Branching ratio for a light Higgs boson to decay into $\mu^+\mu^-$ pairs

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We evaluate the effects of final-state interactions on the decay of a light Higgs boson to two pions. Although the formalism is completely general and can be applied to any strong-interaction decay mode of the Higgs boson, we are particularly interested in the regime where the Higgs-boson mass m_h satisfies the constraint $2m_\pi < m_h < 2m_K$. In this case the dominant decay modes of the Higgs boson are to $\mu^+\mu^-$ and two pions. Final-state interactions tend to enhance the two-pion mode and thus suppress the branching ratio to two muons. Since the two-muon mode is the cleanest signature for identifying the Higgs boson, it is important to obtain a good determination of this branching ratio. We find $B(h\to\mu^+\mu^-)\approx 10^{-2}-10^{-1}$.

I. INTRODUCTION

One of the missing links in the verification of the standard model is the elusive Higgs boson. Since its mass is not fixed by theory, it can only be found by an exhaustive experimental search. The experimental lower limit on the Higgs-boson mass is of order 14 MeV. This limit is set by observing the decay of the 20.1-MeV excitation of ^{4}He (Ref. 1). If the Higgs boson is heavier than a TeV it is necessarily a strong-interacting object.[2] Many reviews have been written on the phenomenology of Higgs bosons with mass greater than several GeV. In this paper we are interested in a light Higgs boson. The theoretical constraints on the mass of a light Higgs boson are discussed in Appendixes A and B.

A light Higgs boson can either be produced directly in high-energy collisions[3] or it can be seen in the decays of other particles. In either case, the experimental signature of the Higgs boson depends crucially on its decay products. A Higgs boson will decay predominantly into the heaviest states which are available. If the Higgs-boson mass m_h satisfies $m_h < 2m_\mu$ then the dominant decay modes are $h\to\gamma\gamma$ or e^+e^- (Ref. 4). Existing data on the processes $\mu^+\to e^+e^-+e^++\bar\nu_\mu+\nu_e$ (Refs. 5 and 6) and $\pi^+\to e^++e^-+e^++\nu_e$ (Ref. 7) have reached very high statistics. (We would like to thank Cy Hoffman for informing us of these data.) They have been obtained as background to the flavor-violating process without the two neutrinos. The branching ratio for the process $\pi^+\to h+e^++\nu_e$ is given by

$$B(\pi^+\to h+e^+\nu_e)=\frac{\sqrt{2}G_F m_\pi^4 f(x)}{48\pi^2 m_\mu^2\left[1-\dfrac{m_\mu^2}{m_\pi^2}\right]^2}\,, \tag{1}$$

where

$$f(x)\equiv(1-8x+x^2)(1-x^2)-12x^2\ln(x)$$

and $x\equiv m_h^2/m_\pi^2$. If there are no excessive losses due to the efficiency of the apparatus, it appears that the SIN data[7] can be used to rule out a Higgs boson with mass less than about 100 MeV. Clearly, the SIN data should be reanalyzed to look for such a light Higgs boson. The Crystal Box Collaboration at Los Alamos is now reanalyzing their data for a light Higgs boson.[5]

If $m_h > 2m_\mu$ then the Higgs boson will predominantly decay to heavier final states. In fact the ratio

$$\frac{\Gamma(h\to\mu^+\mu^-)}{\Gamma(h\to e^+e^-)}=\frac{m_\mu^2(1-4m_\mu^2/m_h^2)}{m_e^2(1-4m_e^2/m_h^2)}$$

is well known. The two-muon decay mode is a very good experimental signature of the Higgs boson. Unfortunately, when $m_h > 2m_\pi$ the branching ratio $B(h\to\mu^+\mu^-)$ is *not* well known. It requires an understanding of the strong-interaction decay modes $h\to\pi\pi$ or, for $m_h > 2m_K$, $h\to KK$. The purpose of this paper is to evaluate these strong-interaction effects and thereby calculate $B(h\to\mu^+\mu^-)$.

The following decays can be used to detect a Higgs boson with a mass greater or less than $2m_\mu$—$K^+\to h+e^+\nu_e$, $K\to\pi+h$, $\eta'\to\eta+h$, $\tau^+\to h+e^++\bar\nu_\tau+\nu_e$, $B\to h+$anything, $B\to K+h$, $\Upsilon\to h+\gamma$. Existing data on the τ can be used to search for a Higgs boson with mass less than $O(m_\tau)$. (We would like to thank K. K. Gan for informing us of the availability of these data.) In all the other cases, strong-interaction dynamics is an essential ingredient. K and η' decays have been studied extensively.[4,8–15] The calculated branching ratios for these decays range from 10^{-3} to 10^{-8}. The problem has been to *correctly* incorporate PCAC (partial conservation of axial-vector current), to evaluate the matrix elements of the effective quark–Higgs-boson Lagrangian responsible for the process and, for the kaon, to include the known $\Delta I=\frac{1}{2}$ enhancements.

$K\to\pi+h$ can be used to find a Higgs boson with mass less than about 350 MeV. The effective Lagrangian includes both flavor-diagonal and flavor-changing couplings. The flavor-diagonal Higgs-boson couplings are given by the Yukawa couplings

$$\mathcal{L}_{\text{mass}} = -\left(1+\frac{h}{\sqrt{2}v}\right)\sum_i m_i \bar{q}_i q_i \ , \tag{2}$$

where m_i is the mass of the quark flavor labeled by the index i and $\sqrt{2}v = 250$ GeV. Note that h is the physical, properly normalized, Higgs-boson field with mass m_h. The flavor-changing couplings (relevant for $K \rightarrow \pi + h$) are determined by the graphs in Figs. 1 and 2. The $\Delta S = 1$ couplings obtained by integrating out the W at the tree level (Fig. 1) are given by

$$\mathcal{L}_{\text{tree}} = -\frac{G_F}{\sqrt{2}} V_{us} V_{ud}^* \left(1+\frac{h}{\sqrt{2}v}\right)^{-2} [\bar{d}\gamma_\mu(1-\gamma_5)u] \times [\bar{u}\gamma^\mu(1-\gamma_5)s] + \text{H.c.} \ , \tag{3}$$

where the V_{ij} are elements of the Kobayshi-Maskawa (KM) matrix, with similar expressions for the other flavor-changing couplings. Note that as discussed by Willey[13] the graphs with h coupled to the external quark lines are suppressed by factors of m_q/λ where $\lambda \sim m_{\text{meson}}$. Finally the flavor-changing couplings induced at the one-loop level (Fig. 2) are given by[13–15]

$$\mathcal{L}_{1\text{ loop}} = +\frac{3\alpha}{32\pi \sin^2\theta_W}\left[\sum_i V_{is}\frac{m_i^2}{M_W^2}V_{id}^*\right]\frac{h}{\sqrt{2}v} \times [m_s\bar{d}(1+\gamma_5)s + m_d\bar{d}(1-\gamma_5)s] + \text{H.c.} \ , \tag{4}$$

in a basis where the quark kinetic-energy terms are diagonal and properly normalized and the quark mass terms are diagonal.

In order to compare with experiment one must take the following effective quark–Higgs-boson Lagrangian

$$\mathcal{L} = i\sum_i \bar{q}_i\gamma^\mu\partial_\mu q_i + \mathcal{L}_{\text{mass}} + \mathcal{L}_{\text{tree}} + \mathcal{L}_{1\text{ loop}} \tag{5}$$

normalized at a scale of order M_W and first run the relevant parameters down to a scale ~ 1 GeV and then take matrix elements between the states K and π. In a recent paper by Chivukula and Manohar[15] a chiral perturbation theory approach was used to obtain the $K-\pi$ matrix elements. This approach presumably takes into account the known $\Delta I = \frac{1}{2}$ enhancement in K decays. They find the amplitude for $K_L \rightarrow \pi^0 h$ to be

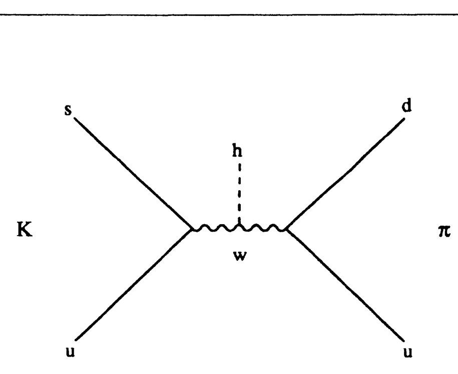

FIG. 1. Tree-level contribution to the decay $K \rightarrow \pi + h$.

