility is that its only jump discontinuity occurs at $z=0$. Then, without loss of generality, we may take $\Theta(0,z)=0$ for $z>0$, and thus arrive at the boundary conditions

$$F(0,0) = k\pi, \quad \Theta(0,z) = \begin{cases} 0, & z>0 , \\ l\pi, & z<0 . \end{cases} \tag{28}$$

Here k is any integer and l is the odd integer specified by Eq. (24).

The value of the integer k is fixed by the total baryon number B. Computing the baryon-number density $B_0(\rho,z)$ for the configuration U_m we find

$$B_0(\rho,z) = -\frac{m}{2\pi^2 \rho} \sin^2 F \sin\Theta (F_{,z}\Theta_{,\rho} - F_{,\rho}\Theta_{,z}) , \tag{29}$$

yielding a total baryon number of

$$B = \frac{mF(0,0)}{\pi} = mk . \tag{30}$$

In deriving (30) we have used the boundary conditions (26) and (28). Since the minimal-energy configurations U_1 and U_2 correspond to $m=1$ and $m=2$, respectively, we infer that $k=1$ in both cases. Finally, the value of the integer l defined by (24) is determined by direct comparison with the known numerical solutions;[10,15] this gives $l=1$ for both U_1 and U_2.

Having fixed the boundary conditions, and consequently the baryon number, of the configurations U_m under consideration, the functions $F(\rho,z)$ and $\Theta(\rho,z)$ are evaluated by minimizing the classical mass:

$$M_m \equiv H[U_m] = \frac{\pi f_\pi}{e} \int_0^\infty \rho\, d\rho \int_{-\infty}^\infty dz\, \epsilon_m(\rho,z) , \tag{31a}$$

$$\begin{aligned} \epsilon_m(\rho,z) = {} & \left[\frac{1}{4} + \frac{m^2}{\rho^2} \sin^2 F \sin^2\Theta \right] [F_{,\rho}^2 + F_{,z}^2 + \sin^2 F (\Theta_{,\rho}^2 + \Theta_{,z}^2)] + \sin^2 F (F_{,z}\Theta_{,\rho} - F_{,\rho}\Theta_{,z})^2 \\ & + \frac{m^2}{4\rho^2} \sin^2 F \sin^2\Theta + \beta^2 \sin^2 \left[\frac{F}{2} \right] , \end{aligned} \tag{31b}$$

where $\beta = m_\pi / e f_\pi$. In the integral, all lengths have been expressed in units of $1/ef_\pi$.

Before giving our results for F and Θ, some comments on our numerical methods are in order. Note that by (24) the classical mass (31a) can be rewritten as an integral over the quarter-plane, $Q = (0 \le \rho < \infty,\ 0 \le z < \infty)$. On Q, the boundary conditions (26) and (28) are supplemented with the condition

$$\Theta(\rho, 0) = \pi/2 \ , \qquad (32)$$

as follows from (24) and $l=1$. To accommodate the asymptotic behavior of F as $\sqrt{\rho^2+z^2}$ goes to infinity, we have found it convenient to map Q onto the unit square $(0,\pi]\otimes(0,\pi]$, using a scaling function that roughly matches the chiral angle $F(r)$ of the Skyrme hedgehog (2). On the unit square, F and Θ are represented by two-dimensional arrays on a uniform 20×20 lattice, and are evaluated by minimizing M_m, $m=1,2$, using the Gauss-Seidel technique with overrelaxation.[18] Numerical accuracy was checked by computing the classical mass, inertia tensors, isoscalar charge radius, and isoscalar magnetic moment of the Skryme hedgehog (corresponding to $m=1$); the known results were reproduced to within 1, 7, 1, and 3 %, respectively.

In Figs. 1–4, we present our results for $F(\rho,z)$, $\Theta(\rho,z)$, the baryon-number density (29), and the energy density (31b). These contour plots agree with the three-dimensional results of Verbaarschot *et al.*[10,15] In particular, a peak in the baryon-number and energy densities is evident near the point $\rho_0 = 1.38/ef_\pi$, $z_0 = 0$ confirming the toroidal structure of U_2. Note that the contours of energy density follow those of the baryon-number density rather closely, except that the energy density does not vanish at $\rho = 0$.

The classical mass M_2 is determined to be 1659 MeV and agrees, to within a few percent, with the values 1620 and 1660 MeV obtained in Refs. 10 and 11, respectively, using the same parameter set (13). By way of comparison, the classical mass of the minimal-energy product *Ansatz* U_{s_0} is 1708 MeV.[8]

Another quantity of interest is

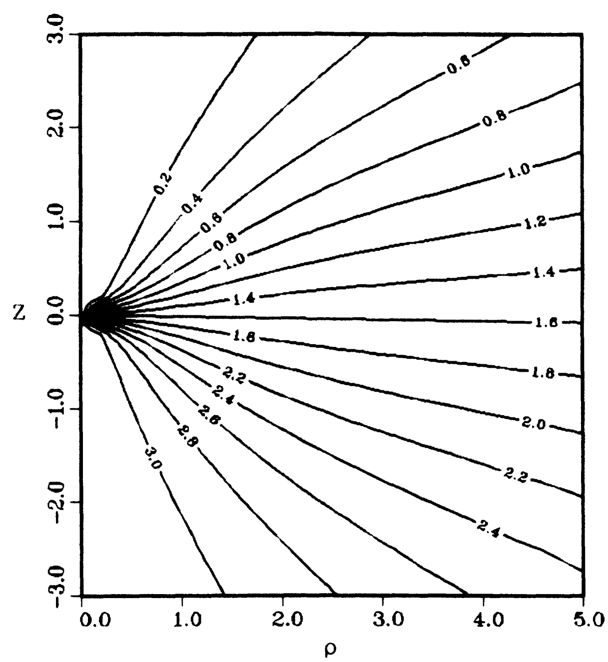

FIG. 2. Contour plot of $\Theta(\rho,z)$ for the $B=2$ Skyrmion $U_2(\mathbf{r})$.

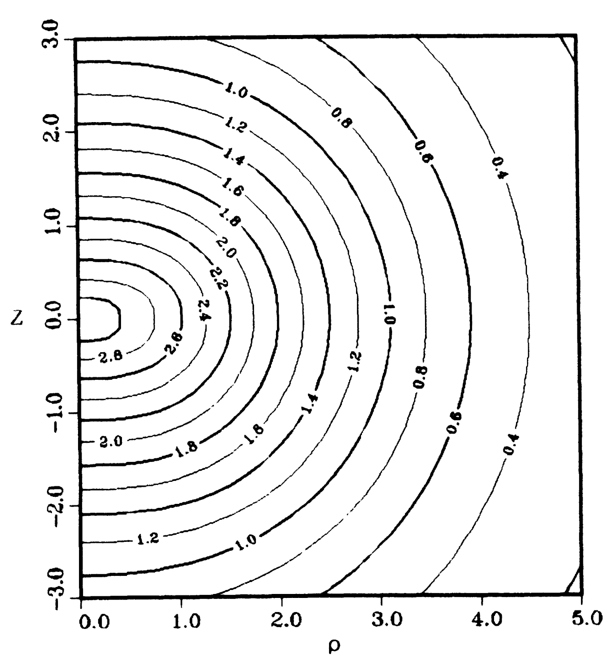

FIG. 1. Contour plot of $F(\rho,z)$ for the $B=2$ Skyrmion $U_2(\mathbf{r})$.

