Dimensional reduction in finite-temperature quantum chromodynamics. II

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(Received 31 May 1988)

The reduction of four-dimensional quantum chromodynamics at finite temperature and chemical potential to an infrared-effective three-dimensional theory is further investigated. The effective couplings are calculated in the manner prescribed by a previously discussed program of quantum dimensional reduction, demonstrating its internal consistency and completing the specification of the effective action. The latter provides a starting point for nonperturbative studies of the infrared behavior of high-temperature quark-gluon plasmas.

I. INTRODUCTION

The study of cooperative phenomena in non-Abelian gauge theories such as quantum chromodynamics (QCD) involves an understanding of their large-scale statistical and thermodynamical properties. At high temperatures ($T \gg \Lambda_{\text{QCD}}$), one can take advantage of a substantial simplification due to the decoupling theorem.[1] As is well known,[2] the leading infrared (IR) behavior of four-dimensional (4D) QCD at high temperatures is governed by its static (zero Matsubara frequency) sector, obtained by integrating out its nonstatic modes to leave behind an effective three-dimensional (3D) theory.[3] This process of "quantum dimensional reduction" was explored in detail in an earlier work[4] (hereafter referred to as Part I) and goes as follows. Classically, nonstatic modes are suppressed at IR momenta $\mathbf{k} \ll T$ by $O(\mathbf{k}^2/T^2)$ relative to their static counterparts. This suppression continues at the quantum level, except for a set of effective 3D couplings induced by the nonstatic integrations. In principle, this set is infinite; in practice, couplings beyond some low order can be ignored since they are suppressed relative to structurally similar static interactions by $O(\mathbf{k}^2/T^2)$ or better. Nonstatic integration is IR finite and at high temperatures, where the running coupling $g(T)$ is small, may be done perturbatively. If the integration were also to include the static modes, then, as is well known, all ultraviolet (UV) divergences would be canceled by the usual $T=0$ counterterms. However, the cancellation is incomplete when these counterterms are applied to the nonstatic integrals alone. The residual UV divergences survive as counterterms of the effective theory, showing up in the guise of its *bare* couplings. They will cancel the corresponding UV divergences arising from the 3D integrals of the effective theory, reflecting the UV finiteness of the original 4D theory.

To summarize Part I, the full theory is approximated, up to $O(\mathbf{k}^2/T^2)$ terms, by a theory dubbed "extended three-dimensional quantum chromodynamics" (EQCD$_3$), described by the following superrenormalizable effective action:

$$S=\int d^3\mathbf{x}\left[\frac{1}{2G^2}\mathrm{Tr}F^2(\mathbf{A})+\mathrm{Tr}(\mathcal{D}\phi)^2 +m_0^2\mathrm{Tr}\phi^2+\frac{\kappa}{2}(\mathrm{Tr}\phi^2)^2\right].$$

Here $\mathbf{A}$ represents the magnetostatic potential and ϕ (an adjoint scalar field proportional to the logarithm of the Polyakov loop operator) the electrostatic potential. G is the 3D gauge coupling which, by superrenormalizability, does not depend on the UV cutoff. EQCD$_3$ becomes effective at IR momentum scales, typically $\mathbf{k}^2 \lesssim g^2T^2$, for which the power-counting analysis of Part I prescribes that the bare mass parameter m_0^2 need only be computed to two loops and the induced quartic coupling κ to one loop in the nonstatic modes. The corresponding integrals are most easily evaluated in the class of gauges characterized by the condition $\partial_4 A_4=0$, the so-called "static gauges," in which static and nonstatic modes are cleanly separated and the electrostatic potential A_4 is proportional to the 3D scalar field ϕ.

The parameters of EQCD$_3$ have so far been computed only partially: the quartic coupling has been estimated to be $O(g^4T)$ with an undetermined coefficient,[4] while the bare mass is known only to one-loop order, in dimensional regularization (where it is UV finite)[4] and in an arbitrary regularization.[5] For the general case of N colors and N_f quark flavors (each having a chemical potential μ_i, $i=1,\ldots,N_f$), the parameters of EQCD$_3$ can be written[4,5]

$$G=g(T,\mu_i)\sqrt{T}\ ,$$

$$m_0^2=m_E^2-2Ng^2T\int^{\Lambda}\frac{d_3\mathbf{k}}{\mathbf{k}^2}+(\text{two-loop terms})\ ,$$

$$\kappa=f(N,N_f)g^4T\ ,$$

where $d_n\mathbf{k}\equiv d^n\mathbf{k}/(2\pi)^n$, Λ is a generic UV regulator, and the one-loop electric mass is given by

$$m_E^2\equiv(N+\tfrac{1}{2}N_f)\frac{g^2T^2}{3}+\frac{g^2}{2\pi^2}\sum_{i=1}^{N_f}\mu_i^2\ .$$

This incomplete knowledge of the EQCD$_3$ parameters is sufficient if one is interested only in low-order perturbative calculations. However, there is substantial and growing evidence that for most quantities of interest the onset of nonperturbativity is almost immediate, i.e., perturbative breakdown occurs at unexpectedly early stages.[5,6] One is then compelled to consider treating EQCD$_3$ nonperturbatively, which requires a knowledge of the full effective action. In this paper we shall complete the specification of the effective theory by computing the two-loop correction to m_0^2 and the coefficient $f(N,N_f)$ of κ. The latter result has been reported in a previous work.[7] Here, the derivation will be presented in some detail, mainly in order to illustrate static-gauge calculations in finite-temperature QCD. In the interest of brevity we shall avoid repeating material already contained in Part I, which should be consulted for notation, formalism, Feynman rules, integrals, and the like.