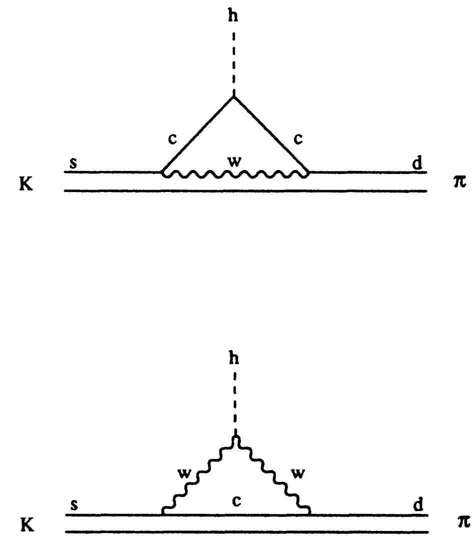

FIG. 2. One-loop contribution to the decay $K \rightarrow \pi + h$.

$$\mathcal{A}(K_L \rightarrow \pi h) = \left[-(1.5\times 10^{-10})\left(1+\frac{m_\pi^2 - m_h^2}{m_K^2}\right) + (0.72\times 10^{-10}) + \sum_{i\neq c}\eta_i^* + B(0.68\times 10^{-10})\right] \text{GeV} \ , \tag{6}$$

where

$$\eta_i = \frac{m_K^2}{2\sqrt{2}v}\frac{3\alpha}{16\pi\sin^2\theta_W}V_{is}\frac{m_i^2}{M_W^2}V_{id}^* \ .$$

The first term in (6) is the contribution of the $\Delta I = \frac{1}{2}$ term and the second is η_c. The third term is negligible and in the fourth B is an undetermined parameter in the chiral Lagrangian. Barring an "accidental" cancellation the $K_L \rightarrow \pi h$ amplitude is therefore presumably at most $\sim 10^{-10}$ GeV. Notice, however, that if $B \approx 1$, then, in fact, this amplitude is $\sim 10^{-11}$ GeV. The experimental limits are[16]

$$B(K_L \rightarrow \pi\mu^+\mu^-) < 1.2\times 10^{-6}$$

and (7)

$$B(K_L \rightarrow \pi e^+e^-) < 2.3\times 10^{-6} \ ,$$

which imply, respectively, that

$$\mathcal{A}(K_L \rightarrow \pi h) < 0.20\times 10^{-10}(m_K/2p_h) \times [B(h\rightarrow \mu^+\mu^-)^{-1/2}] \text{ GeV}$$

and (8)

$$\mathcal{A}(K_L \rightarrow \pi h) < 0.28\times 10^{-10}[B(h\rightarrow e^+e^-)^{-1/2}] \text{ GeV} \ .$$

It is clear that, even if the branching ratios are of ~ 1 (which is the case for $m_h < 2m_\pi$) the presence of the unknown parameter B in Eq. (6) prohibits any definitive conclusion being drawn on the nonexistence of the Higgs boson. For $m_h > 2m_\pi$ we shall show that $B(h \to \mu^+\mu^-)$ is in fact $\leq \frac{1}{10}$. Note that if the experimental branching ratios are improved by at least 2 orders of magnitude, it would then require a considerable fine-tuning of B (≈ 1) in order to avoid the conclusion that the Higgs boson is ruled out with $m_h < 350$ MeV.

There is a claim in the literature[11] that a Higgs boson with $m_h < 409$ GeV can be excluded from the failure to observe the decay $\eta' \to \eta + h$. However, apart from the considerable theoretical uncertainties in calculating this decay, the claim is again based on the assumption that $B(h \to \mu^+\mu^-) \approx 1$. A reanalysis of these results[10,12] leads to the conclusion that they do not rule out a Higgs boson with any mass.

Inclusive B meson decay minimizes the problems associated with strong-interaction matrix elements. In principle, since the available energy for the Higgs boson is of order $m_B/2$, a perturbative calculation of b quark decay $b \to h + s$ should be sufficient to calculate B meson decay. This process has been recently studied[17,18,15] and compared with data for the branching fraction

$$\frac{\Gamma(B \to h(\to \mu^+\mu^-) + \text{anything})}{\Gamma(B \to e + \nu_e + \text{anything})} .$$

The effective one-loop Lagrangian for this process is given by (Fig. 3)

$$\mathcal{L} = + \frac{3\alpha}{32\pi \sin^2\theta_W} V_{tb} \frac{m_t^2}{M_W^2} V_{ts}^* \frac{h}{\sqrt{2}v} [m_b \bar{s}(1+\gamma_5)b] + \text{H.c.} \tag{9}$$

Taking $m_b = 4.5$ GeV and using the experimental branching ratio for $B \to e\nu X$ of 12.3%, we obtain, from Ref. 15,

$$B(B \to h + \text{anything}) = (0.35) \frac{|V_{ts} V_{tb}^*|^2}{|V_{cb}|^2} \left[\frac{m_t}{M_W} \right]^4 \times \left[1 - \frac{m_h^2}{m_b^2} \right]^2 . \tag{10}$$

The experimental limit for this process[19] is

$$B(B \to h + \text{anything}) B(h \to \mu^+\mu^-) < 0.008 \tag{11}$$

for $m_h > 500$ meV. The theoretical value depends crucially on the top-quark mass, mixing angles and, of course, on the $h \to \mu^+\mu^-$ branching ratio. A fourth generation of quarks would also contribute with unknown mixing angles. As discussed in Appendix A, in order to have a light Higgs boson with just one Higgs doublet and three families of quarks, the top-quark mass must be ~ 80 GeV. Combining (10) and (11) this would require $B(h \to \mu^+\mu^-) \leq 2.7 \times 10^{-2}$. However, as can be seen from Fig. 12, this condition is satisfied for m_h greater than approximately 700 MeV. Thus, contrary to the claim of Ref. 15 one cannot entirely rule out even the single Higgs doublet case. Furthermore, if there are more families and/or more Higgs doublets, the situation is even less definitive (see Appendixes A and B).

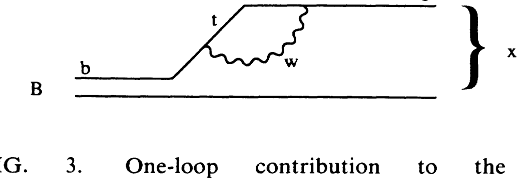

FIG. 3. One-loop contribution to the decay $B \to h + \text{anything}$.

The exclusive process $B \to K + h$ has also been calculated by Godbole, Turke, and Wirbel.[20,18] The result relies on less certain strong-interaction matrix elements. A light Higgs boson cannot at the present be ruled out with this process.

Let us finally consider the important process $\Upsilon \to h + \gamma$. An extensive experimental search for a monochromatic photon has been conducted by the CUSB Collaboration at CESR (Ref. 21). This process circumvents the question of the Higgs-boson branching ratios. However, although no monochromatic photons have been seen, it is difficult to set limits on the Higgs-boson mass using this process, as a result of the unfortunate uncertainties in the theoretical calculation. The process was first calculated by Wilczek.[22] Recently, both radiative QCD corrections and relativistic corrections of the nonrelativistic Υ wave function have been calculated.[23,24] The QCD radiative corrections give[23]

$$\Gamma(\Upsilon \to h + \gamma) = \Gamma_{\text{Wilczek}} \left[1 - \left[\frac{4\alpha_s}{3\pi} \right] a_H(z) \right] , \tag{12}$$

where $z \equiv 1 - m_h^2/m_\Upsilon^2 \approx 1$ and $a_H(1) = 7 + 6\ln 2 - \pi^2/8 \approx 10$. For $\alpha_s(m_\Upsilon^2) = 0.15$ the one-loop correction is 63% of the tree-level result. Thus the use of a perturbation expansion is suspect so no definitive statement can be deduced from the data. Furthermore, the relativistic corrections are just as large, but the two different calculations disagree in the sign of the effect.[24] Hopefully this distressing theoretical situation will improve with closer scrutiny. In the meantime, this process can still be used to search for a light Higgs boson.

As indicated in the above discussion, a major experimental signature for such a particle is its decay into two muons. In many of the previous analyses the branching ratio for this decay seems to have been grossly overestimated leading to false conclusions about the possible existence of a low-mass Higgs boson. The main purpose of this paper is to show how this branching ratio can be calculated and in the next section we begin its evaluation. The studious reader, who wants to know all the details of the calculation, can read this paper straight through. Otherwise, the results of the calculation and estimates for the branching ratio are summarized in Sec. V. It is, however, worth making a few remarks here as to why this branching ratio is so much smaller than one might otherwise think.

Since the Higgs-boson coupling is proportional to mass [see, e.g., Eq. (2)] one might naively guess that

$$\Gamma(h\to\mu^+\mu^-)/\Gamma(h\to\pi^+\pi^-)\sim(m_\mu/m_\pi)^2 \,.$$

However, there are two sources that lead to the suppression of this estimate. The first is a low-energy theorem which requires that the amplitude for the 2π mode behave like $(\frac{11}{9}m_\pi^2+\frac{2}{9}m_h^2)$ for the case of three flavors of quarks.[25] Thus, as m_h increases this amplitude is eventually proportional to m_h^2 rather than m_π^2 (Ref. 26). The second source is, in fact, associated with the extrapolation of this low-energy theorem to larger values of m_h and arises from the enhancement in the S-wave π-π scattering amplitude.[4] In the following sections we set up the formalism for incorporating these effects directly from a phase shift analysis. However, in making our estimates, we have taken some standard resonance fits to the data in order to give an idea of just how large and important such contributions can be. A complete analysis of the $\pi\pi$ data is beyond the scope of this paper but should certainly be done when the data on the K and B decays improve. As already stated, these enhancements in the $h\to2\pi$ amplitude and the other uncertainties discussed above nullify any definitive conclusion concerning the existence or nonexistence of a light Higgs boson.

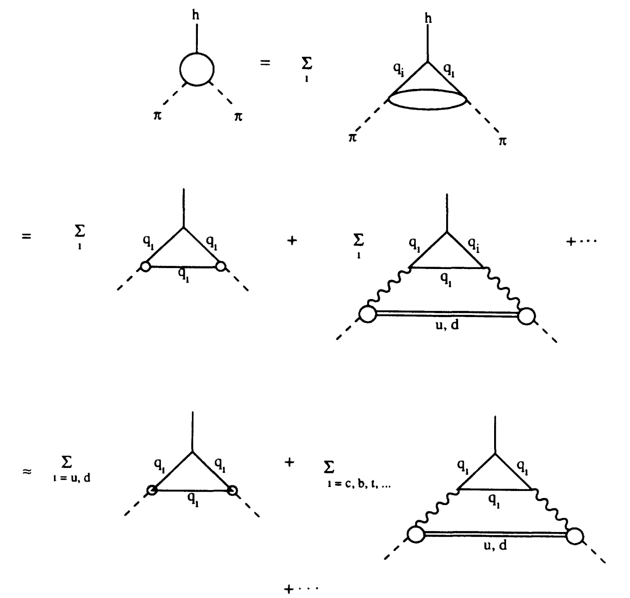

FIG. 4. The heavy-quark contribution to the decay $h\to2\pi$ as a result of the $h\to$two-gluon triangle graph.