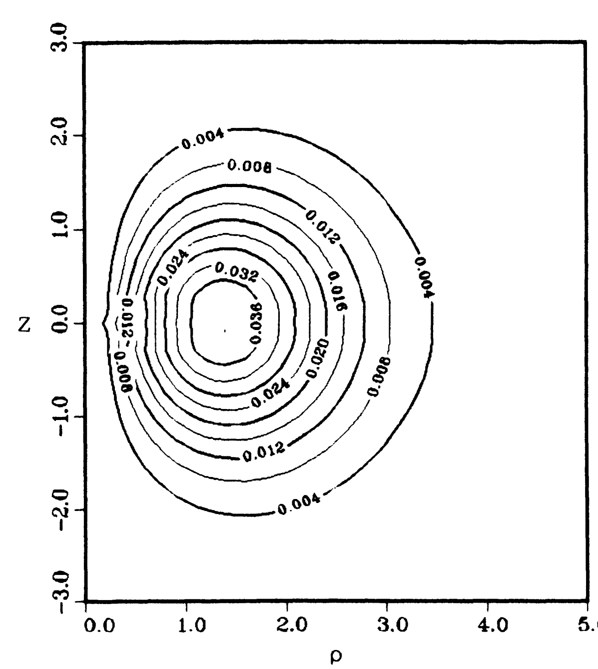

FIG. 3. Baryon-number density $B_0(\rho,z)$ of $U_2(\mathbf{r})$ [cf. Eq. (29)].

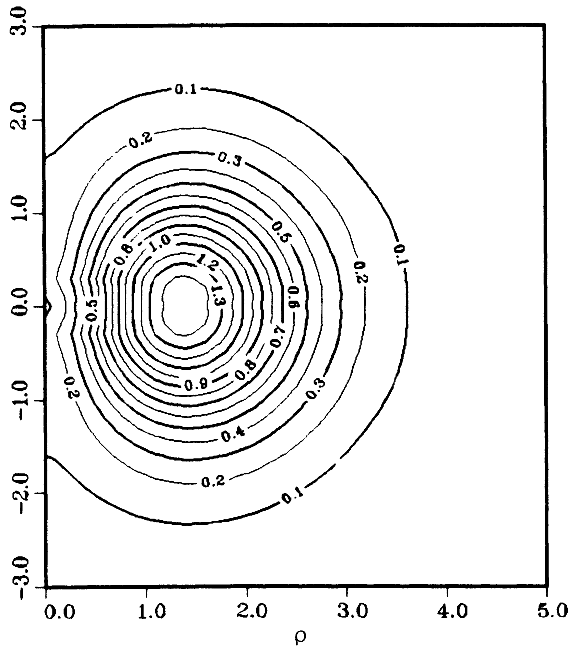

FIG. 4. Energy density $\epsilon_2(\rho,z)$ of $U_2(\mathbf{r})$ [cf. Eq. (31b)].

$$\langle \rho \rangle = \tfrac{1}{2}\int d^3r\, \rho B_0(\mathbf{r}) \tag{33}$$

which measures in a rough sense the mean radius of the toroid. We obtained $\langle \rho \rangle = 2.06/ef_\pi \simeq 0.78$ fm, in reasonable agreement with the value $\pi R_0/4 = 1.95/ef_\pi$ quoted in Ref. 15.

III. SEMICLASSICAL QUANTIZATION

Given a generic static configuration $U(\mathbf{r})$, there exists a nine-parameter set of configurations, all degenerate in energy, obtained from U by some combination of translation, spatial rotation, and isospin rotation:

$$U(\mathbf{r};\mathbf{X},A,B) \equiv AU(R(B)(\mathbf{r}-\mathbf{X}))A^\dagger \,. \tag{34}$$

Here we have chosen to represent spatial rotations by an SU(2) matrix B and

$$R_{ij}(B) = \tfrac{1}{2}\mathrm{Tr}(\tau_i B \tau_j B^\dagger) \tag{35}$$

is the associated rotation matrix. [The distinction between the SU(2) matrix B introduced here and the total baryon number should always be clear by context.] As is well known, the classical degeneracy of the configurations (34) is removed when the theory is quantized. A minimal quantization procedure incorporating this feature promotes the parameters $\mathbf{X}$, A, and B to the status of dynamical variables $\mathbf{X}(t)$, $A(t)$, and $B(t)$. One then quantizes the resulting dynamical system according to standard canonical methods. As we shall only be concerned with the computation of static observables below (e.g., at the limit of zero momentum transfer), we shall ignore the translational degrees of freedom $\mathbf{X}(t)$ and quantize the Skyrme solitons in their rest frame.

Our dynamical *Ansatz* for the chiral field is then

$$\hat{U}(\mathbf{r},t) = A(t)U(R(B(t))\mathbf{r})A(t)^\dagger \,, \tag{36}$$

where the caret serves to distinguish $\hat{U}$, the quantum operator, from U, the static background field. Inserting (36) into the Lagrangian (12), the kinetic energy contribution to the total energy is determined to be

$$T = \tfrac{1}{2}a_i U_{ij} a_j - a_i W_{ij} b_j + \tfrac{1}{2} b_i V_{ij} b_j \,, \tag{37}$$

where

$$a_j = -i\,\mathrm{Tr}\tau_j A^\dagger \dot{A}, \quad b_j = i\,\mathrm{Tr}\tau_j \dot{B}B^\dagger \,, \tag{38}$$

and the inertia tensors U_{ij}, W_{ij}, and V_{ij} expressed as functionals of the background field $U(\mathbf{r})$, are given by

$$U_{ij} = \frac{1}{8e^3 f_\pi}\int d^3r\, \mathrm{Tr}\{U^\dagger[\tfrac{1}{2}\tau_i,U]U^\dagger[\tfrac{1}{2}\tau_j,U] + [U^\dagger\partial_k U, U^\dagger[\tfrac{1}{2}\tau_i,U]][U^\dagger\partial_k U, U^\dagger[\tfrac{1}{2}\tau_j,U]]\} \,, \tag{39a}$$

$$W_{ij} = -U_{ij}\{[\tfrac{1}{2}\tau_j,U] \to -i(\mathbf{r}\times\nabla)_j U\} \,, \tag{39b}$$

$$V_{ij} = -W_{ij}\{[\tfrac{1}{2}\tau_i,U] \to -i(\mathbf{r}\times\nabla)_i U\} \,, \tag{39c}$$

where all lengths are in units of $1/ef_\pi$. In the quantized theory, a_j and b_j are expressed in terms of operators K_i and L_i conjugate to the coordinates A and B, via the linear relations

$$K_i = U_{ij}a_j - W_{ij}b_j, \quad L_i = -W_{ij}^T a_j + V_{ij}b_j \,, \tag{40}$$

where the superscript T denotes transpose. The operators $\mathbf{K}$ and $\mathbf{L}$ are in fact the body-fixed isospin and angular momentum operators, respectively, and are related to the usual coordinate-fixed isospin ($\mathbf{I}$) and spin ($\mathbf{J}$) operators by orthogonal transformations:

$$I_i = -R_{ij}(A)K_j, \quad J_i = -R_{ij}(B)^T L_j \,. \tag{41}$$

The four sets of operators mutually commute and individually satisfy the SU(2) commutation relations

$$\begin{aligned} &[I_i,I_j] = i\epsilon_{ijk}I_k, \quad [K_i,K_j] = i\epsilon_{ijk}K_k \,, \\ &[J_i,J_j] = i\epsilon_{ijk}J_k, \quad [L_i,L_j] = i\epsilon_{ijk}L_k \,. \end{aligned} \tag{42}$$

By (41) the Casimir invariants satisfy the equalities

$$\mathbf{I}^2 = \mathbf{K}^2, \quad \mathbf{J}^2 = \mathbf{L}^2 \,. \tag{43}$$

Thus the operators $\mathbf{I},\mathbf{J},\mathbf{K}$, and $\mathbf{L}$ form the Lie algebra of $\mathrm{O}(4)_{I,K}\otimes\mathrm{O}(4)_{L,J}$. Their action on the coordinates A and B is given by

$$[I_i,A]=-\tfrac{1}{2}\tau_i A\,,\quad [J_i,B]=\tfrac{1}{2}B\tau_i\,,$$
$$[K_i,A]=\tfrac{1}{2}A\tau_i,\quad [L_i,B]=-\tfrac{1}{2}\tau_i B\,, \tag{44}$$

with all other commutators between momenta and coordinates zero.

