

## Skyrme topological soliton coupled to gravity

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Skyrmions of large topological quantum number  $N$  or large energy scale  $f_\pi$  are studied in general relativity. No stable or metastable solutions are found between the nucleon with  $N=1$  and mini black holes with  $N \sim M_{\text{Planck}}/f_\pi$ . On the nuclear scale ( $f_\pi \sim 90$  MeV) this corresponds to  $N \sim 10^{19}$ .

### I. INTRODUCTION

In a recent series of papers, Friedberg, Lee, and Pang<sup>1</sup> carried out an investigation of nontopological soliton stars and black holes. It was shown that, corresponding to a simple Lagrangian for a scalar field together with a meson or fermion field, a variety of interesting structures might exist, ranging from mini to massive soliton stars and black holes. Inspired by these results we investigate the Skyrme topological soliton for large topological quantum number or large-energy scale in general relativity. The Skyrme soliton<sup>2</sup> has attracted much interest in the last several years because of developments in QCD concerning the large- $N_c$  limit and because its properties resemble those of nucleons at the 30–40% level.<sup>3,4</sup> Why might such a soliton possess interesting solutions when coupled to gravity? Consider the expression for the soliton mass without gravity:

$$M = 4\pi \int_0^\infty dr r^2 \left[ \frac{f_\pi^2}{2} \left( F'^2 + 2 \frac{\sin^2 F}{r^2} \right) + \frac{1}{2e^2} \frac{\sin^2 F}{r^2} \left( 2F'^2 + \frac{\sin^2 F}{r^2} \right) \right], \quad (1)$$

where the chiral function  $F(r)$  obeys the boundary conditions

$$F(0) = N\pi, \quad F(\infty) = 0, \quad (2)$$

with the nucleon corresponding to  $N=1$ . As a rough estimate for large  $N$  we take

$$F(r) = N\pi \left[ 1 - \frac{r}{R} \right], \quad r < R, \quad (3)$$

$$F(r) = 0, \quad r > R,$$

and replace the rapidly oscillating  $\sin^2 F(r)$  by  $\frac{1}{2}$ . We obtain, in leading order,

$$M(N) = 2\pi^3 N^2 \left[ \frac{f_\pi^2 R}{3} + \frac{1}{e^2 R} \right] = \frac{4\pi^3 f_\pi}{\sqrt{3} e} N^2. \quad (4)$$

The second equality follows for the radius that minimizes this mass:

$$R = \frac{\sqrt{3}}{ef_\pi}. \quad (5)$$

Thus we see that  $M$  varies as  $N^2$ , as if there were a pairwise interaction of  $N$  particles. As could be expected for strongly interacting particles that all experience each others force, the radius is independent of  $N$  to leading order. Therefore, for large enough  $N$ , gravity must become important, and ultimately the Schwarzschild radius  $R_S$  will exceed  $R$ , for some critical  $N_S$ . The soliton will then become a black hole. In this paper we investigate the possibility of there being a finite range of  $N$  below the critical  $N_S$  for which gravity is strong enough to stabilize the soliton against single-particle ( $N=1$ ) decay. The scale at which gravity becomes strong is

$$\frac{R_S}{R} \equiv \frac{2GM(N)}{R} = \frac{8\pi^3}{3} G(Nf_\pi)^2 \sim 1, \quad (6)$$

or roughly for  $Nf_\pi \sim M_{\text{Planck}}$  (where  $M_{\text{Planck}} = G^{-1/2}$ ). Note that because both mass and size depend on the inverse of  $e$ , this parameter drops out of the scale, Eq. (6).

For the purpose of investigating the above possibility, both at the nucleon scale ( $f_\pi \sim 90$  MeV) or at some other scale, for example, the strongly interacting Higgs sector of the electroweak interaction ( $\sim$  TeV), we develop the coupled equations for the matter and gravitational fields, and discuss the properties of an approximate solution.