II. THE ROLE OF HEAVY QUARKS AND GLUONS

In the minimal model, the coupling of a Higgs boson (h) to hadrons is governed by the Lagrangian [see Eq. (2)]

$$\mathcal{L}_I=-(2^{1/2}G_F)^{1/2}\sum_i m_i\bar{q}_i q_i \,, \tag{13}$$

where q_i are quark fields and the sum runs over all quark flavors. In a multi-Higgs-boson model, where h is to be interpreted as the lightest Higgs boson, the interaction will also have this generic structure; however, it will now depend upon various mixing angles. Although this affects the details of our results it does not affect the basic formalism presented below. As an example, we show in Appendix B how our results generalize to a two-Higgs-boson model constrained by supersymmetry.

For the decay of interest here, $h\to2\pi$, we need to evaluate the matrix element

$$\langle\pi\pi|h\rangle=(2^{1/2}G_F)^{1/2}\sum_i\langle\pi\pi|m_i\bar{q}_iq_i|0\rangle \,. \tag{14}$$

Since the pion is well described as a bound state of u and d quarks, one might have expected from the Zweig rule that the contribution of these quarks dominates the sum over flavors. However, as pointed out by Shifman, Vainshtein, and Zakharov,[8] heavy quarks contribute indirectly to this matrix element by virtue of the triangle anomaly as illustrated in Fig. 4. They couple to the gluon content of the pion via a quark-triangle graph which, in the limit $m_i\to\infty$, loses all explicit dependence on m_i. This is a direct consequence of the fact that the Higgs-boson coupling to quarks grows with mass. For small m_i, on the other hand, this mechanism only gives rise to a small form factorlike contribution and so is unimportant. These "hidden" heavy-quark terms can be formally expressed in a heavy-quark operator-product expansion:

$$\sum_{i=H} m_i\bar{q}_iq_i\approx-\frac{\alpha_s}{12\pi}n_H(F_{\mu\nu}^a)^2+\frac{\alpha_s^2}{30m_H^2}\left[\sum_{i=u,d,s}\bar{q}_i\gamma_\mu\lambda^a q_i\right]^2+\cdots \,. \tag{15}$$

Here n_H is the number of heavy-quark (H) flavors; $\alpha_s\equiv g^2/4\pi$, where g is the quark-gluon coupling and $F_{\mu\nu}^a$ the conventional gluon field tensor. One can think of (15) as an expansion in the parameter $(\alpha_s\Lambda^2/m_H^2)$ where Λ is some typical QCD mass scale (presumably several hundred MeV). Clearly, this expansion is only meaningful when $m_H^2\gg\alpha_s\Lambda^2$ and so excludes the u and d quarks, but includes the c,b,t and any further heavy flavors. The question of the role of the s quark is obviously borderline. For the proton there is some evidence that, despite the Zweig rule, the ratio $\langle p|\bar{s}s|p\rangle/\langle p|\bar{u}u|p\rangle$ may not be small but could be as large as 30% Ref. 27). If that is the case then the s quark contributes significantly ($\sim$40%) to the nucleon mass. For the pion, however, we shall assume that this is not the case so that if s is treated as a "light" quark (in the Shifman, Vainshtein, and Zakharov sense) then

$$\langle\pi|m_s\bar{s}s|\pi\rangle\ll\langle\pi|m_u\bar{u}u|\pi\rangle\approx\langle\pi|m_d\bar{d}d|\pi\rangle=O(m_\pi^2) \,.$$

Actually, for our purposes it will turn out that we shall only require $\langle\pi|m_s\bar{s}s|\pi\rangle\ll m_h^2$, which is certainly valid. If s is a "heavy" quark, in the sense that $m_s^2\gg\alpha_s\Lambda^2$, then, of course, it must be included in the expansion (15) contributing one unit to n_H.

With this in mind we can express Eq. (14) in the form

$$\langle \pi\pi | h \rangle \approx (2^{1/2} G_F)^{1/2} \times \left\langle \pi\pi \left| \left[\sum_{u,d} m_q \bar{q} q - \frac{n_H \alpha_s}{12\pi} (F_{\mu\nu})^2 \right] \right| 0 \right\rangle . \quad (16)$$

As stressed by Shifman, Vainshtein, and Zakhavov,[8] this amplitude is closely related to that of the trace of the energy-momentum tensor θ:

$$F(q^2) \equiv \langle \pi\pi | \theta | 0 \rangle . \quad (17)$$

Here q is the four-momentum carried by θ (and therefore the sum of the pion momenta). Recall that, in QCD,

$$\theta \simeq \sum_q m_q \bar{q} q + \frac{\beta(g)}{g} (F_{\mu\nu})^2 , \quad (18)$$

where $\beta(g)$ is the usual β function. [Note that, more precisely, the mass term enters with a factor $1+\gamma_m(g)$, a small correction, which we shall neglect.] To leading order in g^2 it is given by

$$\beta(g) = -\frac{g^3}{96\pi^2}(33-2n_f) , \quad (19)$$

where n_f is the total number of quark flavors. We remind the reader that the anomalous second term in (18) also has its origins in the triangle graph. Working as before one can straightforwardly derive a formula for $F(q^2)$ analogous to Eq. (16): namely,

$$F(q^2) \approx \left\langle \pi\pi \left| \left[\sum_{u,d} m_q \bar{q} q + \frac{\beta_L(g)}{g} (F_{\mu\nu})^2 \right] \right| 0 \right\rangle . \quad (20)$$

Here $\beta_L(g)$ is given by Eq. (19) but with n_f replaced by n_L ($\equiv n_f - n_H$), the number of "light" quarks.

Now, on general grounds [see Eq. (37)], $F(0) = 2m_\pi^2$. On the other hand, in the chiral limit, where $m_u = m_d \to 0$, $m_\pi^2 \to 0$ and so at $q^2 = 0$, we expect

$$\left\langle \pi\pi \left| \sum_{u,d} m_q \bar{q} q \right| 0 \right\rangle \approx m_\pi^2$$

and (21)

$$\left\langle \pi\pi \left| \frac{\beta_L(g)}{g} (F_{\mu\nu})^2 \right| 0 \right\rangle = m_\pi^2 .$$

The gluon terms, representing the contribution of heavy quarks, can be eliminated from Eq. (16) to give

$$F_h(q^2) \approx \left[\frac{2n_H}{33-2n_L} \right] F(q^2) + \left[1 - \frac{2n_H}{33-2n_L} \right] \left\langle \pi\pi \left| \sum_{u,d} m_q \bar{q} q \right| 0 \right\rangle , \quad (22)$$

where for convenience we have introduced the form factor $F_h(q^2)$ defined via

$$\langle \pi\pi | h \rangle \equiv (2^{1/2} G_F)^{1/2} F_h(q^2) . \quad (23)$$

Here q is the four-momentum of the Higgs boson; for the physical decay we obviously need to know $F_h(m_h^2)$.

Note incidentally, that

$$F_h(0) = \left[1 + \frac{2n_H}{33-2n_L} \right] m_\pi^2 .$$

Now, from general PCAC arguments, we expect the second term in (22) to be $O(m_\pi^2)$ for *all* values of q^2 (not just at $q^2=0$). If the same could be said for the first term $F(q^2)$ then the complete amplitude $F_h(q^2)$ would also be $O(m_\pi^2)$. This would lead to a small branching ratio for this mode and the decay into $\mu^+\mu^-$ would then dominate. However, Novikov and Shifman[25] have claimed that in the chiral limit ($m_\pi^2 \to 0$) $F(q^2) = q^2$ so that $F(m_h^2) = m_h^2$. This result has been used by Voloshin[26] to argue that the $\pi\pi$ mode is thereby considerably enhanced and dominates the $\mu^+\mu^-$ mode. In the next section we shall show that the Novikov-Shifman result is only valid for small q^2 and not for all q^2 as implied by their paper. However, in Sec. III we use a dispersion relation for $F(q^2)$ together with elastic unitarity to make the extrapolation to $q^2 = m_h^2$. We shall find that $F(q^2)$ is, indeed, $O(m_h^2)$ but can be further enhanced by a possible resonance in the S-wave π-π scattering amplitude. Our result is, in fact, in qualitative agreement with that of Ellis, Gaillard, and Nanopoulos[4] even though they identify the Higgs-boson coupling with θ_μ^μ.

III. LOW-ENERGY THEOREMS AND WARD IDENTITIES

This section is devoted to a review of low-energy theorems and Ward identifies relevant to the amplitude (23). We shall give reasonably detailed derivations since there are some misleading and, at times somewhat obscure, statements in the literature which we feel need clarifying. Furthermore, as will become clear below, there are some particularly subtle aspects to the soft-pion ($p \to 0$) and chiral ($m_\pi^2 \to 0$) limits for the three-point function of interest here.

We begin by considering the amplitude

$$T_{\mu\nu}(p,q) \equiv (p^2 - m_\pi^2)(p'^2 - m_\pi^2) \times \int d^4x \int d^4y \, e^{-ip\cdot x + iq\cdot y} \times \langle 0 | T[\phi^*(x) \theta_{\mu\nu}(y) \phi(0)] | 0 \rangle . \quad (24)$$

$\theta_{\mu\nu}$ is the energy-momentum tensor, ϕ the pion field, and p and p' the pion momenta (see Fig. 5). The trace $T_\mu^\mu(p,q)$ is just the off-shell continuation of $F(q^2)$; i.e.,

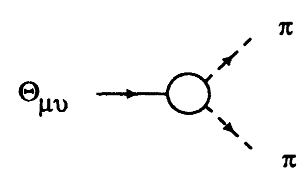

FIG. 5. Pictorial representation of the amplitude $T_{\mu\nu}$ [see Eq. (24)].

$$F(q^2)=\lim_{p^2\to p^2\to m_\pi^2} T^\mu_\mu(p,q) \ . \tag{25}$$

The most general form for $T_{\mu\nu}$ is

$$T_{\mu\nu}(p,q)=A(q^2g_{\mu\nu}-q_\mu q_\nu)+BP_\mu P_\nu + C(P_\mu q_\nu+q_\mu P_\nu)+Dq_\mu q_\nu \ , \tag{26}$$

where $P=(p'-p)$ and the form factors A, B, C, and D are, like T^μ_μ, functions of q^2, p^2, and p'^2 (in this order). Note, incidentally, they are also explicit functions of m_π^2.