In Sec. II the quartic coupling κ is computed and shown to be positive (provided the number of fermion flavors is small enough), a result with important consequences for the consistency of the dimensional-reduction scheme. Section III deals with the UV-divergent part of the two-loop correction to the bare mass parameter m_0^2, which is shown to consist of a small correction to the linear one-loop divergence plus exactly canceling logarithmic divergences. The finite part of this correction is of little consequence and difficult to compute, so we do not attempt its evaluation here. The results are summarized in Sec. IV, which also updates the conclusions of Part I. Appendixes A and B evaluate the various integrals which arise during the course of the calculations.

II. THE INDUCED QUARTIC COUPLING

The induced quartic coupling κ is given by integrals over nonstatic graphs with vanishing external momenta. It is convenient to divide these into gluonic and fermionic parts, which we consider in turn.

A. The gluonic contribution

Although the gluonic contribution κ_G can, in principle, be calculated in any gauge, in practice the calculation is extremely tedious in gauges other than the static gauge, where the electric potential and the Polyakov loop operator are very simply related. The relevant graphs fall into three groups, within each of which they differ only by permutations of the external legs, as shown in Figs. 1(a)–1(c).

For type-(a) graphs we write

$$\sum \text{graphs (a)} = i(2g)^4(f^{afe}f^{deh}f^{chg}f^{bgf} + 2 \text{ permutations}) \times T\sum_{n\neq 0}\int d_3\mathbf{k}\frac{k_4^4}{k^8}\left[\delta_{jl}+\frac{k_jk_l}{k_4^2}\right]\left[\delta_{lm}+\frac{k_lk_m}{k_4^2}\right]\left[\delta_{mn}+\frac{k_mk_n}{k_4^2}\right]\left[\delta_{nj}+\frac{k_nk_j}{k_4^2}\right]$$

and use the reduction

$$(ffff + 2 \text{ permutations}) = \frac{5N^2}{2(N^2+1)}(\delta\delta + 2 \text{ permutations})$$

to find the contribution

$$\kappa_a = -\frac{40N^2g^4T}{N^2+1}\left[\frac{2k_4^4}{k^8}+\frac{1}{k_4^4}\right]'_{T\mu}.$$

For type-(b) graphs we write

$$\sum \text{graphs (b)} = i(2g^2)^2[(f^{aeh}f^{bfh}+f^{afh}f^{beh})f^{deg}f^{cgf} + 5 \text{ permutations}] \times T\sum_{n\neq 0}\int d_3\mathbf{k}\frac{k_4^2}{k^6}\left[\delta_{jl}+\frac{k_jk_l}{k_4^2}\right]\left[\delta_{lm}+\frac{k_lk_m}{k_4^2}\right]\left[\delta_{mj}+\frac{k_mk_j}{k_4^2}\right]$$

and use the reduction

$$[(ff+ff)ff + 5 \text{ permutations}] = -4(ffff + 2 \text{ permutations})$$

to find the contribution

$$\kappa_b = \frac{40N^2g^4T}{N^2+1}\left[\frac{2k_4^2}{k^6}+\frac{1}{k_4^4}\right]'_{T_\mu}.$$

For type-(c) graphs we write

$$\sum \text{graphs } (c) = \frac{ig^4}{2}[(f^{aeg}f^{bfg}+f^{afg}f^{beg})(f^{deh}f^{cfh}+f^{dfh}f^{ceh}) + 2 \text{ permutations}]T\sum_{n\neq 0}\int d_3\mathbf{k}\frac{1}{k^4}\left[\delta_{jl}+\frac{k_ik_l}{k_4^2}\right]^2$$

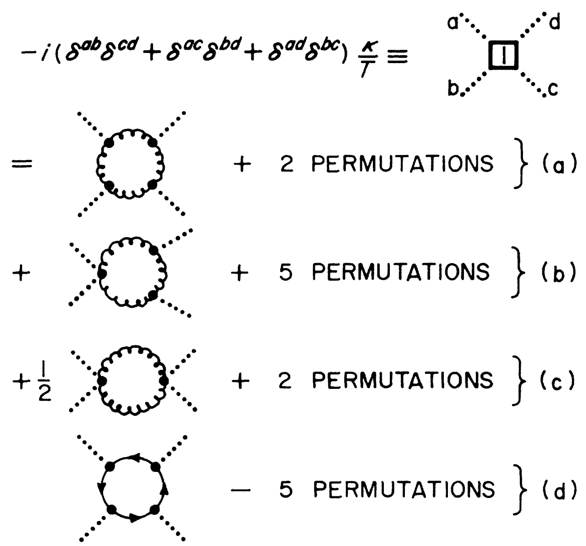

FIG. 1. Contributions to the induced quartic coupling κ: gluonic (a),(b),(c) and fermionic (d).