### II. FIELD EQUATIONS

For the gravitational fields the action density is (cf. Ref. 5)

$$\mathcal{L}_g = -\frac{1}{16\pi G} R \sqrt{-g}, \quad (7)$$

where  $G$  is Newton's constant,  $R$  is the Ricci scalar curvature, and  $g$  is the determinant of the metric  $g_{\mu\nu}$ . We also define  $\mathcal{L}_m = L_m \sqrt{-g}$  for the Lagrangian  $L_m$  of the matter field  $F$ , and construct the total action

$$I = \int (\mathcal{L}_g + \mathcal{L}_m) d^4x. \quad (8)$$

The coupled field equations for the matter and metric functions emerge as the conditions that yield vanishing variation of the action with respect to the metric and matter fields. They can be written as

$$\frac{\delta \mathcal{L}_m}{\delta F} - \partial_\mu \frac{\delta \mathcal{L}_m}{\delta (\partial_\mu F)} = 0, \quad (9)$$

$$G^{\mu\nu} = -8\pi G T^{\mu\nu}, \quad (10)$$

where  $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$  is Einstein's curvature tensor and  $T^{\mu\nu}$  is the matter stress-energy tensor:

$$T^{\mu\nu} \equiv -g^{\mu\nu}L_m + 2 \frac{\partial L_m}{\partial g_{\mu\nu}}. \quad (11)$$

For static, spherical geometry, the metric tensor takes the Schwarzschild form

$$\begin{aligned} g_{00} &= e^{2\nu(r)}, \quad g_{11} = -e^{2\lambda(r)}, \quad g_{22} = -r^2, \\ g_{33} &= -r^2 \sin^2\theta, \quad g_{\mu 0} = 0 \quad (\mu \neq 0), \\ g_{00}g^{00} &= 1, \text{ etc.}, \quad \sqrt{-g} = e^{\nu+\lambda} r^2 \sin\theta, \end{aligned} \quad (12)$$

and the line element is

$$ds^2 = e^{2\nu(r)} dt^2 - e^{2\lambda(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2, \quad (13)$$

where the metric functions  $\nu$  and  $\lambda$  depend only on  $r$ . Then the matter-field equation can be written (writing now for brevity  $L = L_m$ )

$$\frac{\partial L}{\partial F} - \frac{d}{dr} \frac{\partial L}{\partial F'} - \left[ \nu' + \lambda' + \frac{2}{r} \right] \frac{\partial L}{\partial F'} = 0, \quad (14)$$

where primes denote derivatives with respect to  $r$ .

We now specialize to the Skyrme Lagrangian (cf. Ref. 4):

$$\begin{aligned} L &= L_2 + L_4, \\ L_2 &= \frac{f_\pi^2}{4} \text{tr}(\partial_\mu U \partial^\mu U^\dagger), \\ L_4 &= \frac{1}{32e^2} \text{tr}[(\partial_\mu U)U^\dagger, (\partial_\nu U)U^\dagger]^2 \\ &= \frac{1}{32e^2} \text{tr}(F_{\mu\nu} F^{\mu\nu}), \end{aligned} \quad (15)$$

where the field  $U$  is a unitary SU(2) matrix, so that

$$\partial_\mu U = -U(\partial_\mu U^\dagger)U \quad (16)$$

which is used to obtain

$$\begin{aligned} F_{\mu\nu} &\equiv [(\partial_\mu U)U^\dagger, (\partial_\nu U)U^\dagger] \\ &= \partial_\nu U \partial_\mu U^\dagger - \partial_\mu U \partial_\nu U^\dagger. \end{aligned} \quad (17)$$

The hedgehog ansatz is

$$U = \cos F(r) + i\tau \cdot \hat{r} \sin F(r), \quad (18)$$

where the components of  $\tau$  are the Pauli matrices for isospin, and  $\hat{r}$  is a unit radial vector. Because we want to study the Skyrmion in strong gravitational fields and because of the spherical symmetry of the problem we

rewrite the Lagrangian using the metric of gravitating, static, spherically symmetric geometry, i.e., the Schwarzschild metric. The derivatives of  $U$  in spherical coordinates are

$$\begin{aligned} \partial_r U &= (-\sin F + i\tau \cdot \hat{r} \cos F)F', \\ \partial_\theta U &= i\tau \cdot \hat{\theta} \sin F, \\ \partial_\phi U &= i\tau \cdot \hat{\phi} \sin F \sin\theta. \end{aligned} \quad (19)$$