The conservation of $\theta_{\mu\nu}$ implies the existence of a conserved momentum operator

$$P_\mu=\int d^3x\ \theta_{0\mu}(x) \tag{27}$$

with the property that

$$[P_\mu,\phi(x)]=-i\partial_\mu\phi(x) \ . \tag{28}$$

Up to Schwinger terms, the local form of this is

$$\delta(x_0-y_0)[\theta_{0\nu}(x),\phi(y)]=-i\partial_\nu\phi(y)\delta^4(x-y)\cdots \ . \tag{29}$$

This can be used to derive the Ward identity

$$q^\mu T_{\mu\nu}=-(p^2-m_\pi^2)(p'^2-m_\pi^2) \times[p_\nu\Delta_F(p^2)+p'_\nu\Delta_F(p'^2)] \ , \tag{30}$$

where

$$\Delta_F(p^2)\equiv\int d^4x\ e^{ip\cdot x}\langle 0|T[\phi^*(x)\phi(0)]|0\rangle \tag{31}$$

is the full pion propagator. As usual, we assume that the contribution of Schwinger terms is canceled by possible seagull terms in the definition of $T_{\mu\nu}$, Eq. (24). We shall return to this point below. Combining this with (26) then gives

$$B(p^2-p'^2)-Cq^2=-\tfrac{1}{2}(p^2-m_\pi^2)(p'^2-m_\pi^2) \times[\Delta_F(p^2)-\Delta_F(p'^2)] \tag{32}$$

and

$$C(p^2-p'^2)-Dq^2=\tfrac{1}{2}(p^2-m_\pi^2)(p'^2-m_\pi^2) \times[\Delta_F(p^2)+\Delta_F(p'^2)] \ . \tag{33}$$

Various results of interest can be derived from these: for example,

$$B(0,p^2,m_\pi^2)=B(0,m_\pi^2,p^2)=\tfrac{1}{2} \ , \tag{34}$$

$$C(q^2,p^2,p^2)=D(q^2,m_\pi^2,m_\pi^2)=0 \ , \tag{35}$$

and

$$D(q^2,p^2,p^2)=-\frac{(p^2-m_\pi^2)^2}{q^2}\Delta_F(p^2)\approx-\frac{p^2-m_\pi^2}{q^2} \ . \tag{36}$$

The second of these verifies that, on shell, $T_{\mu\nu}$ is conserved and that

$$F(q^2)=3q^2A(q^2,m_\pi^2,m_\pi^2)+(4m_\pi^2-q^2)B(q^2,m_\pi^2,m_\pi^2) \ . \tag{37}$$

Combining this with (34) confirms that $F(0)=2m_\pi^2$.

We next want to exploit PCAC to obtain a normalization constraint on A. To do so, introduce the amplitude

$$T_{\mu\nu\alpha}\equiv\frac{p^2-m_\pi^2}{f_\pi m_\pi^2}\int d^4x\ e^{-ip\cdot x}\langle\pi|T[\theta_{\mu\nu}(0)A_\alpha(x)]|0\rangle \ , \tag{38}$$

where A_α is the axial-vector current and f_π the pion decay coupling constant. In terms of these, PCAC reads

$$\partial^\mu A_\mu(x)=f_\pi m_\pi^2\phi(x) \ . \tag{39}$$

The following Ward identity can now be derived:

$$p^\alpha T_{\mu\nu\alpha}=T_{\mu\nu}-\frac{i(p^2-m_\pi^2)}{f_\pi m_\pi^2}\int d^4x\ e^{-ip\cdot x}\langle\pi|[\theta_{\mu\nu}(0),\ A_0(x)]\delta(x_0)|0\rangle \ . \tag{40}$$

In this equation $T_{\mu\nu}$ is to be evaluated with $p'^2=m_\pi^2$. A more general identity without this constraint and analogous to Eq. (30) can be derived; however, for our purposes it does not seem to contain any further useful information beyond that implied by (40). The equal-time commutator occurring in this equation can be eliminated by setting $p^2=m_\pi^2$ and going completely on shell. In this case neither C nor D contributes to $T_{\mu\nu}$, which now satisfies the simple equation $T_{\mu\nu}=p^\alpha T_{\mu\nu\alpha}$. If we now set $p_\alpha=0$ we can eliminate the unknown amplitude $T_{\mu\nu\alpha}$ to obtain the low-energy theorem $T_{\mu\nu}(0,q)=0$. To state this more precisely, note that when $p=0$, and $p^2=m_\pi^2$ then $p'^2=q^2=m_\pi^2=0$ and $P=p'=q$. In terms of Eq. (26) the low-energy theorem therefore leads to

$$A(0,0,0)=B(0,0,0)=\tfrac{1}{2} \ , \tag{41}$$

the last equality coming from Eq. (34). Notice that the theorem only fixes A and B at one kinematical point which is ultimately related to the chiral limit.

It is instructive to consider an alternative procedure in which limits are taken in a different order. Suppose we first set $p_\alpha=0$ in Eq. (40); then

$$T_{\mu\nu}(0,q)=\frac{i}{f_\pi}\langle\pi|[\theta_{\mu\nu}(0),Q_A(0)]|0\rangle \ , \tag{42}$$

where $Q_A(x_0)$ is the axial charge:

$$Q_A(x_0)\equiv\int d^3x\ A_0(x,x_0) \ . \tag{43}$$

Notice that in Eq. (42) one of the pions is now off shell (since $p^2=0\neq m_\pi^2$) so C and D contribute. However, if we go to the chiral limit by taking $m_\pi^2\to 0$ then this pion goes back on shell (so $C=D\rightarrow 0$) and, moreover, the commutator in (42) vanishes since Q_A is now conserved. Clearly one is led back to the previous result, Eq. (41). However, suppose we keep $m_\pi^2\neq 0$; then PCAC requires the right-hand side (RHS) of (42) to be $O(m_\pi^2)$. For example, the local form of the commutation relation $[P_0,Q_A]=-i\partial_0 Q_A$ can up to Schwinger terms be expressed as

$$\delta(x_0-y_0)[\theta_{00}(x),A_0(y)]=-i\partial_\mu A^\mu(y)\delta^{(4)}(x-y) \tag{44}$$

$$=-if_\pi m_\pi^2\phi(y)\delta^{(4)}(x-y) \tag{45}$$

or

$$[\theta_{00}(x),Q_A(x_0)]=-if_\pi m_\pi^2\phi(x)\ . \tag{46}$$

Using this in (42) gives

$$T_{00}(0,q)=m_\pi^2\ . \tag{47}$$

This is valid only when $p^2=0$ and $p'^2=m_\pi^2\neq 0$. From Eq. (40), this leads to

$$Am_\pi^2+q_0^2(B+2C+D-A)=m_\pi^2\ , \tag{48}$$

which, since q_0 is arbitrary, implies

$$A(m_\pi^2,0,m_\pi^2)=1 \tag{49}$$

and (at the same kinematic point)

$$B+2C+D-A=0\ . \tag{50}$$

Using Eqs. (32) and (33) it is straightforward to check that (50) is equivalent to (49). Note that Eqs. (41) and (49) disagree in the chiral limit.

The nature of the ambiguity can be clearly illustrated by examining Eq. (36) which is the $p^2=p'^2$ limit of Eq. (33). If p^2 is first set equal to m_π^2 then this gives $D(q^2,m_\pi^2,m_\pi^2)=0$ in agreement with (35). If, however, one first sets $p_\alpha=0$ (so that $p^2=0$ and $q^2=m_\pi^2$), then $D(m_\pi^2,0,0)=1$. When the chiral limit is taken we clearly end up with an ambiguous result for $D(0,0,0)$.

These ambiguities in D stem from combining the Ward identities with the right kinematical constraints of the three-point function. Unlike the more familiar case of a scattering amplitude, where setting the analog of p_α to zero is relatively harmless, here it leads to stringent kinematic constraints especially when $m_\pi^2\rightarrow 0$. It is our belief that the appropriate value for $D(0,0,0)$ is, in fact, zero since this is the "smooth" analytic continuation of the general result, Eq. (35). After all, $p^2=p'^2=0$ is the *on*-shell condition for a massless pion. The problem is that if $p_\alpha=0$ then for a massless pion $q^2=0$ forces D to be eliminated from the Ward identity, thereby making it indeterminate. This being the case we believe that the first derivation presented above, in which the pions are both put on shell *before* taking $p_\alpha=0$, is the appropriate one. This automatically eliminates the ambiguous form factor D from the discussion.

It seems likely that the source of the problem is the assumed cancellation between seagulls and Schwinger terms. For example, in Eq. (40) the naive local form of the commutator, analogous to Eqs. (44)–(46), is

$$\delta(x_0)[\theta_{0i}(x),A_0(0)]=-i\partial_i A_0(0)\delta^{(4)}(x)\ . \tag{51}$$

This leads to an extension of Eq. (47) to other components:

$$T_{0i}=q_0q_i\ . \tag{52}$$

Using (26), this gives back Eq. (50). However, this equation is surely incorrect in that it violates PCAC, for, from Eq. (42) we know that T_{0i} must vanish when $m_\pi^2=0$. This suggests that there are further Schwinger-type contributions to the relevant equal-time commutators which ensure the consistency of the Ward identities with PCAC.