Let us now apply this formalism to the semiclassical quantization of the minimal-energy $B=2$ configuration U_2, whose explicit form (25) we developed in the previous section. The discrete symmetries (9) and cylindrical symmetry (11) of U_2 have immediate consequences for the inertia tensors (39). Applying first the discrete symmetries (9b) and (9c), one may easily deduce that U_{ij}, W_{ij}, and V_{ij} are all diagonal. Symmetry (9a) then implies that $W_{11}=W_{22}=0$. [Contrast this last condition with the case of the Skyrme hedgehog, which obeys $U_1(-x,y,z)=\tau_1 U_1(x,y,x)^\dagger\tau_1$ instead of (9a) and whose inertia tensor satisfies $W_{11}=W_{22}=W_{33}\neq 0$.] Next, by virtue of cylindrical symmetry, one has $U_{11}=U_{22}$ and $V_{11}=V_{22}$ for the perpendicular components while for the parallel components the identity

$$-i(\mathbf{r}\times\boldsymbol{\nabla})_3 U_2=-i\frac{\partial U_2}{\partial\phi}=-2[\tfrac{1}{2}\tau_3,U_2]\,, \tag{45}$$

implies that

$$U_{33}=\tfrac{1}{2}W_{33}=\tfrac{1}{4}V_{33}\,. \tag{46}$$

Finally we may evaluate the independent components U_{11}, U_{33}, and V_{11} as explicit functions of F and Θ. We obtain

$$U_{11}=\frac{\pi}{e^3 f_\pi}\int\rho\,d\rho\,dz\,\sin^2F\left[\tfrac{1}{4}+F_{,\rho}^2+F_{,z}^2+\frac{4}{\rho^2}\sin^2F\sin^2\Theta+\cos^2\Theta[\tfrac{1}{4}+F_{,\rho}^2+F_{,z}^2+\sin^2F(\Theta_{,\rho}^2+\Theta_{,z}^2)]\right]\,, \tag{47a}$$

$$\begin{aligned}V_{11}=\frac{\pi}{e^3 f_\pi}\int\rho\,d\rho\,dz\Bigg[&[(zF_{,\rho}-\rho F_{,z})^2+\sin^2F(z\Theta_{,\rho}-\rho\Theta_{,z})^2]\left[\frac{1}{4}+\frac{4}{\rho^2}\sin^2F\sin^2\Theta\right]\\&+\frac{4z^2}{\rho^2}\sin^2F\sin^2\Theta[\tfrac{1}{4}+F_{,\rho}^2+F_{,z}^2+\sin^2F(\Theta_{,\rho}^2+\Theta_{,z}^2)]+(\rho^2+z^2)\sin^2F(F_{,z}\Theta_{,\rho}-F_{,\rho}\Theta_{,z})^2\Bigg]\,,\end{aligned} \tag{47b}$$

$$U_{33}=\frac{2\pi}{e^3 f_\pi}\int\rho\,d\rho\,dz\,\sin^2F\sin^2\Theta[\tfrac{1}{4}+F_{,\rho}^2+F_{,z}^2+\sin^2F(\Theta_{,\rho}^2+\Theta_{,z}^2)]\,. \tag{47c}$$

Numerical values for U_{11}, V_{11}, and U_{33}, in units of $1/e^3f_\pi$, are found to be 127.8, 200.2, and 86.9, respectively.

The equalities (46), inserted in Eqs. (40), lead to the following constraint on the physical Hilbert space:

$$(2K_3+L_3)|\text{phys}\rangle=0\,. \tag{48}$$

This is simply the statement of cylindrical symmetry in operator form: namely, that a spatial rotation of α radians about the axis of symmetry can be compensated by an isorotation of -2α radians about the τ_3 axis. Incorporating (48) and the results above for the inertia tensors, and also utilizing the Casimir equalities (43), the kinetic energy operator is found to be

$$T=\frac{1}{2U_{11}}\mathbf{I}^2+\frac{1}{2V_{11}}\mathbf{J}^2+\frac{1}{2}\left[\frac{1}{U_{33}}-\frac{1}{U_{11}}-\frac{4}{V_{11}}\right]K_3^2\,. \tag{49}$$

Finkelstein-Rubinstein constraints

Heretofore, we have implicitly been working in the coordinate representation $|A,B\rangle$ of the Hilbert space, with state vectors $|\psi\rangle$ represented by wave functions $\psi(A,B)=\langle A,B|\psi\rangle$. A more convenient basis is given by the direct products

$$|ii_3k_3\rangle|jj_3l_3\rangle\,, \tag{50}$$

where $-i<i_3$, $k_3<i$ and $-j<j_3$, $l_3<j$. In basis (50), the operator T is diagonal and constraint (48) is easily satisfied by eliminating all states (50) save those with $l_3=-2k_3$. Not all the remaining states are physical, however. In addition to (48) we must specify constraints on states which implement the requirements of the Pauli exclusion principle. In particular these constraints must be such that an isolated $B=1$ Skyrmion is a fermion: when adiabatically rotated by 2π radians, its wave function should pick up a phase of -1.

In the context of soliton physics, the necessary constraints were first formulated by Finkelstein and Rubinstein[12] and later elaborated by Williams.[13] They observed that a 2π rotation of a single Skyrmion is a closed path in the Hilbert space of configurations $|U(\mathbf{r})\rangle$ which is not continuously deformable to a point. Furthermore, according to the homotopy $\pi_4(\mathrm{SU}(2))=Z_2$, there are only two topologically inequivalent types of closed path, those that are contractible to a point and those that are not. Therefore, a general rule consistent with the quantization of single Skyrmions as fermions may be stated as follows:

to the end points of every closed path in the space of states, assign a relative phase of $+1$ if the path is contractible and -1 if it is noncontractible.

As an application of this rule to our quantum system, consider isorotations about any axis $\hat{\mathbf{n}}$ by an angle θ that increases from 0 to 2π radians. This one-parameter set of transformations is implemented by the operators $e^{i\theta\hat{\mathbf{n}}\cdot\mathbf{I}}$, $0\le\theta\le 2\pi$, and corresponds to a path in the Hilbert space of zero modes given by

$$A(\theta)=e^{i\theta\hat{\mathbf{n}}\cdot\mathbf{I}}Ae^{-i\theta\hat{\mathbf{n}}\cdot\mathbf{I}}=e^{-i\theta\hat{\mathbf{n}}\cdot\tau/2}A\ ,$$
$$B(\theta)=e^{i\theta\hat{\mathbf{n}}\cdot\mathbf{I}}Be^{-i\theta\hat{\mathbf{n}}\cdot\mathbf{I}}=B,\quad 0\le\theta\le 2\pi\ . \tag{51}$$

In deriving (51) use has been made of the commutation relations (44). Now at the end point $\theta=2\pi$, the coordinate A has been transformed to $-A$, and $A^\dagger$ to $-A^\dagger$. This path is nevertheless closed since

$$e^{2\pi i\hat{\mathbf{n}}\cdot\mathbf{I}}\hat{U}(\mathbf{r},t)e^{-2\pi i\hat{\mathbf{n}}\cdot\mathbf{I}}=\hat{U}(\mathbf{r},t)\ . \tag{52}$$

As argued for above, the closed path (51) must be associated with a Finkelstein-Rubinstein constraint. To determine it we quote the result of Witten,[2] who demonstrated that path (51) is contractible if the baryon number B is even and noncontractible if B is odd. Consequently,

$$e^{2\pi i\hat{\mathbf{n}}\cdot\mathbf{I}}|\psi\rangle=(-1)^B|\psi\rangle\ , \tag{53}$$

for physical states $|\psi\rangle$. Analogous considerations apply to spatial rotations, giving

$$e^{2\pi i\hat{\mathbf{n}}\cdot\mathbf{J}}|\psi\rangle=(-1)^B|\psi\rangle\ , \tag{54}$$

For baryon number $B=2$, Eqs. (53) and (54) imply that $\psi(A,B)=\psi(-A,B)=\psi(A,-B)$, and therefore the isospin and spin quantum numbers i and j are integral. [We remind the reader that the argument B of ψ represents an SU(2) matrix, and should not be confused with the baryon number.]