This allows us to express  $L_2$ . We also need the  $F_{\mu\nu}$  in spherical coordinates to express  $L_4$ :

$$\begin{aligned} F_{r\theta} &= 2i(\tau \cdot \hat{\theta} \sin^2 F + \tau \cdot \hat{\phi} \sin F \cos F)F', \\ F_{r\phi} &= 2i(\tau \cdot \hat{\phi} \sin^2 F - \tau \cdot \hat{\theta} \sin F \cos F)F' \sin\theta, \\ F_{\theta\phi} &= 2i\tau \cdot \hat{r} \sin\theta \sin^2 F. \end{aligned} \quad (20)$$

Hence, after some algebra,

$$\begin{aligned} L_2 &= \frac{f_\pi^2}{2} (g^{11} F'^2 + g^{22} \sin^2 F + g^{33} \sin^2\theta \sin^2 F), \\ L_4 &= -\frac{1}{2e^2} [g^{11} F'^2 \sin^2 F (g^{22} + g^{33} \sin^2\theta) \\ &\quad + g^{22} g^{33} \sin^2\theta \sin^4 F]. \end{aligned} \quad (21)$$

We can now compute the four nonvanishing components of the matter stress-energy tensor,  $T_\mu{}^\nu = T_{\mu\rho} g^{\rho\nu}$ , and then insert the Schwarzschild metric to obtain Einstein's field equations, which together with the matter-field equation, are coupled differential equations in the two metric functions  $\lambda$  and  $\nu$  and the matter field  $F$ . Einstein's equations are

$$\begin{aligned} r^2 G_0^0 &\equiv e^{-2\lambda}(1 - 2r\lambda') - 1 = 8\pi G r^2 L, \\ r^2 G_1^1 &\equiv e^{-2\lambda}(1 + 2r\nu') - 1 \\ &= 8\pi G r^2 \left[ L + \left[ f_\pi^2 + \frac{2}{e^2} \frac{\sin^2 F}{r^2} \right] F'^2 e^{-2\lambda} \right], \\ G_2^2 &\equiv e^{-2\lambda} \left[ \nu'' + \nu'^2 - \lambda'\nu' + \frac{\nu' - \lambda'}{r} \right] \\ &= 8\pi G \left\{ L + \left[ f_\pi^2 + \frac{1}{e^2} \left[ e^{-2\lambda} F'^2 + \frac{\sin^2 F}{r^2} \right] \right] \frac{\sin^2 F}{r^2} \right\}, \\ G_3^3 &= G_2^2, \quad T_3^3 = T_2^2, \end{aligned} \quad (22)$$

where Eqs. (21) give

$$\begin{aligned} L &= -\frac{f_\pi^2}{2} \left[ e^{-2\lambda} F'^2 + 2 \frac{\sin^2 F}{r^2} \right] \\ &\quad - \frac{1}{2e^2} \frac{\sin^2 F}{r^2} \left[ 2e^{-2\lambda} F'^2 + \frac{\sin^2 F}{r^2} \right]. \end{aligned} \quad (23)$$

The matter-field equation, Eq. (14), can now be written

$$\left[1 + \frac{2 \sin^2 F}{x^2}\right] \dot{F} + \frac{\sin 2F}{x^2} \dot{F}^2 + \left[\frac{2}{x} + (\dot{v} - \dot{\lambda})\right] \left[1 + \frac{2 \sin^2 F}{x^2}\right] \dot{F} - \left[\frac{\sin 2F}{x^2} + \frac{\sin^2 F \sin 2F}{x^4}\right] e^{2\lambda} = 0, \quad (24)$$

$$x \equiv e f_{\pi} r, \quad \dot{F} \equiv dF/dx.$$

In summary we have to solve the coupled field equations, Eqs. (22) and (24).