Now let us apply these results to $F(q^2)$ given in Eq. (37). We have already seen that $F(0)=2m_\pi^2$. Let us now evaluate

$$\frac{\partial F}{\partial q^2}(0)=3A(0,m_\pi^2,m_\pi^2)-B(0,m_\pi^2,m_\pi^2)+4m_\pi^2\frac{\partial B}{\partial q^2}(0,m_\pi^2,m_\pi^2) \tag{53}$$

$$=1+\left[3[A(0,m_\pi^2,m_\pi^2)-A(0,0,0)]+4m_\pi^2\frac{\partial B}{\partial q^2}(0,m_\pi^2,m_\pi^2)\right]\ . \tag{54}$$

Thus, for $q^2\rightarrow 0$,

$$F(q^2)=2m_\pi^2+q^2+O(q^4,m_\pi^2q^2)\ . \tag{55}$$

From Eqs. (34)–(36), this implies that the physical amplitude, defined in Eq. (23), has the following expansion when $q^2\rightarrow 0$:

$$F_h(q^2)=\left[1+\frac{2n_H}{33-2n_L}\right]m_\pi^2+\left[\frac{2n_h}{33-2n_L}\right]q^2+O(q^4,m_\pi^2q^2)\ . \tag{56}$$

This result summarizes the low-energy theorem constraints arising from both PCAC and the conservation of $\theta_{\mu\nu}$. In the next section we shall show how to incorporate it into a dispersion relation which can then be exploited to calculate final-state interactions in the outgoing π-π system.

Before doing so, however, we should make some remarks concerning the relationship of these results to earlier work. We disagree in one major respect with the work of Novikov and Shifman.[25] They claim that in the chiral limit the form factors A and B are constants, independent of q^2. This they obtain from Eq. (42) which, in this limit, reads $T_{\mu\nu}(0,q)=0$. When setting $p=0$ they appear to have ignored energy-momentum conservation since they imply that, although $p^2=m_\pi^2=0$, q^2 is still arbitrary. If this were valid, then, indeed, it would follow that $A=B=\frac{1}{2}$ for *all* q^2. However, as emphasized above, when $p=0$, necessarily one must have $q^2=m_\pi^2=0$.

IV. DISPERSION RELATIONS AND THE FINAL-STATE INTERACTION

In Sec. III we used the Ward identities coupled with PCAC to derive the small-q^2 expansion of $F(q^2)$—see Eq. (55). We now want to explore the possibility of extrapolating this result to $q^2=m_h^2$. The natural framework for this is to write a dispersion relation for F and then make some reasonable approximations for its imaginary part.

Our first observation is that if one postulates unsubtracted dispersion relations for A and B, then Eq. (37) suggests that $F(q^2)$ be once subtracted. Basically this is because (37) requires F to be one less power of q^2 convergent for large q^2 than A or B. Thus it is natural to write

$$F(q^2)=2m_\pi^2+\frac{q^2}{\pi}\int_{4m_\pi^2}^{\infty} dq'^2\frac{\operatorname{Im}F(q'^2)}{q'^2(q'^2-q^2)} . \tag{57}$$

We want to derive a general expression for F in terms of the S-wave π-π phase shift (δ) in order to estimate the corrections to a naive extrapolation of Eq. (55) to the physical-Higgs-boson mass. We shall first discuss the structure of $F(q^2)$ for the case when only the 2π intermediate state is kept in the unitarity sum for $\operatorname{Im}F(q^2)$. $F(q^2)$ can thereby be expressed in terms of an Omnès function involving an integration over $\delta(q^2)$. We shall begin by reviewing this formalism stressing those aspects which are peculiar to this problem. This will then be extended to the two-channel case in which the $2K$ intermediate state is also allowed to contribute. Since we are only interested in the case where $m_h \le 1$ GeV the inclusion of further states is unwarranted.

A. The Omnès-Muskeleshvili representation (Ref. 28)

As implied by Eq. (57), $F(z)$ is an analytic function of z except for cuts on the positive real axis beginning at $\operatorname{Re}z_0=q_0^2=4m_\pi^2$. Notice that in the chiral limit, this threshold is driven to the origin and care must be taken in evaluating the dispersive contribution. Suppose that along the upper edge of the cut [i.e., along the line of integration of Eq. (57)] the phase of $F(z)$ is $\alpha(z)$; so, for $z=q^2+i\epsilon$ and $q^2\ge 4m_\pi^2$,

$$F(z)=|F(z)|e^{i\alpha(z)} . \tag{58}$$

This obviously implies that

$$\operatorname{Im}\ln F(q^2+i\epsilon)=\alpha(q^2+i\epsilon)\theta(q^2-4m_\pi^2) . \tag{59}$$

By writing a dispersion relation for $\ln F(z)$ analogous to Eq. (57) and assuming that F has no zeros we can deduce the following representation for $F(q^2)$:

$$F(q^2)=F(0)\Omega[\alpha(q^2)] , \tag{60}$$

where

$$\Omega[\alpha(q^2)]\equiv \exp\left[\frac{q^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{dz\,\alpha(z)}{z(z-q^2)}\right] . \tag{61}$$

The function Ω will be referred to as the Omnès function. As with any dispersion relation, this representation can be subtracted any number of times and, in fact, (60) is the once-subtracted form. Now, recall that $F(0)=2m_\pi^2$ and, that when $m_\pi^2\to 0$, $F'(0)=1$. From Eqs. (60) and (61) these lead to the sum rule

$$\lim_{m_\pi^2\to 0}\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{dq^2}{q^4}\alpha(q^2)=\frac{1}{2m_\pi^2} , \tag{62}$$

which requires $\alpha(0)=2\pi$. However, in that case, the $m_\pi^2\to 0$ limit of (60) and (61) gives

$$F(q^2)=\frac{2m_\pi^2}{\left[1-\dfrac{q^2}{4m_n^2}\right]^2} , \tag{63}$$

a result which is clearly unphysical. Thus the representation (60) and (61) cannot, as it stands, be consistent with the low-energy constraints of the Sec. III. It is straightforward to verify that further subtractions to (60) and (61) do not circumvent this problem.

The way out of this apparent dilemma is to recognize that, unlike an ordinary dispersion relation, the Omnès representation does not uniquely specify F if we are given only Eq. (58). For example, as should already be clear from the above discussion, a change in $\alpha(z)$ by $2\pi n$ which leaves (58) invariant, changes Ω to $(1-q^2/4m_\pi^2)^{-2n}\Omega$. Such an additional factor would imply that $F(q^2)$ either vanishes or diverges at $q^2=4m_\pi^2$ and so one is forced to choose $n=0$. More relevant to our situation is the observation that $\Omega(q^2)$ can be multiplied by an arbitrary polynomial in q^2 with real coefficients. For example, consider the function $F(q^2)=(A+Bq^2)\Omega$; clearly the phase $\ln F(q^2)=\alpha(q^2)\bmod(2\pi)$ if A and B are real. This obviously does not change Eq. (58). More generally, multiplying Ω by a polynomial of degree n (with real coefficients) simply shifts $\alpha(q^2)$ by $2\pi n$. For the case of interest here we can use this freedom to write

$$F(q^2)=(2m_\pi^2+q^2)\Omega[\alpha(q^2)] . \tag{64}$$

It is straightforward to check that the problems associated with the "naive" solution, Eqs. (60) and (61), are now averted provided that $\alpha(0)$ vanishes. Finally, it should be emphasized that the derivation of (64) requires $\Omega(q^2)$ to have no zeros, so any zeros of F must be put in by hand. Equation (64) therefore presumes that the only zero of F is that required by the low-energy theorem, namely, at $q^2=-2m_\pi^2$.

We now want to use unitarity, shown symbolically in Fig. 6 to obtain an expression for $\alpha(q^2)$. Below the $2K$ threshold, only the 2π intermediate state contributes to $\operatorname{Im}F$. So in the region $4m_\pi^2\le q^2<4m_K^2$,

$$\operatorname{Im}F(q^2)=F^*(q^2)T(q^2) , \tag{65}$$

where $T(q^2)=e^{i\delta(q^2)}\sin\delta(q^2)$, $\delta(q^2)$ being the $I=0$, S-wave π-π phase shift. Note that $\delta(q^2)$ is real and that the π-π S matrix is given by

$$S(q^2)=1+2iT(q^2)=e^{2i\delta(q^2)} .$$

Equation (65) immediately tells us that $\alpha(q^2)=\delta(q^2)$, which is simply a statement of the Watson final-state

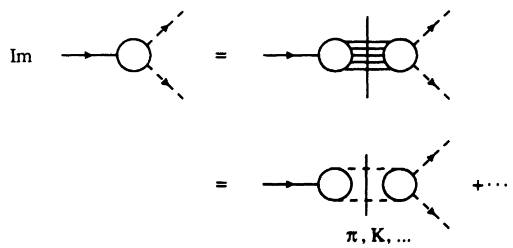

FIG. 6. A pictorial representation of the unitarity relation for $F(q^2)$ [see Eq. (65)].

theorem. Thus, in the single-channel elastic unitarity approximation,

$$F(q^2)=(2m_\pi^2+q^2)\Omega[\delta(q^2)] \ . \tag{66}$$

In principle, one can insert the existing experimental data for $\delta(q^2)$ into Eqs. (60) and (61) to determine Ω. A useful approximation is to take a resonance form

$$T(q^2)=\frac{k\Gamma}{M_R^2-q^2-ik\Gamma} \ . \tag{67}$$

M_R is the position of the resonance, Γ its width, and $k=(\frac{1}{4}q^2-m_\pi^2)^{1/2}$. This is equivalent to

$$\tan\delta(q^2)=\frac{k\Gamma}{M_R^2-q^2} \ , \tag{68}$$

which when inserted in (66) gives the resonance approximation for F:

$$F(q^2)=\frac{(2m_\pi^2+q^2)(M_R^2+m_\pi\Gamma)}{M_R^2-q^2-ik\Gamma} \ . \tag{69}$$

Note, incidentally, that Eq. (68) incorporates the correct threshold behavior of δ, namely, $\delta \propto k$, so that δ vanishes when $m_\pi^2=0$ as required for Eq. (66) to be valid. For the physical decay rate we need $F_h(q^2)$ with $q^2 \equiv m_h^2$, which is given by

$$F_h(q^2)=\left[\left(1+\frac{2n_H}{33-2n_L}\right)m_\pi^2+\left(\frac{2n_H}{33-2n_L}\right)q^2\right] \times\Omega[\delta(q^2)] \ . \tag{70}$$

In the resonance approximation we have

$$F_h(q^2)=\left[\left(1+\frac{2n_h}{33-2n_L}\right)m_\pi^2+\left(\frac{2n_H}{33-2n_L}\right)q^2\right] \times\frac{M_R^2+m_\pi\Gamma}{M_R^2-q^2-ik\Gamma} \ . \tag{71}$$

Clearly if there is a resonance in the range $M_R \sim m_h$ one can expect a considerable enhancement. In Sec. V we shall discuss the numerical aspects of these equations.