and use the reduction

$$[(ff+ff)(ff+ff)+2 \text{ permutations}] = 4(ffff+2 \text{ permutations})$$

to find the contribution

$$\kappa_c = -\frac{5N^2 g^4 T}{N^2+1}\left[\frac{2}{k^4}+\frac{1}{k_4^4}\right]'_{T\mu}.$$

Adding the contributions from graphs (a), (b), and (c), the total gluonic contribution is given by

$$\kappa_G \equiv \kappa_a + \kappa_b + \kappa_c = -\frac{10N^2 g^4 T}{N^2+1}\left[\frac{1}{k^4} - 8\frac{k_4^2}{k^6} + 8\frac{k_4^4}{k^8} + \frac{1}{2k_4^4}\right]'_{T\mu}.$$

The last term, which is proportional to $\delta^3(\mathbf{x}=0)$, is the singular ghost term found in any unitarity gauge calculation[8] and, as usual, is exactly canceled by a corresponding contribution from the effective measure in this gauge. The remaining integrals are evaluated in Appendix A; UV divergences cancel out (as they must) leaving behind the finite result

$$\kappa_G = \frac{5N^2}{N^2+1}\frac{g^4 T}{6\pi^2} = \begin{cases} 2g^4T/3\pi^2 & (N=2)\,, \\ 3g^4T/4\pi^2 & (N=3)\,. \end{cases}$$

The $N=2$ result has been obtained previously by several authors[9], using altogether different approaches.

B. The fermionic contribution

The fermionic contribution κ_F is the same in any gauge and for each fermionic flavor is given by six permutations of a basic one-loop graph, Fig. 1(d). As shown in Appendix A, the integrals we will encounter are independent of the chemical potentials running through the internal propagators, so κ_F is just N_f times the contribution of a single flavor.

For type-(d) graphs we write

$$\sum \text{graphs (d)} = -ig^4 \mathrm{Tr}(T^aT^bT^cT^d + 5 \text{ permutations})\, T \sum_n \int d_3\mathbf{p}\, \mathrm{tr}\left[\gamma_4 \frac{1}{\not p}\right]^4$$

and use the reductions

$$\mathrm{Tr}(T^aT^bT^cT^d + 5 \text{ permutations}) = \frac{2N^2-3}{2N(N^2+1)}(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc})$$

and

$$\mathrm{tr}\left[\gamma_4 \frac{1}{\not p}\right]^4 = \frac{4}{p^8}(p^4 - 8p_4^2p^2 + 8p_4^4)$$

to get

$$\frac{\kappa_F}{N_f T} = g^4 \frac{2(2N^2-3)}{N(N^2+1)}\left[\frac{1}{p^4} - 8\frac{p_4^2}{p^6} + 8\frac{p_4^4}{p^8}\right]_{T\mu}.$$

On substituting the values of the integrals from Appendix A, it is seen that the UV divergences again cancel, leaving the finite result

$$\kappa_F = -\frac{2N^2-3}{N(N^2+1)}\frac{N_f g^4 T}{6\pi^2} = -\frac{N_f g^4 T}{12\pi^2} \quad (N=2,3)\,.$$

C. The total contribution

Adding the fermionic and gluonic contributions to κ we finally arrive at the desired result:

$$\kappa \equiv \kappa_G + \kappa_F = \frac{5N^3 - (2N^2-3)N_f}{N(N^2+1)}\frac{g^4 T}{6\pi^2} = \begin{cases} (8-N_f)g^4T/12\pi^2, & \mathrm{SU}(2)\,, \\ (9-N_f)g^4T/12\pi^2, & \mathrm{SU}(3)\,, \end{cases}$$

which is positive for a sufficiently small number of fermion flavors.

III. TWO-LOOP CORRECTIONS TO THE BARE ELECTRIC MASS

It is necessary to check that for the bare mass parameter m_0^2, the one-loop value is stable against two-loop corrections. Since this is trivially true for the finite part, we need only calculate the UV-divergent part of the two-loop contribution. [As pointed out in Part I, contributions to m_0^2 higher than two-loop are too weak to be important in the dimensionally reduced theory for distance scales $O(1/gT)$.] Since there are no mass divergences in the full 4D theory, the two-loop UV divergences in m_0^2 will cancel those of the $O(g^4)$ graphs of EQCD$_3$, which are much easier to calculate. Consider therefore Π_{44} (i.e., the scalar self-energy) in EQCD$_3$, up to $O(g^4)$. The UV-divergent parts are obtained by setting the external three-momenta to zero. The net UV divergence is independent of the choice of gauge in EQCD$_3$, because the nonstatic contribution it must cancel is gauge independent. We are, therefore, free to choose any suitable gauge. The most convenient choice is the Landau gauge, in which most graphs vanish because they contain the vertex shown in Fig. 2. The surviving graphs are shown in Fig. 3; of these, graph (a) yields the previously computed $O(g^2)$ linear divergence.[5] Here we calculate the $O(g^4)$ graphs and show that graph (b) is just a small correction to graph (a), while graphs (c) and (d) have logarithmic UV divergences which cancel against each other.