Constraints (53) and (54) hold for any static configuration $U(\mathbf{r})$ of given baryon number B. Additional Finkelstein-Rubinstein constraints may result whenever $U(\mathbf{r})$ possesses any special nontrivial symmetries, since these same symmetries could be associated with the end points of closed paths in the Hilbert space of zero modes. This is indeed the case for symmetries (10) of the configuration U_2. The closed paths associated with each of these three symmetries are given by

$$A(\theta)=Ae^{i\theta\tau_1/2}=e^{iK_1\theta}Ae^{-iK_1\theta}\ ,$$
$$B(\theta)=e^{-i\theta\tau_1/2}B=e^{iL_1\theta}Be^{-iL_1\theta}\ ; \tag{55a}$$

$$A(\theta)=Ae^{i\theta\tau_1/2}=e^{iK_1\theta}Ae^{-iK_1\theta}\ ,$$
$$B(\theta)=e^{-i\theta\tau_2/2}B=e^{iL_2\theta}Be^{-iL_2\theta}\ ; \tag{55b}$$

$$A(\theta)=A,\quad B(\theta)=e^{-i\theta\tau_3/2}B=e^{iL_3\theta}Be^{-iL_3\theta}\ , \tag{55c}$$

where θ runs from 0 to π in each case. One may verify that the paths (55) are closed in the sense of returning the transformed dynamical field $\hat{U}$, at $\theta=\pi$, to its original value at $\theta=0$:

$$\hat{U}(\mathbf{r},t)=e^{i\pi(K_1+L_1)}\hat{U}(\mathbf{r},t)e^{-i\pi(K_1+L_1)}$$
$$=e^{i\pi(K_1+L_2)}\hat{U}(\mathbf{r},t)e^{-i\pi(K_1+L_2)}$$
$$=e^{i\pi L_3}\hat{U}(\mathbf{r},t)e^{-i\pi L_3}\ . \tag{56}$$

Consequently, when acted upon by any of the end-point operators $e^{i\pi(K_1+L_1)}$, $e^{i\pi(K_1+L_2)}$, or $e^{i\pi L_3}$, physical states must acquire a phase depending on whether or not the associated paths (55) are contractible. Actually, we may deduce the contractibility of path (55c) by indirect argument: using constraint (48) we have

$$e^{i\pi L_3}|\text{phys}\rangle=e^{-2\pi iK_3}|\text{phys}\rangle=|\text{phys}\rangle\ , \tag{57}$$

where the last equality follows from the fact that, for $B=2$, the quantum number $k\,(=i)$, and hence the eigenvalues of K_3, are integral. That (57) is not an independent constraint is not surprising, since symmetry (10c) is after all only a special instance of the cylindrical symmetry (11).

Turning to the remaining end-point operators let us first use constraint (48) and the fact that K_1 has integer eigenvalues to derive

$$e^{i\pi(K_1+L_2)}e^{i\pi(K_1+L_1)}|\text{phys}\rangle=e^{2\pi iK_1}e^{i\pi L_3}|\text{phys}\rangle$$
$$=|\text{phys}\rangle\ . \tag{58}$$

This shows that the operators $e^{i\pi(K_1+L_1)}$ and $e^{i\pi(K_1+L_2)}$ must give the same phase when acting on physical states. To show that this common phase is -1 we must show that path (55a) is noncontractible. Since contractibility is a topological invariant we can deform path (55a) into another path which is easier to analyze. A particularly convenient path is obtained by replacing the configuration U_2 by the product *Ansatz* configuration U_s given by Eq. (4):

$$A(\theta)U_2(R(B(\theta))\cdot\mathbf{r})A(\theta)^\dagger$$
$$\to A(\theta)U_s(R(B(\theta))\cdot\mathbf{r})A(\theta)^\dagger\ . \tag{59a}$$

To show that a deformation between these two paths is possible we must exhibit a continuous one-parameter family of configurations $U(\mathbf{r})$ which have the symmetry $U(x,-y,-z)=\tau_1U(\mathbf{r})\tau_1$ and which interpolate between $U_2(\mathbf{r})$ and $U_s(\mathbf{r})$. The existence of such a deformation is implicit in the work of VWWW: in order to define a potential $V(R)$, they explicitly constructed configurations with the symmetries (9) and with $\langle\rho\rangle=\pi R/4$, where $\langle\rho\rangle$ is defined in (33). At $R=R_0$ the configuration is $U_2(\mathbf{r})$. For very large R, the configuration represents two very distant Skyrmions and can be well approximated by a product *Ansatz* $U_s(\mathbf{r})$ with $R=s$. Thus we can study the contractibility of the path (55a) by applying it to the product *Ansatz*. It then takes the form

$$A(\theta)U_s(R(B(\theta))\mathbf{r})A(\theta)^\dagger = Ae^{i\theta\tau_1/2}U_1(R(e^{-i\theta\tau_1/2})R(B)\mathbf{r}+s\hat{\mathbf{x}})\tau_3 U_1(R(e^{-i\theta\tau_1/2})R(B)\mathbf{r}-s\hat{\mathbf{x}})\tau_3 e^{-i\theta\tau_1/2}A^\dagger$$
$$= AU_1(R(B)\mathbf{r}+s\hat{\mathbf{x}})e^{i\theta\tau_1/2}\tau_3 e^{-i\theta\tau_1/2}U_1(R(B)\mathbf{r}-s\hat{\mathbf{x}})e^{i\theta\tau_1/2}\tau_3 e^{-i\theta\tau_1/2}A^\dagger$$
$$= AU_1(R(B)\mathbf{r}+s\hat{\mathbf{x}})A^\dagger A\tau_3 e^{-i\theta\tau_1}U_1(R(B)\mathbf{r}-s\hat{\mathbf{x}})e^{i\theta\tau_1}\tau_3 A^\dagger \ . \tag{59b}$$