Before concluding this subsection, it is worth noting that if $\delta(q^2)\geq n\pi$ for $q^2\geq M^2 > m_h^2$ then the integral in Ω from M^2 to ∞ contributes a multiplicative factor to $F(m_h^2)$ which is smaller than

$$(1-m_h^2/M^2)^{-n}\approx(1+nm_h^2/M^2) \ .$$

This contribution can therefore be neglected when $M \gg n^{1/2}m_h$. Experimentally, $\delta\sim 3\pi/2$ at $M\sim 1.3$ GeV/c; if there are no resonances in this channel beyond this, then one can choose $n=2$. In that case one would safely cut off the integral in Ω at a value of $M \gg 1.5$ GeV. This means that one should certainly take into account the $2K$ intermediate state. In the following subsection we discuss how this can be done. Although the method presented can be generalized to more than two channels, the effort hardly seems worth it.

B. The two-channel Omnès-Muskeleshvili representation

This subsection is devoted to a generalization of the previous discussion to include the $2K$ intermediate state. If we label the pion variables by a suffix 1 and the kaon ones by a suffix 2, then the generalization of the unitarity constraint, Eq. (65), reads

$$\mathrm{Im}F_i(q^2)=F_j^*(q^2)T_{ji}(q^2) \ , \tag{72}$$

where i and j both run from 1 to 2 and T is the $I=0$, S-wave scattering amplitude. This can conveniently be expressed as

$$F_i(q^2)=S_{ij}(q^2)F_j^*(q^2) \ . \tag{73}$$

Note that S is symmetric (i.e., $S_{ij}=S_{ji}$ or $S=S^T$) and that it can be diagonalized by a unitary transformation

$$S'=USU^\dagger=US(U^*)^T \ . \tag{74}$$

S' can be expressed in the form

$$S'=\begin{bmatrix} e^{2i\sigma_1} & 0 \\ 0 & e^{2i\sigma_2}\end{bmatrix} \ , \tag{75}$$

where the σ_i are real. The transpose of Eq. (74) reads

$$S'=U^*SU^T \ . \tag{76}$$

By comparing Eqs. (74) and (76) we see that we can choose U to be real (i.e., these equations can be simultaneously satisfied by setting $U=U^*$). This allows us to diagonalize Eq. (73): introduce $F'\equiv UF$; then Eq. (73) transforms to $F'=S'F'^*$ whose elements read

$$F_i'(q^2)=e^{2i\sigma_i(q^2)}F_i'^*(q^2) \ . \tag{77}$$

Thus σ_i can be identified as the phase of F_i' which can therefore be represented in Omnès form as

$$F_i'(q^2)=F_i'(0)\Omega[\sigma_i(q^2)] \ . \tag{78}$$

Since $F=U^{-1}F'=U^TF'$, a solution for F_i can then be obtained. Although the above discussion was given within the context of a two-channel problem, it is clear that the strategy applies to the general multichannel case.

Let us now apply this formalism directly to the problem at hand. The unitarity of the S matrix

$$\sum_n S_{in}S_{jn}^*=\delta_{ij} \tag{79}$$

constrains the $I=0$, S-wave T matrix to satisfy

$$\sum_n T_{in} T_{jn}^* = \mathrm{Im} T_{ij} \ . \qquad (80)$$

When combined with the symmetry property $T_{ij} = T_{ji}$ (which is a consequence of PT invariance), this leads to the following general structure for S *provided one is above the* $2K$ *threshold* (i.e., that $q^2 > 4m_K^2$):

$$S = \begin{bmatrix} \eta e^{2i\delta_1} & i(1-\eta^2)^{1/2} e^{i(\delta_1+\delta_2)} \\ i(1-\eta^2)^{1/2} e^{i(\delta_1+\delta_2)} & \eta e^{2i\delta_2} \end{bmatrix} , \qquad (81)$$

where η and the δ_i are all real and η (the inelasticity) ≤ 1. Introduce $\eta = \cos\theta$, $\phi = \delta_1 + \delta_2$, and $\gamma = \delta_1 - \delta_2$; then S can be reexpressed in the form

$$S = e^{i\phi} \begin{bmatrix} \cos\theta e^{i\gamma} & i\sin\theta \\ i\sin\theta & \cos\theta e^{-i\gamma} \end{bmatrix} . \qquad (82)$$

Its eigenvalues are $e^{i(\phi+\rho)}$ and $e^{i(\phi-\rho)}$ where ρ is given by

$$\cos\rho = \cos\theta \cos\gamma \ . \qquad (83)$$

Thus

$$\sigma_1 = \tfrac{1}{2}(\phi+\rho) \qquad (84)$$

and

$$\sigma_2 = \tfrac{1}{2}(\phi-\rho) \ . \qquad (85)$$

The matrix U is now easily determined to be

$$U = \frac{1}{1+y^2} \begin{bmatrix} 1 & y \\ y & -1 \end{bmatrix} , \qquad (86)$$

where $y = (\sin\rho - \cos\theta \sin\gamma)/\sin\theta$. Notice that this confirms that U is indeed a real matrix and that $U = U^{-1} = U^\dagger = U^*$.

Below the $2K$ *threshold* the general form of S changes, since the $2K$ intermediate states do not contribute to the sum over intermediate states in Eq. (80). Thus, for example, even though $T_{12} \neq 0$ below $4m_K^2$, it makes no contribution to the left-hand side of (80). The set of unitarity relations therefore reads

$$\mathrm{Im} T_{11} = |T_{11}|^2, \quad \mathrm{Im} T_{12} = T_{11} T_{12}^* \ ,$$

and (87)

$$\mathrm{Im} T_{22} = |T_{12}|^2 \ ,$$

as illustrated in Fig. 7.

Notice, therefore, that for this region, the matrix multiplication implied in Eqs. (79) and (80) is no longer valid. From these equations we can now straightforwardly deduce the general structure of S: namely,

$$S = \begin{bmatrix} e^{2i\delta_1} & 2iae^{i\delta_1} \\ 2iae^{i\delta_1} & 2ib + 1 - 2a^2 \end{bmatrix} , \qquad (88)$$

where a and b are real functions of q^2. It is important to recognize that although unitarity has been used to determine this form, S is *not* a unitary matrix since only pions contribute to the intermediate states. Only above the $2K$ threshold where the kaon contribution comes into effect on the RHS of Eq. (87) does the unitarity of S_{ij} and, concomitantly, matrix multiplication get reinstated.

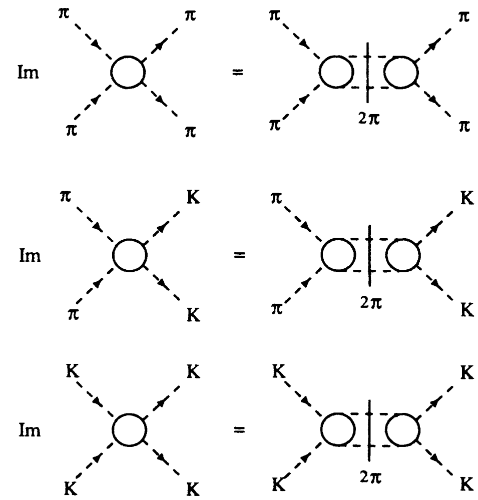

FIG. 7. Unitarity relation for the two-channel scattering amplitude below the $2K$ threshold [see Eq. (87)].

The analog to Eq. (73) with the sum over intermediate states excluding the kaon, now reads

$$F_1 = |F_1| e^{i\delta_1}, \quad \mathrm{Im} F_2 = a|F_1| \ . \qquad (89)$$

Again, matrix multiplication implied by (73) does not apply. However, it is possible to express this in matrix form, analogous to Eq. (73), by writing

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} e^{2i\delta_1} & 0 \\ 2iae^{i\delta_1} & 1 \end{bmatrix} \begin{bmatrix} F_1^* \\ F_2^* \end{bmatrix} . \qquad (90)$$

Let us denote the matrix occurring in this equation as $\bar{S}$ (to be distinguished from the S matrix). Then the solution of this equation can be obtained by defining $F' = \bar{U}F$ such that $\bar{U}\bar{S}\bar{U}^{-1} = \bar{S}'$, $\bar{S}'$ being diagonal and $\bar{U} = \bar{U}^*$. Note that in this case $\bar{S}$ is not symmetric so $\bar{U}$ is not *unitary*. The general solution has the form

$$\bar{U} = \begin{bmatrix} 1 & 0 \\ u & v \end{bmatrix} \text{ and } \bar{S}' = \begin{bmatrix} e^{2i\delta_1} & 0 \\ 0 & 1 \end{bmatrix} , \qquad (91)$$

where u and v are real functions of q^2 satisfying $v/u = -\sin\delta_1/a$. Hence, for $q^2 < 4m_K^2$, we have $F_1'(q^2) \equiv F_1(q^2)$. Note that continuity of the real part of the T matrix forces $\eta = 1$ and $a = \delta_2 = b = 0$ at $q^2 = 4m_K^2$ which implies that y vanishes at this threshold.