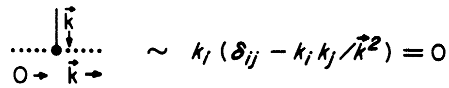

FIG. 2. Graphical element which vanishes in the Landau gauge.

With κ as given in Sec. II, we easily calculate the second graph to be

$$\text{graph (b)} = -(i\delta^{ab})2Ng^2T\left[\frac{(N^2+1)\kappa}{4Ng^2T}\right]\int^{\Lambda}\frac{d_3\mathbf{k}}{\mathbf{k}^2},$$

where the term in large parentheses is $O(g^2)$, while the rest of the expression is just graph (a).

The third graph is given by

$$\text{graph (c)} = -(i\delta^{ab})2Ng^2T\int\frac{d_3\mathbf{k}}{\mathbf{k}^4}\Pi^{(1)}(\mathbf{k}^2),$$

where we have, from Part I,

$$\Pi^{(1)}(\mathbf{k}^2) = [(\xi+1)^2+8]\frac{Ng^2T\,|\mathbf{k}|}{64}.$$

In the Landau gauge ($\xi=0$), this gives

$$\text{graph (c)}\,|^{\text{Landau}} = -(i\delta^{ab})\frac{9N^2g^4T^2}{32}\int\frac{d_3\mathbf{k}}{|\mathbf{k}|^3} = -(i\delta^{ab})\left[\frac{3Ng^2T}{8\pi}\right]^2\ln\frac{\Lambda}{m_{\text{mag}}},$$

where m_{mag} regulates the magnetostatic IR divergences.

For vanishing external momentum and Landau gauge, the fourth graph is given by

$$\text{graph (d)}\,|^{\text{Landau}} = \frac{ig^4T^2}{2}(f^{adf}f^{cef}+f^{aef}f^{cdf})(f^{bdg}f^{ceg}+f^{beg}f^{cdg}) \times\int\frac{d_3\mathbf{k}\,d_3\mathbf{l}}{(\mathbf{k}+\mathbf{l})^2+m_{\text{el}}^2}\frac{1}{\mathbf{k}^2}\left[\delta_{ij}-\frac{k_ik_j}{\mathbf{k}^2}\right]\frac{1}{\mathbf{l}^2}\left[\delta_{ij}-\frac{l_il_j}{\mathbf{l}^2}\right],$$

where m_{el} regulates the electrostatic IR divergences. Using the reduction $(ff+ff)(ff+ff)=3N^2\delta$, we get

$$\text{graph (d)}\,|^{\text{Landau}} = (i\delta^{ab})\frac{3N^2g^4T^2}{2}\int\frac{d_3\mathbf{k}\,d_3\mathbf{l}}{(\mathbf{k}+\mathbf{l})^2+m_{\text{el}}^2}\left[\frac{1}{\mathbf{k}^2\mathbf{l}^2}+\frac{(\mathbf{k}\cdot\mathbf{l})^2}{\mathbf{k}^4\mathbf{l}^4}\right].$$

The integrals are calculated in Appendix B, and give

$$\text{graph (d)}\,|^{\text{Landau}} = (i\delta^{ab})\left[\frac{3Ng^2T}{8\pi}\right]^2\left[\ln\frac{\Lambda}{m_{\text{el}}}+\frac{1}{6}\right].$$

The UV divergences cancel between (c) and (d), leaving for the EQCD$_3$ bare parameter m_0^2 the expression

$$m_0^2 = m_E^2[1+O(g^2)] - 2Ng^2T\left[1+\frac{5N^3-(2N^2-3)N_f}{N^2}\frac{g^2}{24\pi^2}\right]\int^{\Lambda}\frac{d_3\mathbf{k}}{\mathbf{k}^2},$$

which is just the one-loop expression with $O(g^2)$ corrections to its coefficients. The finite correction to m_E^2 has

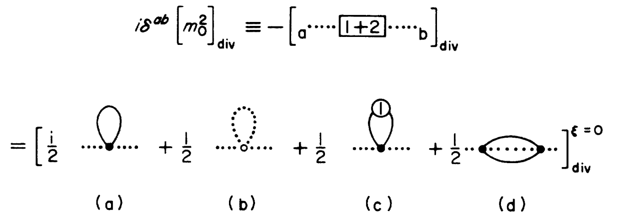

FIG. 3. Divergent part of the induced electrostatic bare mass parameter m_0^2, reexpressed in terms of EQCD$_3$ graphs in Landau gauge, showing (a) $O(g^2)$ and (b),(c),(d) $O(g^4)$ contributions.

not been evaluated explicitly; to do so would require an evaluation of the full two-loop nonstatic contribution to Π_{44}, a formidable task. Fortunately, an explicit value is not required for present or foreseeable purposes.

IV. CONCLUSION

We have completed the specification of the effective theory $EQCD_3$, under the dimensional reduction program outlined in Part I. For distance scales $O(1/gT)$, the effective action S has the form given in Sec. I, with the parameters κ and m_0^2 as given at the end of Secs. II and III, respectively. $EQCD_3$, which is only valid at very high temperatures ($T \gg \Lambda_{QCD}$), forms a convenient starting point for nonperturbative studies of cooperative phenomena in hot QCD. The results of such studies should serve as useful guidelines for the more difficult task of analyzing the full 4D theory at the lower temperatures ($T \sim \Lambda_{QCD}$) relevant to realistic quark-gluon plasmas.