In the second line we have used the definition of path (55a) and the definition of the product *Ansatz* in terms of the Skyrme hedgehog U_1. In the third line we employed the special property of U_1 that $U_1(R(B)\mathbf{r})=BU_1(\mathbf{r})B^\dagger$ and the identity $R(e^{i\theta\tau_1/2})\hat{\mathbf{x}}=\hat{\mathbf{x}}$. Continuously deforming the two Skyrmions to infinite separation ($s\rightarrow\infty$), we now recognize (59b) as the rotation of a single Skyrmion by 2π radians, which is a noncontractible path. We conclude that physical states must satisfy the constraints

$$e^{i\pi(K_1+L_1)}|\text{phys}\rangle = -|\text{phys}\rangle \ , \tag{60a}$$

$$e^{i\pi(K_1+L_2)}|\text{phys}\rangle = -|\text{phys}\rangle \ . \tag{60b}$$

The solution of constraints (60) are most easily worked out in the basis (50) with $l_3=-2k_3$ in accordance with (48). Consider the subspace of fixed total isospin ($i=k$) and spin ($j=l$). We require the identities[19]

$$\begin{aligned} e^{i\pi K_1}|ii_3k_3\rangle &= (-1)^i|ii_3-k_3\rangle \ , \\ e^{i\pi L_1}|jj_3l_3\rangle &= (-1)^j|jj_3-l_3\rangle \ , \\ e^{i\pi L_2}|jj_3l_3\rangle &= (-1)^{j+l_3}|jj_3-l_3\rangle \\ &= (-1)^j|jj_3-l_3\rangle \ , \end{aligned} \tag{61}$$

where, in the last equality, we have used the fact that l_3 is an even integer. Comparing (60) with (61) we see that the two constraints (60) are equivalent and describe a single condition on physical states. This condition is implemented by an operator (e.g., $e^{i\pi(K_1+L_1)}$), which simultaneously flips the sign of the k_3 and l_3 quantum numbers while introducing a phase $(-1)^{i+j}$. As such it commutes with the kinetic energy operator T, as required by the consistency of the theory. Condition (60a) also partitions the basis (50) into classes, depending on $\kappa=|k_3|$. Working out constraint (60a) in basis (50), the allowed states are determined to be

(i) $|ii_30\rangle|jj_30\rangle$, provided $i+j$ is odd, $\kappa=0$

and (62)

(ii) $\frac{1}{\sqrt{2}}[|ii_3\kappa\rangle|jj_3-2\kappa\rangle-(-1)^{i+j}|ii_3-\kappa\rangle|jj_32\kappa\rangle]$,

$\kappa=1,\ldots,\min\{i,[j/2]\}$.

Their corresponding energy eigenvalues are

$$E=M_2+\frac{i(i+1)}{2U_{11}}+\frac{j(j+1)}{2V_{11}} + \left[\frac{1}{U_{33}}-\frac{1}{U_{11}}-\frac{4}{V_{11}}\right]\frac{\kappa^2}{2} \ . \tag{63}$$

Note that the second set of states in (62) with $\kappa>0$ is possible only if $i\geq1$ and $j\geq2$.

From the calculated values of M_2, U_{11}, U_{33}, and V_{11} we may construct using (62) and (63) the entire mass spectrum of allowed states. The masses of the lowest-lying states are given in Table I in physical units using the values (13) for f_π, e, and m_π. We observe, in particular, that the ground state of the soliton,

$$i=0,\quad j=1,\quad \kappa=0:\ E=M_2+\frac{1}{V_{11}} \ , \tag{64a}$$

is the unique state with the quantum numbers of the deuteron. On the other hand, the first excited state

$$i=1,\quad j=0,\quad \kappa=0:\ E=M_2+\frac{1}{U_{11}} \ , \tag{64b}$$

may be identified with the isovector $^1S_0$ state of the two-nucleon system; it lies higher in mass than the deuteron since $V_{11}>U_{11}$. Note that the calculated mass spectrum predicts that the deuteron and $^1S_0$ state are stable against decay into two nucleons, with binding energies of 158 and 123 MeV, respectively. However, it is well-established experimentally that the deuteron is marginally bound by only 2 MeV while the isovector $^1S_0$ state is marginally

TABLE I. Mass spectrum (in MeV).

Classification	i	j	κ	Parity	Theory	Experiment
Deuteron ($^3S_1$)	0	1	0	+	1720	1876
NN ($^1S_0$)	1	0	0	+	1755	1880[a]
? ($^3P_2$)	1	2	1	−	1838	
$N\Delta$ ($^5S_2$)	1	2	0	+	1938	
$N\Delta$ ($^3S_1$)	2	1	0	+	2007	
? ($^5P_3$)	1	3	1	−	2022	
$\Delta\Delta$ ($^7S_3$)	0	3	0	+	2026	
? ($^3P_2$)	2	2	1	−	2030	
? ($^5D_4$)	2	4	2	+	2159	
? ($^5P_3$)	2	3	1	−	2213	
$\Delta\Delta$ ($^1S_0$)	3	0	0	+	2233	

[a]As extracted from Ref. 23.

unbound. The clear discrepancy of the calculations with experiment can be understood as a particular limitation of the semiclassical quantization procedure we have applied; see the discussion in Sec. V.

There is a remarkable correspondence between the lowest-lying $\kappa=0$ states and the states expected in the nonrelativistic quark model for the six-quark system. Specifically, the states of Table I with $\kappa=0$ are found to fill out the 50-dimensional irreducible representation of the SU(4) group of spin flavor. This irreducible representation can be interpreted as a six-quark composite, with each quark transforming as the fundamental representation **4** of SU(4). Under decomposition of the **4** with respect to the SU(2) isospin $\otimes$SU(2) spin subalgebra, one finds $\mathbf{4}=(\frac{1}{2},\frac{1}{2})$. The corresponding decomposition of the 50-dimensional representation is

$$\mathbf{50}=(1,0)+(0,1)+(2,1)+(1,2)+(3,0)+(0,3)\ . \tag{65}$$

It is straightforward, by direct-product decomposition, to assign each set of states in the **50** with fixed i and j to NN, $N\Delta$, and $\Delta\Delta$ pairs of three-quark subcomposites. This assignment is also listed in Table I where the assignment of the deuteron as an NN composite is understood. The dibaryon resonances of the **50** (e.g., the $N\Delta$ and $\Delta\Delta$ states) also appear in theoretical calculations based on the spherical bag model.[20]

We have been unable, however, to find an analogous quark model interpretation for any of the $\kappa>0$ states, and have indicated this lack of interpretation with question marks (?) in Table I. The principal difficulty here is that, for states with $\kappa>0$, the total isospin and spin are bounded from below by a nonzero integer ($i\geq\kappa$, $j\geq 2\kappa$). This property is simply not shared by any six-quark SU(4) representation, even if the intrinsic spins of quarks are coupled to nonzero orbital angular momentum. Thus the spectrum of dibaryon resonances in this model includes states which could not appear in the spherical MIT bag model.

Next we construct the conserved parity operator P which yields the parity quantum numbers of the allowed states (62). P is defined by the transformation

$$P\hat{U}(\mathbf{r},t)P^{-1}=\hat{U}(-\mathbf{r},t)^{\dagger}\ . \tag{66}$$

But the configuration U_2 satisfies

$$U_2(-\mathbf{r})^{\dagger}=\tau_3 U_2(\mathbf{r})\tau_3\ , \tag{67}$$

as may be verified using Eqs. (9). Thus P must transform the coordinates A and B according to

$$PAP^{-1}=A(i\tau_3),\quad PBP^{-1}=B\ , \tag{68}$$

implying $P=e^{i\pi K_3}$. Applying P to the allowed states (62) we find that states have even (odd) parity if κ is even (odd). In particular, the lowest-mass state (64a) has the correct parity (+) of the physical deuteron.