The solution to the Omnés equation for the pion form factor below the K threshold is now given by $F_1(q^2) = \Omega[\sigma_1(q^2)] F_1(0)$, where σ_1 is given by

$$\sigma_1(q^2) = \begin{cases} \delta_1(q^2), & 4m_\pi^2 < q^2 < 4m_K^2 \ , \\ \tfrac{1}{2}(\phi+\rho), & q^2 > 4m_K^2 \ . \end{cases} \qquad (92)$$

In a similar way, we can write down the solutions for both the pion and kaon form factors above the K threshold. In the next section, we shall apply these results to evaluate the branching ratio for the decay $h \to \mu^+\mu^-$. As expected F_1 is determined from the measurable quantities δ_1, η, and ϕ; in principle, one need "only" feed these into Eqs. (92) and (61). However, this requires data analysis which is beyond the scope of the present paper. Instead we shall follow what has typically been done in the past and use resonant approximations, as given in Eqs. (67)–(69).

V. RESULTS FOR THE BRANCHING RATIO $h \to \mu^+\mu^-$

The relevant amplitudes for this process are given by

$$T(h \to \mu^+\mu^-) = (\sqrt{2}G_F)^{1/2} m_\mu \bar{u}(p') v(p) , \tag{93}$$

where p and p' are the outgoing four-momenta and u and v are Dirac spinors, and

$$T(h \to \pi_i \pi_i) = (\sqrt{2}G_F)^{1/2} F_h(m_h^2) , \tag{94}$$

where

$$F_h(m_h^2) = \left[\left[1 + \frac{2n_H}{33-2n_L} \right] m_\pi^2 + \left[\frac{2n_H}{33-2n_L} \right] m_h^2 \right] \times \Omega[\sigma_1(m_h^2)] \tag{95}$$

and i is an isospin index. For the definition of $\Omega[\sigma_1(q^2)]$, see Eqs. (61) and (92). The constants n_L (n_H) are the number of light (heavy) quarks, respectively. If we take the u, d, and s quarks as light, then $n_L = 3$ and $n_H = 3$ for three generations of quarks or $n_H = 5$ for four generations. We shall leave n_H as a free parameter. In the single-resonance approximation, the function Ω is given by the expression [see Eqs. (67) and (69)]

$$\Omega[\sigma_1(m_h^2)] = \frac{M_R^2 + m_\pi \Gamma}{M_R^2 - m_h^2 - ik\Gamma} , \tag{96}$$

where R denotes the $I=0$, $J^{PC}=0^{++}$ resonances with mass (M_R) and width (Γ) and $k = (m_h^2/4 - m_\pi^2)^{1/2}$. The final form for the branching ratio is given by

$$B(h \to \mu^+\mu^-) \equiv \frac{1}{1+f}, \quad f \equiv \frac{\sum_{i=1}^{3} \Gamma(h \to \pi_i \pi_i)}{\Gamma(h \to \mu^+\mu^-)} . \tag{97}$$

The branching fraction f is given by

$$f = \frac{3 \left[\left[1 + \frac{2n_H}{33-2n_L} \right] m_\pi^2 + \left[\frac{2n_H}{27} \right] m_h^2 \right]^2 \frac{(M_R^2 + m_\pi \Gamma)^2}{(M_R^2 - m_h^2)^2 + k^2\Gamma^2} \left[\frac{2k}{m_h} \right]}{4 m_\mu^2 m_h^2 \left[1 - \frac{4m_\mu^2}{m_h^2} \right]^{3/2}} . \tag{98}$$

In the following examples we take $n_H = 3$. Now we must face the question, what values should we use for the resonance parameters? In Fig. 8 we have reproduced the most recent compilation on $\delta_1(q^2)$. Earlier data did not reach a sufficiently high energy for δ_1 to reach 90° and the S-wave enhancement was parametrized by a broad resonance referred to as the σ with a mass $M_R \sim 700$ MeV and width ~ 300 MeV.

Recent data as illustrated in Fig. 8 shows no dramatic sign of a resonance; they do show that δ_1 passes through 90° at roughly 850 MeV (rather than at 700 MeV). A fit of Eq. (68) to the data in this region leads to a corresponding width $\Gamma \sim 1.3$ GeV. These parameters may not conform to one's natural intuition as to what a resonance should look like, but they do fit the data. We have therefore used them to evaluate (97) and the results for this case are shown in Fig. 9. The parameters for this case are, in fact, quite close to a resonance fit made by Au, Morgan, and Pennington[29] in their relatively exhaustive treatment of the problem. They have attempted to fit all relevant $\pi\pi$ data (including the inelasticities) with a series of resonances in order to try to isolate a scalar glueball. They find three resonances below 1.6 GeV which couple to $\pi\pi$: a broad one at 900 MeV with a width of 1.4 GeV (obviously the analog of the one used to obtain Fig. 9), a relatively narrow one (the glueball candidate) at 991 MeV

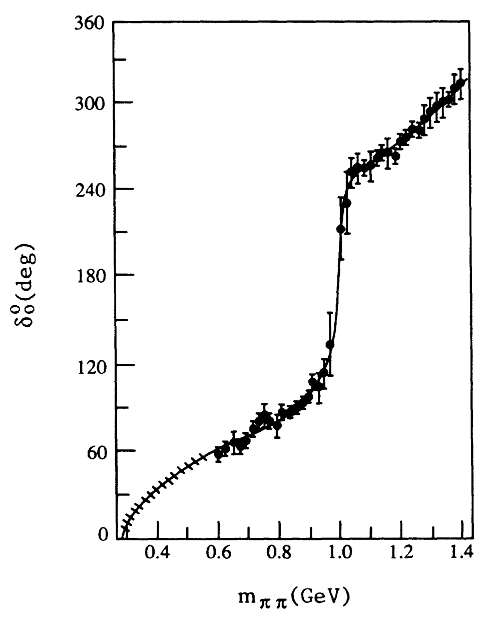

FIG. 8. The S-wave π-π phase-shift data (see Ref. 29).

with a width of 80 MeV, and another broad one at 1.43 GeV with a width of 800 MeV. Note, incidentally, that the Particle Data Group tables[16] list a narrow resonance at 975 MeV with a width of 33 MeV (presumably the analog of the Au, Morgan, and Pennington glueball candidate) and a broad resonance at 1300 MeV with a width of 150–400 MeV.

The effect of a narrow resonance is considerably larger in the vicinity of the pole than a broad resonance. This is illustrated in Figs. 10 and 11 where, as an example, we have taken the parameters from the Particle Data Group tables: ($M_R = 975$ MeV, $\Gamma = 33$ MeV). Figure 11 shows a blowup of the shoulder and tail of Fig. 10. The final result for f is the sum of the contributions of all of these resonances. We shall approximate this by the incoherent sum of resonances as given by the product of Omnés functions [Eq. (96)] for the individual resonances. This is shown in Fig. 12. Clearly, in order to do a better job on this problem it will be necessary to include the results of phase shift analyses for the π-π and K-K systems. We hope our formalism will be useful in this endeavor.

To conclude, it is clear that final-state interactions will enhance the π-π decay mode of the Higgs boson. As a result, the $\mu^+\mu^-$ decay mode may be greatly suppressed leading to values $\sim 10^{-1}$–10^{-2}. This result has significant consequences for experiments designed to search for a light Higgs boson via its two-muon decay channel.

Note added in proof

Willey (Ref. 30) has recently emphasized that η_t may be the dominant contribution to Eq. (6). Using mixing-angle limits obtained from the Particle Data Group, one finds $\eta_t \geq 0.2\eta_c[m_t/(40\ \mathrm{GeV}/c^2)]^2$. Thus for $m_t < 80\ \mathrm{GeV}/c^2$, $\eta_t < \eta_c$. However, using B_d^0-$\bar{B}_d^0$ mixing, he finds $\eta_t \geq \eta_c[m_t/(20\ \mathrm{GeV}/c^2)]$ and η_t dominates for $m_t \sim 80\ \mathrm{GeV}/c^2$. Our conclusion is nevertheless unchanged. Willey has not kept the arbitrary parameter B [Eq. (6)] in his analysis. He has used vacuum insertion to fix the value of B.

If no Higgs boson is found in K decay, we can at most claim to have measured the value of the arbitrary param-

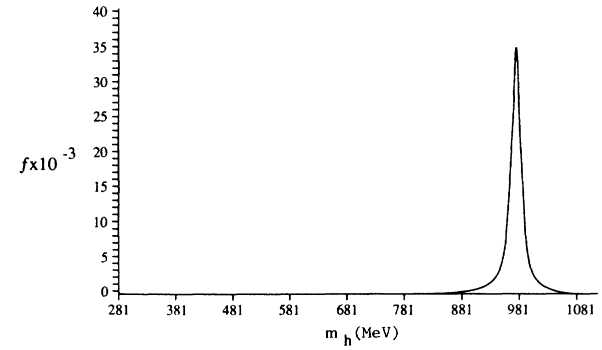

FIG. 10. Contribution to f of the π-π resonance with mass $M_R = 975$ MeV and width $\Gamma = 33$ MeV.

eter B. On the other hand, if a Higgs boson is found in K decay, we would learn a great deal about the standard model.