### III. TOPOLOGICAL CHARGE

Define, in close analogy to Skyrme,

$$B^{\mu} = \frac{c}{\sqrt{-g}} \epsilon^{\mu\alpha\beta\gamma} S_{\alpha} S_{\beta} S_{\gamma}, \quad (25)$$

$$S_{\alpha} \equiv U^{\dagger} \partial_{\alpha} U, \quad c \equiv \frac{1}{24\pi^2}.$$

The covariant divergence is

$$B^{\mu}{}_{;\mu} \equiv \frac{1}{\sqrt{-g}} \partial_{\mu} \sqrt{-g} B^{\mu} = \frac{c}{\sqrt{-g}} \partial_{\mu} \epsilon^{\mu\alpha\beta\gamma} S_{\alpha} S_{\beta} S_{\gamma} = 0, \quad (26)$$

where  $\epsilon^{\mu\alpha\beta\gamma}$  is the (constant) fully antisymmetric tensor. The second line vanishes for the same reason that the divergence of the Skyrme current vanishes (i.e., it vanishes as an algebraic identity and not as a Noether current arising from an invariance of the Lagrangian). Corresponding to the vanishing of the covariant divergence, the conserved charge is

$$Q = \int \sqrt{-g} B^0 d^3x = c \epsilon^{0\alpha\beta\gamma} \int S_{\alpha} S_{\beta} S_{\gamma} d^3x. \quad (27)$$

For the hedgehog ansatz, Eq. (18), this gives  $Q = N$  for boundary condition Eq. (2).

### IV. APPROXIMATE SOLUTION

The first of Einstein's equations can be written

$$-\frac{d}{dr} [r(1 - e^{-2\lambda(r)})] = 8\pi G r^2 L = -8\pi G r^2 T_0^0 \quad (28)$$

we can show, after some algebra, and using the second of Einstein's equations in Eq. (22), that the vanishing of the functional variation of  $M$  with respect to  $F(r)$ , implies that the matter-field equation, Eq. (24), is satisfied. We shall use this principle to get an approximation to  $M$ .

According to Eq. (6), gravitational binding will be important either for large  $N$ , or large  $f_{\pi}$  or both. The analysis is dictated by the size of  $N$ . Consider first large  $N$ . In that case,

$$q(x) \ll p(x), \quad \sin^2 F \approx \frac{1}{2} \quad (\text{for large } N). \quad (35)$$

Then

with  $L$  given by Eq. (23). Because the metric function is contained in  $L$ , this cannot be used to calculate  $\lambda$ , but its "solution"

$$e^{-2\lambda(r)} = 1 - \frac{8\pi G}{r} \int_0^r T_0^0 r^2 dr \quad (29)$$

suggests the identification of the mass:

$$M(r) = 4\pi \int_0^r T_0^0 r^2 dr. \quad (30)$$

For weak gravitational fields, this with Eq. (23) yields Eq. (1). While Eq. (30) does not provide a means of calculating  $M$ , it shows that it requires only a partial solution to the metric to calculate it: namely,  $\lambda(r)$ ,

$$M(r) = \frac{r}{2G} (1 - e^{-2\lambda(r)}). \quad (31)$$

Keeping in mind the scale, Eq. (6), at which we want to solve this problem, we introduce, in addition to the dimensionless length  $x$  of Eq. (24), the dimensionless mass

$$\mu(x) \equiv e f_{\pi} G M(x). \quad (32)$$

Then we can write Eq. (28) as

$$\frac{d\mu(x)}{dx} = \left[1 - \frac{2\mu(x)}{x}\right] p(x) + q(x),$$

$$p(x) \equiv \frac{3}{2} \frac{R_S}{R} \left[\frac{x^2}{2} + \sin^2 F\right] \left[\frac{\dot{F}}{N\pi}\right]^2, \quad (33)$$

$$q(x) \equiv \frac{3}{2\pi^2} \frac{1}{N^2} \frac{R_S}{R} \left[x^2 + \frac{\sin^2 F}{2}\right] \frac{\sin^2 F}{x^2}.$$