In Table I we have also endeavored to provide the standard spectroscopic classification $^{2s+1}L_j$ of states. These assignments are somewhat arbitrary, however, since in our treatment of $B=2$ nuclei as quantum states of a Skyrme soliton the nucleon constituents of these nuclear states are not identifiable. Consequently, while the total angular momentum j is well defined, it is not decomposable into separate contributions due to the total intrinsic spin s of nucleon constituents and the relative orbital angular momentum L. To arrive at the provisional assignments of Table I we have input the known dominance of the S-wave component of the deuteron to infer, by SU(4) symmetry, that the remaining states of the **50** are also predominantly S wave. The intrinsic spin s is then fixed by the total angular momentum, $s=j$. For the remaining $\kappa>0$ states, the spectroscopic assignments are determined by the lowest orbital angular momentum L consistent with the quantum numbers j and P and the fact that s cannot exceed 3 for a six-quark system. Of course, nothing said above precludes the admixture of higher orbital momenta, such as the well-known D-wave component of the deuteron, which are consistent with the selection rules of parity and angular momentum addition.

IV. STATIC ELECTROMAGNETIC PROPERTIES

Having confirmed that the lowest-lying state (64a) is the unique state with the quantum numbers of the deuteron, we may check the identification with a calculation of its static electromagnetic properties. Below we evaluate the mean charge radius $\langle r^2\rangle_d^{1/2}$, magnetic moment μ_d, and quadrupole moment Q, all known experimentally for the deuteron to quite high precision.[24] We also present a calculation of the photodisintegration transition moment $\mu_{d\to np}$ associated with the process $\gamma d\to {}^1S_0$. The electromagnetic form factors of the deuteron at nonzero momentum transfer are the subject of a forthcoming paper.[16]

The electromagnetic current $J_\mu^{(\mathrm{em})}(x)$ in the Skyrme model consists of two pieces, isoscalar and isovector, given by

$$J_\mu^{(\mathrm{em})}(x)=\tfrac{1}{2}B_\mu(x)+I_\mu^3(x)\ . \tag{69}$$

$B_\mu(x)$ is the baryon current density given by Eq. (16) and $I_\mu^3(x)$ is the third component of the isospin current density:

$$I_\mu^a(x)=\frac{if_\pi^2}{2}\mathrm{Tr}\left\{U^{\dagger}\left[\frac{\tau^a}{2},U\right]U^{\dagger}\partial_\mu U-\frac{1}{(ef_\pi)^2}\left[U^{\dagger}\left[\frac{\tau^a}{2},U\right],U^{\dagger}\partial_\nu U\right][U^{\dagger}\partial_\mu U,U^{\dagger}\partial^\nu U]\right\}\ . \tag{70}$$

Equation (69), evaluated for the dynamical field $\hat{U}$, gives the electromagnetic current operator $\hat{J}_\mu^{(\mathrm{em})}$ of the semiclassical theory.

The deuteron charge radius $\langle r^2\rangle_d^{1/2}$ is defined as the square root of

$$\langle r^2\rangle_d \equiv \langle d|\int d^3r\, r^2 \hat{J}_0^{(\mathrm{em})}(\mathbf{r},t)|d\rangle \;, \tag{71}$$

where $|d\rangle$ represents any of the spin states of the deuteron. Now only the isoscalar piece of $\hat{J}_0^{(\mathrm{em})}$ contributes to the matrix element, in which case the indicated integral is a pure c number. Thus the matrix element is trivial, yielding

$$\langle r^2\rangle_d = \frac{\pi}{(ef_\pi)^2}\int \rho\, d\rho\, dz(\rho^2+z^2)B_0(\rho,z) \;. \tag{72}$$

(Here and below all integrands expressed in cylindrical coordinates have lengths in units of $1/ef_\pi$.) On the other hand, the isoscalar parts of the magnetic and quadrupole moments,

$$\hat{\mu}_i = \tfrac{1}{2}\int d^3r\, \epsilon_{ijk} r_j \hat{J}_k^{(\mathrm{em})}(\mathbf{r},t) \;, \tag{73a}$$

$$\hat{Q}_{ij} = \int d^3r(3r_ir_j - r^2\delta_{ij})\hat{J}_0^{(\mathrm{em})}(\mathbf{r},t) \;, \tag{73b}$$

are nontrivial q-number operators. Inserting the expression (36) for $\hat{U}$ we find

$$\hat{\mu}_i|_{I=0} = \tfrac{1}{2}\{R_{ij}(B)^T, \mu_{jk}a_k + \mu'_{jk}b_k\} \;, \tag{74a}$$

$$\hat{Q}_{ij}|_{I=0} = R_{ia}(B)^T Q_{ab} R_{bj}(B) \;. \tag{74b}$$

Here $R_{ij}(B)$ is the rotation matrix (35), a_k and b_k are operators related to $\mathbf{K}$ and $\mathbf{L}$ via (40), and the c-number tensors μ_{jk}, μ'_{jk}, and Q_{ab} are given by

$$\mu_{jk} = \frac{1}{(ef_\pi)^2}\int d^3r\, \tfrac{1}{2}\epsilon_{jlm} r_l \tfrac{1}{2}\mathrm{Tr}[U^\dagger[\tfrac{1}{2}\tau_k, U]C_m(\mathbf{r})] \;, \tag{75a}$$

$$\mu'_{jk} = \frac{1}{(ef_\pi)^2}\int d^3r\, \tfrac{1}{2}\epsilon_{jlm} r_l \tfrac{1}{2}\mathrm{Tr}[U^\dagger(-i\mathbf{r}\times\nabla)kU \times C_m(\mathbf{r})] \;, \tag{75b}$$

$$Q_{ab} = \frac{1}{(ef_\pi)^2}\tfrac{1}{2}\int d^3r(3r_ar_b - r^2\delta_{ab})B_0(\mathbf{r}) \;, \tag{75c}$$

where $B_0(\mathbf{r})$ is the baryon-number density (29) and

$$C_m(\mathbf{r}) = \frac{i}{8\pi^2}\epsilon_{mnp}(U^\dagger\partial_n U)(U^\dagger\partial_p U) \;. \tag{76}$$

In (74a) we have introduced an anticommutator to ensure that $\hat{\mu}_i$ is a Hermitian operator.

As in the case of the inertia tensors, the work involved in evaluating μ_{jk}, μ'_{jk}, and Q_{ab} is significantly reduced by exploiting the particular symmetries of the configuration U_2. The discrete symmetries (9) imply that all three tensors are diagonal and furthermore that $\mu_{11}=\mu_{22}=0$. Cylindrical symmetry then requires $\mu'_{11}=\mu'_{22}$ and $Q_{11}=Q_{22}$, and produces the relation

$$\mu'_{33} = -2\mu_{33} \;. \tag{77}$$

Finally the tracelessness of Q_{ab} relates Q_{11} and Q_{22} with Q_{33}. Altogether, this leaves three independent quantities: μ'_{11}, μ'_{33}, and Q_{33}.

Using (77) in Eq. (74a) we first note that since $a_3-2b_3=K_3/U_{33}$, the terms of $\hat{\mu}_i$ involving μ'_{33} multiply the operator K_3, which annihilates the deuteron state. Using (41) and constraint (48), the remaining terms may be cast in the form

$$\hat{\mu}_i|_{I=0} = -\frac{\mu'_{11}}{V_{11}}J_i + \text{terms proportional to } K_3 \;, \tag{78}$$

where V_{11} is given by (39c). The symmetry relations for Q_{ab} also reduce the quadrupole moment operator to the expression

$$\hat{Q}_{ij}|_{I=0} = Q_{33}[\tfrac{3}{2}R_{i3}(B)^T R_{3j}(B) - \tfrac{1}{2}\delta_{ij}] \;. \tag{79}$$

Consequently, only the quantities μ'_{11} and Q_{33} are required; as functionals of $F(\rho,z)$ and $\Theta(\rho,z)$ they are evaluated to be

$$\mu'_{11} = -\frac{1}{(ef_\pi)^2}\frac{\pi}{4}\int \rho\, d\rho\, dz(2z^2+\rho^2)B_0(\rho,z) \;, \tag{80a}$$

$$Q_{33} = \frac{1}{(ef_\pi)^2}\pi\int \rho\, d\rho\, dz(2z^2-\rho^2)B_0(\rho,z) \;. \tag{80b}$$