APPENDIX A: THE LINDE-WEINBERG BOUND (REF. 31)

In the standard single-Higgs-doublet model, the one-loop effective potential is

$$V_1(h) = -\mu^2 h^2 + \lambda h^4 + C h^4 \ln\left[\frac{h^2}{M^2}\right], \tag{A1}$$

where μ, λ, and M are constants which can be related to the physical mass and vacuum expectation value of the Higgs boson. The constant C is given by

$$C = \frac{1}{16\pi^2 v^4}\left[3\sum_V M_V^4 + m_h^4 - 4\sum_f m_f^4\right], \tag{A2}$$

where V runs over the vector bosons in the theory ($W^\pm$ and Z^0) and f over the fermions (quarks and leptons). Generically $V_1(h)$ has three minima: one at $h = 0$ and two at $h = \pm h_0$. By definition $v = \sqrt{2}h_0$ and

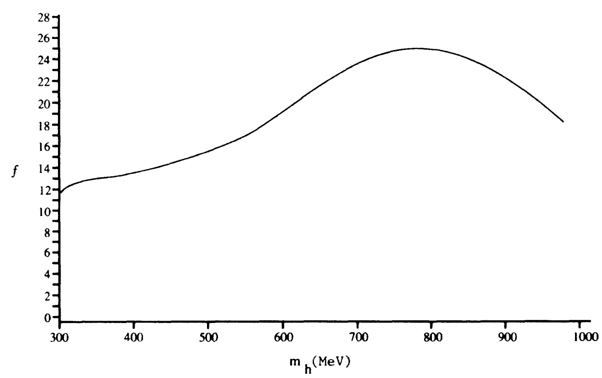

FIG. 9. Contribution to f of the π-π resonance with mass $M_R = 850$ MeV and width $\Gamma = 1.3$ GeV.

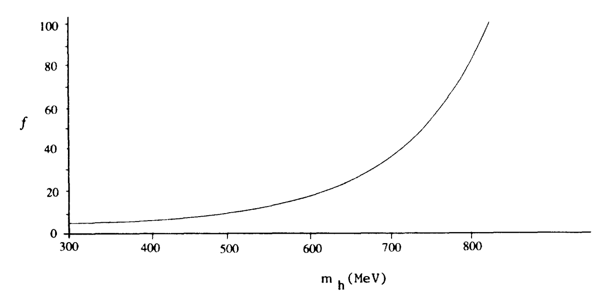

FIG. 11. Blow up of the region described in Fig. 11 below 780 MeV.

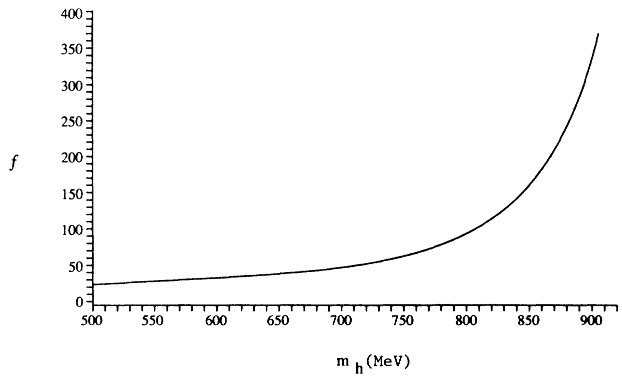

FIG. 12. Incoherent sum of the resonance contributions for $M_R = 850$ MeV $\Gamma = 1.3$ GeV, and $M_R = 975$ MeV, $\Gamma = 33$ MeV.

$$m_h^2 = \frac{1}{2}\frac{\partial^2 V_1}{\partial h^2}\bigg|_{h_0} = 2\mu^2 + 4Ch_0^2 \ . \tag{A3}$$

The Linde-Weinberg bound arises from the stability requirement that $V_1(h_0) < V_1(0) = 0$. This condition ensures that the spontaneous symmetry breaking takes place in a vacuum which is an absolute minimum. This leads to the constraint

$$\mu^2 + Ch_0^2 > 0 \ , \tag{A4}$$

which, when combined with (A3), gives

$$m_h^2 > Cv^2 \ . \tag{A5}$$

If there are no heavy fermions then C is dominated by the $W^{\pm}$ and Z^0 bosons with the result that

$$m_h^2 > \frac{3(2M_W^4 + M_Z^4)}{16\pi^2 v^2} \approx 7 \text{ GeV} \ . \tag{A6}$$

This is the Linde-Weinberg bound. If, however, there is a heavy fermion such as the top quark with mass of approximately 80 GeV then C and concomitantly the bound vanishes. Thus a lighter Higgs boson is allowed. For fermion masses greater than 80 GeV the lower bound on the Higgs-boson mass rises rapidly.

A natural way to avoid the bound which was, in fact, already pointed out in the original papers is to have more than one Higgs doublet. In that case, no general statement can be made. The reader is referred to Ref. 32 for a complete discussion of the problem. Appendix B is devoted to a discussion of some aspects of having two doublets, which is relevant to the problem of Higgs-boson decay.

APPENDIX B: REMARKS CONCERNING THE GENERALIZATION TO TWO HIGGS DOUBLETS

In a two-Higgs-boson model the interaction with hadrons is governed by the Lagrangian

$$\mathcal{L}' = g_u^{ij}\bar{u}_i q_j \phi_1 + g_d^{ij}\bar{d}_i g_j \phi_2 \quad (i,j = 1,2,3) \ . \tag{B1}$$

To avoid dangerous strangeness-changing neutral currents we have coupled ϕ_1 to "up"-type quarks and ϕ_2 to "down" type. The mass eigenstates (h_1 and h_2) are related to the ϕ_i by a mixing angle α:

$$h_1 = \cos\alpha \, \mathrm{Re}\phi_1 + \sin\alpha \, \mathrm{Re}\phi_2 \tag{B2}$$

and

$$h_2 = \cos\alpha \, \mathrm{Re}\phi_2 - \sin\alpha \, \mathrm{Re}\phi_1 \ . \tag{B3}$$

In such models there are, in general, two neutral-scalar-mass eigenstates (h_1 and h_2), one neutral pseudoscalar (χ^0), and two charged scalars ($\chi^{\pm}$). Their masses remain arbitrary since they are governed by arbitrary parameters in the potential. However, for the supersymmetric case, the potential is highly constrained with the result that, at the tree level, the physical masses are given by

$$m_{\chi^0}^2 = 2m_3^2 \csc 2\beta \ , \tag{B4}$$

$$m_{\chi^\pm}^2 = m_{\chi^0}^2 + M_W^2 \ , \tag{B5}$$

and

$$m_{h_1,h_2}^2 = \tfrac{1}{2}\{(m_{\chi^0}^2 + M_Z^2) \pm [(m_{\chi^0}^2 + M_Z^2)^2 - 4m_{\chi^0}^2 M_Z^2 \cos^2 2\beta]^{1/2}\} \ , \tag{B6}$$

where $\tan\beta \equiv v_2/v_1$ with $v_i \equiv \langle 0|\phi_i|0\rangle$. Note that m_3 is arbitrary and that $v_1^2 + v_2^2 = v^2 = 2^{-3/2}G_F^{-1}$. Thus $m_{\chi^\pm}$ and m_{h_2} are all greater than M_W, whereas m_{h_1} can, in principle, be as light as one pleases. Thus, even in this special situation, a light Higgs boson (h_1) is permissible. However, this only obtains with some fine-tuning as can be seen by rewriting (B6) in the form

$$m_{h_1} \approx \frac{m_{\chi^0} M_Z}{(m_{\chi^0}^2 + M_Z^2)^{1/2}} \left[\frac{v_1^2 - v_2^2}{v_1^2 + v_2^2}\right] \ . \tag{B7}$$

This clearly requires either a light m_{χ^0} or $v_1 \approx v_2$ (i.e., $\beta \sim 45°$). If we now consider one-loop corrections to the Higgs-boson potential, we find that a light Higgs boson is still possible providing we fine-tune certain relations between scalar-quark and -lepton masses, gaugino masses, in addition to the usual quark, lepton, and gauge-boson masses. Thus, there is no theoretical lower limit on m_{h_i}.

From $\mathcal{L}'$, quark masses are generated in the usual way through the v_i giving rise to mass matrices $m_{ij}^u = v_1 g_{ij}^u$ and $m_{ij}^d = v_2 g_{ij}^d$. Equations (B2) and (B3) can be inverted to give

$$\mathrm{Re}\phi_1 = h_1 \cos\alpha - h_2 \sin\alpha \ , \tag{B8}$$

$$\mathrm{Re}\phi_2 = h_1 \sin\alpha + h_2 \cos\alpha \ . \tag{B9}$$

$\mathcal{L}'$ can now be diagonalized and written in terms of the physical Higgs boson. The interaction terms involving the light Higgs boson h_1 thereby read

$$\mathcal{L}' = h_1 \left[\frac{\cos\alpha}{v_1}\sum_i m_i \bar{u}_i u_i + \frac{\sin\alpha}{v_2}\sum_i m_i \bar{d}_i d_i + \cdots\right] , \tag{B10}$$

where the sums are over the "up-" and "down"-type quarks separately. The minimal Higgs-boson model is the special case where $\phi_1 = \phi_2^*$ in Eq. (B1) and $\alpha = \beta = 45°$ (so that $v_1 = v_2$). In that case (B10) reduces to Eq. (13).

In terms of the coupling to pions, the light-quark (u and d) contribution can no longer be directly identified with the $q^2 = 0$ behavior of $F(q^2)$ as in Eq. (22) since they no longer contribute in the same way as in the stress tensor. On the other hand, from PCAC they must still remain $O(m_\pi^2)$. The heavy quarks, however, contribute much as before; since their contribution is mass independent they are simply incorporated by replacing n_H (the number of heavy-quark flavors) by

$$\bar{n}_H \equiv \left[\frac{\cos\alpha}{\cos\beta} \right] n_H^u + \left[\frac{\sin\alpha}{\sin\beta} \right] n_H^d , \tag{B11}$$

where $n_H^{u(d)}$ is the number of up (down) quarks.

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