Dimensional reduction in hot QCD is made possible essentially by the infrared suppression of certain nonstatic contributions. By consistently discarding all nonstatic effects at or weaker than the classical suppression level of $O(\mathbf{k}^2/T^2)$, one arrives at the effective theory $EQCD_3$. Clearly, the validity of the dimensional reduction strategy rests on the field-theoretical consistency of $EQCD_3$. It is important for $EQCD_3$ to be well defined as it stands, without requiring, for example, additional terms to achieve stability of its vacuum. Induced couplings higher than the fourth order are suppressed at levels weaker than the classical level, and if it turned out that they were needed to provide stability, there would no longer be any justification for the discarding of any other nonleading nonstatic effects and one would be back to the original 4D theory. The sign of the quartic coupling κ might be expected to play an important role in this regard. The effective potential for ϕ with a positive κ ensures a stable vacuum at the classical level. Only a nonperturbative analysis can determine if this continues to hold at the quantum level. For the gauge group SU(2), such an analysis has already been carried out in Ref. 7, where the results of this paper were first reported. It has confirmed that the positivity of κ is a sufficient condition for the consistency of the effective theory.

One of the major problems of high-temperature QCD has been to understand the physical meaning of the IR divergences that plague perturbative computations. If they cure themselves by generating a magnetic screening mass, how does this mass manifest itself in terms of the gauge-dependent gluon propagator? In Part I it was suggested that a nonperturbative study of $EQCD_3$ could provide some answers and that has indeed turned out to be the case.[7] In the process, a shadow has been cast on the old electrostatic-decoupling scenario[2,4] which posits that at even larger distance scales, $O(1/g^2T)$, the electrostatic modes described by ϕ should also decouple from the effective action, leaving behind only the magnetostatic sector, described by pure 3D Yang-Mills theory, QCD_3. This scenario is based on the belief that at such scales ϕ behaves like a heavy particle, with a Debye screening mass $O(gT) \gg O(g^2T)$. However, this result is true only in lowest-order perturbation theory and may not be stable against higher-order corrections. In fact, attempts to compute the mass gauge invariantly revealed IR-divergent corrections.[5]

It was speculated[5] that the perturbative incalculability of quantities such as the Debye mass could be due to a gauge-symmetry-breaking condensation of the field ϕ. This would cure all IR divergences by giving screening masses to the magnetostatic gluons, rearranging the large-scale degrees of freedom in the quark-gluon plasma, and thereby drastically altering our conventional picture of it. This instability of the perturbative vacuum has been confirmed for SU(2) by nonperturbative analysis of $EQCD_3$ (Ref. 7); a similar picture has emerged from numerical simulations of 4D SU(3) (Ref. 10). The old scenario of dimensional reduction followed by perturbative electrostatic decoupling must therefore be updated to the new scenario of dimensional reduction combined with a nonperturbative gauge-symmetry-breaking Higgs mechanism.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation under Grants Nos. NSF-PHY-8415534 and NSF-RII-8610671 and by the Department of Energy under Grant No. DE-FG05-84ER40154.

APPENDIX A: NONSTATIC INTEGRALS

We evaluate the nonstatic integrals required in Sec. II; the relevant notation and formalism can be found in the Appendixes of Part I. Remarkably, bosonic and fermionic contributions to κ both contain the same combination of integrals:

$$I_{\rm NS} \equiv \left[\frac{1}{q^4} - 8\frac{q_4^2}{q^6} + 8\frac{q_4^4}{q^8} \right]'_{T\mu} ,$$

where the prime indicates that in the bosonic case the zero mode is to be omitted from the sum over Matsubara frequencies (this only affects the first term). The integrals within $I_{\rm NS}$ are linear combinations of the following equivalent set, which we find more convenient to work with:

$$\left[\frac{1}{q^4} \right]'_{T\mu}, \quad \left[\frac{\mathbf{q}^2}{q^6} \right]'_{T\mu}, \quad \left[\frac{\mathbf{q}^4}{q^8} \right]'_{T\mu} .$$

Neglecting for the moment the zero mode subtraction (which will be incorporated later), these can be split up into the usual vacuum and matter parts, which we shall compute separately:

$$[f(q)]_{T\mu} \equiv T \sum_n \int d_3\mathbf{q}\, f(\mathbf{q}, \omega_n - i\mu) = \int d_4 q\, f(q) + \Delta_{T\mu} f(q) .$$