The moments μ_d and Q of the deuteron are defined to be the expectation values of $\hat{\mu}_3$ and $\hat{Q}_{33}$ between deuteron states with $j_3=1$. One finds

$$\mu_d = \langle\hat{\mu}_3\rangle = -\frac{\mu'_{11}}{V_{11}} \;, \tag{81a}$$

$$Q = \langle\hat{Q}_{33}\rangle = -\tfrac{1}{5}Q_{33} \;. \tag{81b}$$

Another observable of interest is the transition moment $\mu_{d\to np}$ which parametrizes the amplitude, at low-momentum transfer, for photodisintegration of the deuteron into the isovector ${}^1S_0$ state. It is computed in terms of the matrix element, between the $j_3=0$ state of (64a) and the $i_3=0$ state of (64b), of the magnetic moment operator $\hat{\mu}_3$ defined by Eq. (73a):

$$\mu_{d\to np} = \langle {}^1S_0, i_3=0|\hat{\mu}_3|d, j_3=0\rangle \;. \tag{82}$$

Evaluating the isovector part of $\hat{\mu}_i$ for the dynamical field $\hat{U}$ we find

$$\hat{\mu}_i|_{I=1} = -\tfrac{1}{2}R_{3j}(A)W_{jk}R_{ki}(B) \;, \tag{83}$$

where W_{jk} is the inertia tensor defined in (39b). Its only nonzero component is $W_{33}=2U_{33}$, so the operator $\hat{\mu}_3$ reduces to

$$\hat{\mu}_3|_{I=1} = -U_{33}R_{33}(A)R_{33}(B) \;. \tag{84}$$

Evaluating the matrix element (82) one obtains

$$\mu_{d\to np} = -\tfrac{1}{3}U_{33} \;. \tag{85}$$

Results for $\langle r^2\rangle_d^{1/2}$, μ_d, Q, and $\mu_{d\to np}$ are given in

TABLE II. Static electromagnetic properties.

	Theory	Experiment
$\langle r^2\rangle_d^{1/2}$ (fm)	0.92	2.095[a]
μ_d (nm)	0.74	0.8574[a]
Q (fm^2)	0.082	0.2859[a]
$\mu_{d\to np}$ (nm)	-4.4	-5 ± 1[b]

[a]Reference 24.
[b]As extracted from Refs. 23 and 25.

Table II in physical units, together with their experimental values. The results for these properties are a mixed success. The magnetic and transition moments μ_d and $\mu_{d\to np}$ are in reasonable accord with experiment. On the other hand, the mean charge radius is at least a factor of 2 smaller than its experimental value while the discrepancy for the quadrupole moment is even larger (a factor of 3 to 4).

These discrepancies could be an artifact of the Skyrme-model parameters f_π and e. Recall that the parameter set (13) was adjusted to optimize the predictions of the model in the $B=1$ sector at the expense of the $B=0$ sector which requires $f_\pi=186$ MeV. It should come as no surprise that this is also not the optimal parameter set to describe the $B=2$ sector. Observe that the mean charge radius and quadrupole moment scale as $(ef_\pi)^{-1}$ and $(ef_\pi)^{-2}$, respectively, while the magnetic moment scales as e/f_π. Thus the disagreement with experiment could be assuaged by an adjustment of e and f_π so that ef_π decreases by about a factor of 2 while the ratio e/f_π is held fixed.

In our view, however, it is more meaningful to consider quantities that are as insensitive as possible to the choice of these parameters. The relevant quantities are ratios of observables constructed so that dependence on the Skyrme-model parameters f_π and e cancels out except for a mild dependence on the dimensionless ratio $\beta=m_\pi/ef_\pi$. A sampling of such quantities is given in Table III. Here we observe that the ratio, $\langle r^2\rangle_d^{1/2}/(\langle r^2\rangle_p+\langle r^2\rangle_n)^{1/2}$, is approximately 1.4 in our calculations while experimentally it is 2.91. Thus, independent of the scale provided by f_π, it appears that the toroidal Skyrmion U_2 in this model is smaller than the physical deuteron by half.

TABLE III. Parameter-independent results. The quantities $\langle r^2\rangle_p^{1/2}$ and $\langle r^2\rangle_n^{1/2}$ are the charge radii of the proton and neutron, respectively, while μ_p and μ_n are their magnetic moments. Theoretical values for these nucleon observables are taken from Ref. 4.

	Theory	Experiment
$\dfrac{M_{^1S_0}-M_d}{M_\Delta-M_N}$	0.12	0.014
$\dfrac{\langle r^2\rangle_d^{1/2}}{(\langle r^2\rangle_p+\langle r^2\rangle_n)^{1/2}}$	1.35	2.91
$\dfrac{Q}{\langle r^2\rangle_d}$	0.097	0.065
$\dfrac{\mu_d}{\mu_p+\mu_n}$	1.01	0.97
$\dfrac{\mu_{d\to np}}{\mu_p-\mu_n}$	-1.37	-1.1 ± 0.2

The agreement of the remaining parameter-independent quantities in Table III with experiment is good, with the exception of the ratio $(M_{^1S_0}-M_d)/(M_\Delta-M_N)$. The theoretical values for this ratio are an order of magnitude larger than experiment. The discrepancy is perhaps not surprising since the theoretical splitting $M_{^1S_0}-M_d$ is proportional to the difference $1/U_{11}-1/V_{11}$ and thus is highly sensitive to the detailed structure of the soliton. Nevertheless, the calculated splitting of 35 MeV represents an improvement over the values of 70–80 MeV obtained in the bag model.[20]

V. DISCUSSION

The success of the Skyrme model in describing nucleons as quantum states of the Skryme soliton makes it natural to also apply the model to bound states of nucleons: namely, nuclei. Attempts to describe nuclei within the Skryme model have followed two very different approaches. In the "potential approach," it is assumed that the interactions of Skyrmions can be described by an effective potential; nuclei would then arise as bound states in this potential. This approach is very close in spirit to traditional nuclear physics and was pioneered by Jackson, Jackson, and Pasquier.[7] They constructed a Skyrmion-Skyrmion potential under the assumption that the field configurations for two interacting Skyrmions could be approximated by the product *Ansatz* (1). Unfortunately, after projection onto the quantum numbers of the nucleon, the resulting nucleon-nucleon potential fails to provide any intermediate-range attraction. This clearly spells disaster for the potential approach, since an intermediate-range attraction is necessary to bind nucleons into nuclei. The work of Jackson, Jackson, and Pasquier has been extended by adding more terms to the Lagrangian and by using more complicated *Ansätze* for the 2-Skyrmion configuration, but these efforts have not succeeded in generating a significant intermediate range attraction.

We argue that the lack of an intermediate-range attraction is simply an artifact of the potential approach. Skyrmions are extended fluid objects and they have particle-like behavior only when they are well separated. When they overlap they can completely lose their individuality. This is demonstrated dramatically by the minimal-energy static $B=2$ solution $U_2(\mathbf{r})$, in which the two Skyrmions have merged into a toroidal configuration. As explained in the Introduction, such a configuration is not describable as a product *Ansatz*. Since it is unable to probe the lowest-energy $B=2$ configurations, the product *Ansatz* underestimates the attractive force between Skyrmions. While one might be able to find indications of a stronger attraction by using a more elaborate *Ansatz*, it is unlikely that the effects of these toroidal configurations could be adequately represented by a potential between two individual Skyrmions.