In order of magnitude,  $\dot{F}/N$  is independent of  $N$  [see Eq. (2)]. So far we have just rewritten Einstein's first equation in terms of the mass. However, by writing

$$\mu(x) = \exp\left[-2 \int^x p dx/x\right] \int_0^x (p+q) \exp\left[2 \int^x p dx''/x''\right] dx' \quad (34)$$

$$\frac{d\mu(x)}{dx} = \left[1 - \frac{2\mu(x)}{x}\right] \bar{p}(x),$$

$$\bar{p}(x) \equiv \frac{3}{4} \frac{R_S}{R} (1+x^2) \left[\frac{\dot{F}}{N\pi}\right]^2. \quad (36)$$

We see that in the large- $N$  limit, the problem depends only on  $R_S/R \propto G(f_{\pi} N)^2$ , and not on  $e$ , or separately on  $f_{\pi}$  or  $N$  provided that  $F(x)$  scales with  $N$ . Therefore it can be solved universally for all energy scales so long as Eq. (35) is satisfied. We solve this equation using for  $F(x)$  an ansatz whose parameters we vary to get a minimum

*M*. Numerical integration of Eq. (24) in the absence of gravity shows that  $F(r)$  is a concave function,<sup>4</sup> and we have verified for  $N \sim 10$  that it approximately scales with  $N$ . Therefore two improved forms for the ansatz over Eq. (3) are

$$F(x) = N\pi \left(1 - \frac{x}{X}\right)^\alpha, \quad X \equiv ef_\pi R, \quad (37)$$

or

$$F(x) = N\pi e^{-x/X} \quad (38)$$

and we vary  $X$  and  $\alpha$ . The second proves to yield a slightly lower mass.

The dependence of the mass on the size is shown in Fig. 1 for several values of the parameter  $R_S/R$  in the vicinity where radial stability is lost to collapse. To interpret this on the nucleon scale we adopt constants for the Lagrangian that yield a nominal ‘‘nucleon’’ of mass  $M(1) = 1$  GeV and radius  $R = 1$  fm. These are  $f_\pi = 69.1$  MeV and  $e = 4.95$ , according to Eqs. (4) and (5). Then loss of radial stability occurs at a little over  $N = 10^{19}$  (where ‘‘little’’ is measured in  $10^{18}$  units of baryon charge). In contrast, neutron stars loose stability at a few times  $10^{57}$  baryons. Also shown on this figure is the limit  $R = 2GM$ , the Schwarzschild radius of the mass  $M$ . We note that instability already occurs at  $R_S/R \approx 0.26$ .

In Fig. 2 we show the mass as a function of  $R_S/R$ , compared to the linear behavior in the absence of gravity. Gravity provides a moderate binding of roughly 15% of the mass in its absence (comparable to neutron stars). However, the condition for stability against single-particle emission is

$$\frac{dM(N)}{dN} < M(1) \quad (39)$$

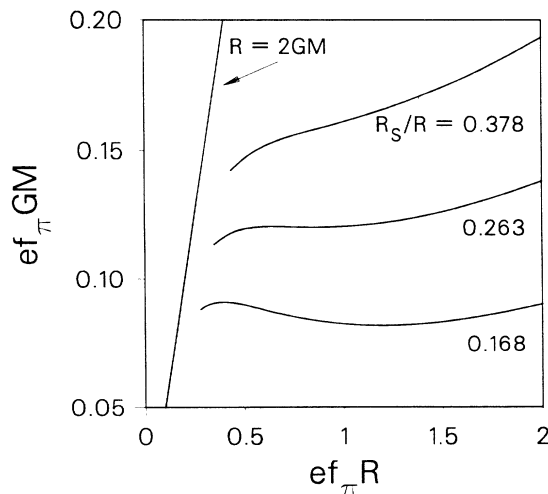


FIG. 1. Skyrmion mass as a function of variational radius parameter, for several values of  $R_S/R \propto G(f_\pi N)^2$  in the vicinity of loss of stability to gravitational collapse. At the nuclear scale ( $f_\pi = 69.1$  MeV), the three curves correspond to  $N/10^{19} = 0.8, 1.0, 1.2$ , in order of increasing  $R_S/R$ . The Schwarzschild relation is also plotted.