1. *Vacuum parts*

The logarithmically divergent vacuum parts are evaluated by regulating the three-momentum integrations by an IR cutoff $T\epsilon$ (for later convenience, ϵ has been scaled to be dimensionless) and a UV cutoff Λ. The former will cancel out when the corresponding matter contribution is added, reflecting the IR finiteness of individual nonstatic integrals; the latter will cancel out in the combination of integrals occurring in $I_{\rm NS}$, reflecting the UV finiteness of κ. We calculate as follows:

$$\int \frac{d_4q}{q^4}=\int \frac{d_3\mathbf{q}}{2\pi}\int_{-\infty}^{\infty}\frac{dq_4}{(q_4^2+\mathbf{q}^2)^2}=\frac{1}{8\pi^2}\int_0^\infty \frac{dq}{q}\to\frac{1}{8\pi^2}\int_{T\epsilon}^{\Lambda}\frac{dq}{q}=\frac{1}{8\pi^2}\ln\frac{\Lambda}{T\epsilon}\ ,$$

$$\int \frac{d_4q\,\mathbf{q}^2}{q^6}=\int \frac{d_3\mathbf{q}\,\mathbf{q}^2}{2\pi}\int_{-\infty}^{\infty}\frac{dq_4}{(q_4^2+\mathbf{q}^2)^3}=\frac{3}{32\pi^2}\int_0^\infty \frac{dq}{q}\to\frac{3}{32\pi^2}\int_{T\epsilon}^{\Lambda}\frac{dq}{q}=\frac{3}{32\pi^2}\ln\frac{\Lambda}{T\epsilon}\ ,$$

$$\int \frac{d_4q\,\mathbf{q}^4}{q^8}=\int \frac{d_3\mathbf{q}\,\mathbf{q}^4}{2\pi}\int_{-\infty}^{\infty}\frac{dq_4}{(q_4^2+\mathbf{q}^2)^4}=\frac{5}{64\pi^2}\int_0^\infty \frac{dq}{q}\to\frac{5}{64\pi^2}\int_{T\epsilon}^{\Lambda}\frac{dq}{q}=\frac{5}{64\pi^2}\ln\frac{\Lambda}{T\epsilon}\ .$$

Here, as elsewhere, we use q to denote both the four-vector $(\mathbf{q},q_4)$ and the magnitude of the three-vector $|\mathbf{q}|$; which one is meant is contextually clear.

2. *Matter parts*

To evaluate the matter parts we use the contour-integral representation of frequency sums.[4] Only the IR cutoff $T\epsilon$ is needed here since all matter parts are UV finite. The prototype matter part

$$\Delta_{T\mu}\left[\frac{1}{q^2}\right]=\int\frac{d_3\mathbf{q}}{2\pi i}\left[\oint_{\Gamma_\mu}\frac{d\alpha}{\mathbf{q}^2-\alpha^2}+\oint_{\Gamma}\frac{d\alpha}{1\mp e^{\alpha/T}}\left[\frac{1}{\mathbf{q}^2-(\alpha+\mu)^2}+\frac{1}{\mathbf{q}^2-(\alpha-\mu)^2}\right]\right]$$

has been evaluated in Part I. The integrals needed presently can be expressed in terms of the derivatives of its integrand, which is readily evaluated to be

$$\frac{1}{2\pi i}\left[\oint_{\Gamma_\mu}\frac{d\alpha}{\mathbf{q}^2-\alpha^2}+\oint_{\Gamma}\frac{d\alpha}{1+\sigma e^{\alpha/T}}\left[\frac{1}{\mathbf{q}^2-(\alpha+\mu)^2}+\frac{1}{\mathbf{q}^2-\alpha-\mu)^2}\right]\right]=-\frac{\rho_\sigma(x,\Delta)}{2q}\ .$$

Here we have defined $\sigma\equiv\mp1$ (upper sign for bosons, lower for fermions), $x\equiv q/T$, $\Delta\equiv\mu/T$, and the function

$$\rho_\sigma(x,\Delta)\equiv\frac{1}{1+\sigma e^{(x-\Delta)}}+\frac{1}{1+\sigma e^{(x+\Delta)}}\ .$$

We express the matter parts in terms of ρ and its x-derivatives (denoted by primes) as follows, using integration by parts to reduce the integrals to their final form:

$$\Delta_{T\mu}\left[\frac{1}{q^4}\right]=\int d_3\mathbf{q}\left[\frac{1}{4}\right]\left[\frac{d}{q\,dq}\right]\frac{\rho_\sigma(x,\Delta)}{q}$$
$$=-\frac{1}{8\pi^2}\rho_\sigma(\epsilon,\Delta)-\frac{1}{8\pi^2}\int_\epsilon^\infty\frac{dx}{x}\rho_\sigma(x,\Delta)\ ,$$

$$\Delta_{T\mu}\left[\frac{\mathbf{q}^2}{q^6}\right]=\int d_3\mathbf{q}\left[-\frac{\mathbf{q}^2}{16}\right]\left[\frac{d}{q\,dq}\right]^2\frac{\rho_\sigma(x,\Delta)}{q}$$
$$=-\frac{1}{8\pi^2}\rho_\sigma(\epsilon,\Delta)-\frac{3}{32\pi^2}\int_\epsilon^\infty\frac{dx}{x}\rho_\sigma(x,\Delta)+\frac{\epsilon\rho_\sigma'(\epsilon,\Delta)}{32\pi^2}\ ,$$

$$\Delta_{T\mu}\left[\frac{\mathbf{q}^4}{q^8}\right]=\int d_3\mathbf{q}\left[\frac{\mathbf{q}^4}{96}\right]\left[\frac{d}{q\,dq}\right]^3\frac{\rho_\sigma(x,\Delta)}{q}$$
$$=-\frac{23}{192\pi^2}\rho_\sigma(\epsilon,\Delta)-\frac{5}{64\pi^2}\int_\epsilon^\infty\frac{dx}{x}\rho_\sigma(x,\Delta)+\frac{\epsilon\rho_\sigma'(\epsilon,\Delta)}{24\pi^2}-\frac{\epsilon^2\rho_\sigma''(\epsilon,\Delta)}{192\pi^2}\ .$$