The alternative approach to describing nuclei in the Skyrme model is the "soliton approach" which was pioneered in Ref. 9. Although this is a very natural approach to follow in a field theory, it is quite radical from the point of view of conventional nuclear physics. Not only does it abandon the concept of a potential but it also abandons the conventional picture of nuclei as bound states of individual nucleons. Instead a nucleus is identified with the lowest quantum states of the static soliton solution with the appropriate topological charge. That is, they are treated in exactly the same way as the nucleon in the $B=1$ sector. While this provides a rather counterintuitive model for the nucleus, it should not be immediately dismissed. Rather one should study the properties of the soliton and determine whether it successfully reproduces the physical properties of the nucleus. In particular, it should have the same spectrum of excited states, it should have the same static properties, and it should scatter electrons, pions, and nucleons in the same way as the physical nucleus.

In this paper we have made the first steps in applying the soliton approach to the simplest compound nucleus, the deuteron. The resulting model for the deuteron is indeed rather counterintuitive. The static $B=2$ soliton solution is a toroidal condensation of pion fields and the deuteron is a quantum superposition of these solutions with isospin 0 and total angular momentum 1. This seems at first to bear very little resemblance to the conventional model in which the deuteron is interpreted as a loosely bound state (which is mostly S wave) of a neutron and a proton. A possible connection is that the baryon density of a proton and a neutron orbiting classically in the nuclear potential is in fact toroidal due to the short-range repulsion between the two nucleons. However, the real test of the model is to examine its predictions for the physical properties of the deuteron. The ground state of the toroidal soliton automatically has the quantum numbers of the deuteron due to the symmetries of the $B=2$ solution, and its first excited state has the quantum numbers of the unbound spin-singlet state of the deuteron. At the qualitative level, our calculations of the static electromagnetic properties of the ground state of this soliton are also compatible with its identification with the deuteron. Although these predictions are not quantitatively accurate, this is not disturbing since we are using a crude effective Lagrangian whose parameters have been adjusted to optimize the properties of the nucleon in the $B=1$ sector. This set of parameter values is not the optimal choice for the $B=0$ sector, however, and it is therefore not surprising that it is also not the optimal choice for describing the $B=2$ sector. Of course, an accurate effective Lagrangian should be able to describe all the sectors with the same set of parameters.

We now comment on the limitations of the calculations that were presented in this paper. These limitations apply equally well to the existing treatments of the nucleon in the Skyrme model. We have worked within the framework of the semiclassical approximation in which the fluctuations of the pion fields around the classical static solution are quantized. These fluctuations include the collective modes for translations, rotations, and isospin rotations; we will refer to the remaining degrees of freedom as vibrational modes.

We have followed previous calculations of the spectrum in the $B=1$ sector by quantizing the collective modes for rotations and isorotations, while ignoring the vibrational modes. This will not yield a consistent semiclassical expansion for the energy levels, because each of the vibrational modes contributes a zero-point energy to the mass of the soliton which is of lower order in the semiclassical expansion (order $\hbar$) than the energy splitting between different quantum states (order $\hbar^2$). These contributions are ignored simply because they are plagued by ultraviolet divergences due to the nonrenormalizability of the model. They could only be calculated by imposing an ultraviolet cutoff on the field theory. For the same reason, the calculations of static properties have not been carried out within a strict semiclassical expansion.

Because we have ignored the vibrational modes we are unable to calculate the binding energy of the deuteron. To determine the binding energy we would have to calculate the energy of the ground state of the $B=2$ soliton, including the sum of the zero-point energies from all its vibrational modes, and subtract twice the energy of the nucleon including the sum of all of its zero-point energies. The difference would depend sensitively on the ultraviolet cutoff as well as on the parameters of the field theory. This also explains why the spin-singlet state of the two-nucleon system appears along with the deuteron as a bound quantum state of the $B=2$ soliton. Only by including the vibrational modes could one determine whether or not it is really bound.

The identification of the singlet state as an excited state of the soliton has implications for calculations of nucleon-nucleon scattering in the Skyrme model. A procedure for calculating the semiclassical limit of the scattering amplitudes has been outlined in Ref. 21. However, the splitting between the deuteron and the singlet state is of higher order in the semiclassical expansion than the binding energy, so in the semiclassical limit both states would be bound. This binding should be reflected in the scattering amplitudes in the form of positive scattering lengths for both the singlet and triplet channels. A negative sign for the singlet channel could only be recovered after including the effects of vibrational modes.

Finally, we comment on previous attempts at describing the deuteron within the Skyrme model. Nyman and Riska[22] have computed static properties of the deuteron under the assumption that the deuteron field configurations could be represented by a product *Ansatz*. This assumption has been discredited by the discovery that the lowest energy $B=2$ configurations cannot be accommodated by a product *Ansatz*. The product *Ansatz* configurations were singled out for algebraic convenience only, and any predictions based on them are simply artifacts of the *Ansatz* and not predictions of the Skyrme model. Of course, this criticism also holds for our earlier calculation.[8]

VWWW have also tried to identify the static $B=2$ soliton with the deuteron. In addition to quantizing the collective modes of the soliton, they have also considered a "radial" degree of freedom R. They defined a potential $V(R)$ by calculating the minimal energy among field configurations constrained to have the first moment of the baryon distribution equal to R. The configuration corresponding to the minimum of the potential is in fact the static $B=2$ solution. VWWW found that, within the WKB approximation, the potential $V(R)$ did not support a bound state. However, the prescription of VWWW for the potential $V(R)$ singles out a one-parameter family of configurations which is completely *ad hoc*, and other choices would have given different results. Our experience with the product *Ansatz* has shown that it is not a good idea to isolate certain degrees of freedom merely for calculational convenience. In the framework of the semiclassical expansion, a more consistent procedure would be to consider all the vibrational modes at once. This is clearly a difficult task and we have therefore made no attempt to consider any of the vibrational degrees of freedom.

In this paper we have taken the first step toward showing that the ground state of the toroidal $B=2$ Skyrmion provides a reasonable model for the deuteron. We have shown that the spectrum and static properties of this soliton are in fact consistent with this interpretation. Further support is provided by calculations of the electromagnetic form factors, which will be described elsewhere.[16] The next step in this program would be to compute pion-deuteron scattering amplitudes. This will be far more difficult than the pion-nucleon calculations[5] because of the reduced symmetry of the static solution. This reduced symmetry is not a serious complication for numerical calculations of nucleon-deuteron scattering amplitudes, so these calculations should not be much more difficult the nucleon-nucleon scattering amplitudes.[21]

If the soliton approach is in fact the proper way to apply the Skyrme model to nuclear physics, then heavier nuclei should also be identified with quantum states of solitons. At this time, the static solutions for $B \geq 3$ are not yet known. In most cases, these solutions will probably not have any continuous symmetries and a fully three-dimensional calculation will be required. We are confident however that these solutions will in fact display some of the systematic features of physical nuclei. In particular, we expect the strong binding of the α particle to be reflected in a classical binding energy for the $B=4$ Skyrmion which is significantly larger than that for $B=3$ or 5.

Note added. After this paper was completed, the papers of Schramm, Dothan, and Biedenharn[26] came to our attention. Independent of the work of Verbaarschot *et al.*[10] and Kopeliovich and Shtern,[11] they calculated the minimal-energy solution $U_2(\mathbf{r})$ numerically and evaluated its charge radius and quadrupole moment. While we agree with their values for these static properties, one should note that their value of the quadrupole moment is that of the *classical solution,* and is denoted by $-Q_{33}$ in Eq. (80b). On the other hand, the quadrupole moment of the deuteron *quantum state* is an expectation value of a quantum operator and in smaller than $-Q_{33}$ by the factor of 5 given in Eq. (81b).

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