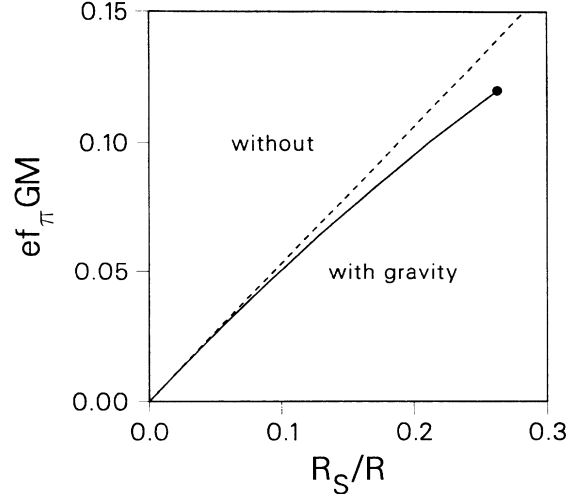


FIG. 2. Skyrmion mass as a function of  $R_S/R$  with and without gravity. The dot marks the end of radial stability against collapse (see Fig. 1).

or in terms of the dimensionless analysis in which the elementary mass is

$$\mu(1) = \frac{\sqrt{3}}{2N^2} \frac{R_S}{R}, \quad (40)$$

we have

$$\frac{d\mu}{d(R_S/R)} < \frac{\sqrt{3}}{4} \frac{1}{N}. \quad (41)$$

This condition is not satisfied for large  $N$  even up to the point of loss of stability against collapse to a black hole where from the figure we estimate  $d\mu/d(R_S/R) \approx 0.38$ . [Our conclusion on the instability to single-particle decay is not absolutely rigorous, since we obtain the mass by a variational principle. However the condition expressed in Eq. (39) is so far from being satisfied that we believe the opposite inequality holds (by many orders of magnitude at the nucleon scale).] It is satisfied nominally for  $N = 1$ , which does not satisfy Eq. (35). We turn now to the domain of small  $N$ .

TABLE I. Soliton mass as a function of  $N$  for various critical  $N = N_S$  at which  $R_S/R = 1$ . No entry for  $\mu$  means radial instability.

$N_S$	$N$	$\mu(N)$	$R_S/R$
3	1	0.0575	0.11
	2		0.44
4	1	0.0322	0.06
	2	0.1179	0.25
	3		0.56
5	1	0.0215	0.04
	2	0.0804	0.16
	3		0.36

We can systematically search for particle stable solutions of Eqs. (33) in the small  $N$  domain by a series of calculations in which the critical  $N=N_S$  for which  $R_S/R=1$  is varied from 3,4, . . . , in each case calculating the mass for  $N=1,2, \dots, N_S$ , and testing whether Eq. (39) is ever satisfied. Again we find that radial stability is lost at  $R_S/R \approx 0.25$ , before gravitational binding is very strong. Consequently, no stability with respect to particle emission exists. The results are summarized in Table I.

## V. SUMMARY

We have investigated Skyrmions in general relativity at all energy scales for the purpose of learning whether stable or metastable stars consisting of a single topological soliton might exist. The size of Skyrmions is independent of the topological quantum number  $N$ , so that they must become black holes for sufficiently large  $N$ . Below the critical point, gravity contributes about 15% binding. However, loss of radial stability against collapse occurs before the binding is great enough to provide single-particle stability. On the nucleon scale where  $f_\pi \approx 70$  MeV, the critical topological number is  $N_c \approx 10^{19}$  in contrast with neutron stars where the critical baryon number is  $\approx 10^{57}$ . The critical mass on the nucleon scale is  $M_c \approx 0.5 \times 10^{-20} M_\odot$ . We do not find stable or metasta-

ble Skyrmions between  $N=1$  and the black hole, at any energy scale  $f_\pi$ . Such mini black holes could have been created at the beginning of the Universe. It is unlikely that conditions favorable for their creation would occur at later times.

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