3. *The total contribution*

By adding the corresponding vacuum and matter contributions, and then taking the appropriate linear combinations, we arrive at the following expressions for the integrals within $I_{\rm NS}$:

$$\left[\frac{1}{q^4}\right]'_{T\mu}=\frac{R_\sigma(\Lambda/T,\Delta)}{8\pi^2}-\frac{\rho_\sigma(\epsilon,\Delta)}{4\pi^2}+\frac{\sigma-1}{2}T\left[\int\frac{d_3\mathbf{q}}{q^4}\right]_{n=0}\ ,$$

$$\left[\frac{q_4^2}{q^6}\right]_{T\mu}=\frac{R_\sigma(\Lambda/T,\Delta)}{32\pi^2}-\frac{\rho_\sigma(\epsilon,\Delta)+\epsilon\rho_\sigma'(\epsilon,\Delta)}{32\pi^2}\ ,$$

$$\left[\frac{q_4^4}{q^8}\right]_{T\mu}=\frac{R_\sigma(\Lambda/T,\Delta)}{64\pi^2}-\frac{2\rho_\sigma(\epsilon,\Delta)+4\epsilon\rho_\sigma'(\epsilon,\Delta)+\epsilon^2\rho_\sigma''(\epsilon,\Delta)}{192\pi^2}\ ,$$

where we have defined the IR-finite function

$$R_\sigma(\Lambda/T,\Delta)\equiv\ln\left[\frac{\Lambda}{T}\right]+\left[\rho_\sigma(\epsilon,\Delta)-\ln\epsilon-\int_\epsilon^\infty\frac{dx}{x}\rho_\sigma(x,\Delta)\right]_{\epsilon\to0}\ .$$

The infrared limit of ρ *and its derivatives*: We need to calculate the limits of the function $\rho_\sigma(x,\Delta)$ and its single and double derivatives with respect to x, at $x=\epsilon \ll 1$. The calculation is straightforward but algebraically tedious, so we only give the final results:

$$\rho_\sigma(\epsilon,\Delta)=\begin{cases}(\sigma-1)/\epsilon+1+O(\epsilon) & (\Delta=0)\ ,\\ 1+O(\epsilon) & (\Delta\neq 0)\ ,\end{cases}$$

$$\epsilon\rho'_\sigma(\epsilon,\Delta)=\begin{cases}(\sigma-1)^2/2\epsilon+O(\epsilon) & (\Delta=0)\ ,\\ O(\epsilon) & (\Delta\neq 0)\ ,\end{cases}$$

$$\epsilon^2\rho''_\sigma(\epsilon,\Delta)=\begin{cases}(\sigma-1)^3/2\epsilon+O(\epsilon) & (\Delta=0)\ ,\\ O(\epsilon^3) & (\Delta\neq 0)\ .\end{cases}$$

The zero-mode contribution. For the bosonic integral $[1/q^4]'_{T\mu}$ we need to subtract the following zero-mode contribution:

$$T\left[\int\frac{d_3\mathbf{q}}{q^4}\right]_{n=0}=\frac{1}{2\pi^2}\int_\epsilon^\infty\frac{x^2dx}{(x^2-\Delta^2)^2}=\begin{cases}1/2\pi^2\epsilon & (\Delta=0)\ ,\\ O(\epsilon^3) & (\Delta\neq 0)\ .\end{cases}$$

On substituting these values into the expressions for the individual nonstatic integrals, we get

$$\left[\frac{1}{q^4}\right]_{T\mu}=\frac{R_\sigma(\Lambda/T,\Delta)}{8\pi^2}-\frac{1}{4\pi^2}\ ,$$

$$\left[\frac{q_4^2}{q^6}\right]_{T\mu}=\frac{R_\sigma(\Lambda/T,\Delta)}{32\pi^2}-\frac{1}{32\pi^2}\ ,$$

$$\left[\frac{q_4^4}{q^8}\right]_{T\mu}=\frac{R_\sigma(\Lambda/T,\Delta)}{64\pi^2}-\frac{1}{96\pi^2}\ ,$$

where all dependence on σ, Δ, and Λ is contained in the function $R_\sigma(\Lambda/T,\Delta)$. This function need not be calculated, since it cancels out in the combination of integrals which enter $I_{\rm NS}$. Interestingly enough, $I_{\rm NS}$ is not only independent of T and μ but is the same for bosons and fermions:

$$I_{\rm NS}=-\frac{1}{12\pi^2}\ .$$

APPENDIX B: STATIC INTEGRALS

We evaluate the static integrals required in Sec. III; the relevant notation and formalism can be found in the Appendixes of Part I. The integrals occur in the combination

$$I_S\equiv\int\frac{d_3\mathbf{k}\,d_3\mathbf{l}}{(\mathbf{k}+\mathbf{l})^2+m_{\rm el}^2}\left[\frac{1}{\mathbf{k}^2\mathbf{l}^2}+\frac{(\mathbf{k}\cdot\mathbf{l})^2}{\mathbf{k}^4\mathbf{l}^4}\right]\equiv I_1+I_2\ .$$

The calculation of I_1 is straightforward. We have, on shifting the momentum $\mathbf{k}\to-(\mathbf{k}+\mathbf{l})$,

$$I_1\equiv\int\frac{d_3\mathbf{k}\,d_3\mathbf{l}}{[(\mathbf{k}+\mathbf{l})^2+m_{\rm el}^2]\mathbf{k}^2\mathbf{l}^2}\to\int\frac{d_3\mathbf{k}}{\mathbf{k}^2+m_{\rm el}^2}\int\frac{d_3\mathbf{l}}{(\mathbf{k}+\mathbf{l})^2\mathbf{l}^2}$$
$$=\int\frac{d_3\mathbf{k}}{(\mathbf{k}^2+m_{\rm el}^2)}\frac{1}{8|\mathbf{k}|}$$
$$\to\frac{1}{16\pi^2}\int_0^\Lambda\frac{k\,dk}{k^2+m_{\rm el}^2}\ ,$$

where we have introduced the UV regulator Λ. The integral is elementary; for $\Lambda \gg m_{\rm el}$ it reduces to

$$I_1=\frac{1}{16\pi^2}\ln\frac{\Lambda}{m_{\rm el}}\ .$$

The calculation of I_2 is somewhat trickier. We first express the factor $(\mathbf{k}\cdot\mathbf{l})^2$ in terms of the factors in the denominator, then use momentum shifts and symmetric integration to simplify the result. Omitting some tedious algebra and using the formula

$$\int\frac{d_3\mathbf{l}}{[(\mathbf{l}+\mathbf{k})^2+m_{\rm el}^2]\mathbf{l}^2}=\frac{1}{4\pi|\mathbf{k}|}\arctan\frac{|\mathbf{k}|}{m_{\rm el}}\ ,$$

we arrive at the form

$$I_2\equiv\int\frac{d_3\mathbf{k}\,d_3\mathbf{l}(\mathbf{k}\cdot\mathbf{l})^2}{[(\mathbf{k}+\mathbf{l})^2+m_{\rm el}^2]\mathbf{k}^4\mathbf{l}^4}$$
$$=\frac{I_1}{2}-\frac{m_{\rm el}}{4\pi}\int\frac{d_3\mathbf{k}}{\mathbf{k}^4}\left[1-\frac{m_{\rm el}}{|\mathbf{k}|}\arctan\frac{|\mathbf{k}|}{m_{\rm el}}\right]$$
$$+\frac{m_{\rm el}}{8\pi}\int\frac{d_3\mathbf{k}}{(\mathbf{k}^2+\epsilon^2)^2}-\frac{m_{\rm el}^2}{4}\left[\int\frac{d_3\mathbf{k}}{(\mathbf{k}^2+\epsilon^2)^2}\right]^2$$
$$+\frac{m_{\rm el}^4}{4}\int\frac{d_3\mathbf{k}}{(\mathbf{k}^2+\epsilon^2)^2}\int\frac{d_3\mathbf{l}}{[(\mathbf{l}+\mathbf{k})^2+\epsilon^2]^2(\mathbf{l}^2+m_{\rm el}^2)}\ .$$

Here a magnetic IR regulator ϵ has been introduced in the last three integrals and will eventually cancel out. Using

$$\int\frac{d_3\mathbf{k}}{(\mathbf{k}^2+\epsilon^2)^2}=\frac{1}{8\pi\epsilon}\ ,$$

$$\int\frac{d_3\mathbf{l}}{[(\mathbf{l}+\mathbf{k})^2+\epsilon^2]^2(\mathbf{l}^2+m_{\rm el}^2)}=\frac{1}{8\pi\epsilon[\mathbf{k}^2+(m_{\rm el}+\epsilon)^2]}\ ,$$

$$\int\frac{d_3\mathbf{k}}{(\mathbf{k}^2+\epsilon^2)^2[\mathbf{k}^2+(m_{\rm el}+\epsilon)^2]}=\frac{1}{8\pi\epsilon(m_{\rm el}+2\epsilon)^2}\ ,$$

and

$$\int_0^\infty\frac{dx}{x^2}\left[1-\frac{1}{x}\arctan x\right]=\frac{\pi}{4}\ ,$$

we secure the IR finite result

$$I_2=\frac{I_1}{2}-\frac{1}{32\pi^2}+\frac{3}{64\pi^2}=\frac{I_1}{2}+\frac{1}{64\pi^2}\ .$$

Adding the results for I_1 and i_2, we finally obtain

$$I_S=\frac{3}{32\pi^2}\left[\ln\frac{\Lambda}{m_{\rm el}}+\frac{1}{6}\right]\ .$$

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