Quantum theory of nonlocal vortex fields

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Nonlocal vortex field operators are introduced in continuum (2+1)-dimensional quantum field theory, through an order-disorder algebra. A general method for the evaluation of vortex correlation functions is established and an explicit operator realization of the vortex field is obtained.

An important problem in quantum field theory (QFT) is the full description of the quantum theory (mass spectrum, correlation functions, scattering amplitudes, . . .) of the topological excitations of a given system.[1] In the case of sine-Gordon solitons in (1+1)-dimensional space-time, this problem was solved in the 1970s (Refs. 2 and 3). More recently, we considered the kinks of ϕ^4-type theories, also in 1+1 dimensions, and established a general method of quantization, based on the statistical mechanical concept of order-disorder duality (this method also applied to sine-Gordon solitons) (Ref. 4). In this framework, the topological excitation creation operator emerged as the disorder variable, defined so as to satisfy a certain dual algebra with the basic fields of the theory.[4,5] This dual algebra in 1+1 dimensions was such that the commutation of the disorder (kink) field $\mu(x)$ with the basic Lagrangian field $\phi(y)$ applied to the latter the large-distance behavior of the classical kink: namely,[4]

$$\mu(x,t)\phi(y,t)=e^{i\pi\theta(x-y)}\phi(y,t)\mu(x,t) \ . \tag{1}$$

An order-disorder algebra of this kind was first introduced in Ref. 6, in the context of the Ising model. In that work, a method is provided for the computation of correlation functions involving μ. Our method[4] is a generalization of their procedure for continuum QFT.

In this work, we consider the vortices of the Abelian Higgs model in 2+1 dimensions. In what follows, we establish a general method for the computation of arbitrary vortex correlation functions and obtain an explicit operator realization for the vortex field. The method is a generalization of the one of Ref. 4 and is, again, based on a dual algebra similar to (1). Some related ideas on the lattice framework are described in Ref. 7.

The theory is defined by the Lagrangian density

$$\mathcal{L}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+(D_\mu\phi)^*(D^\mu\phi)+m^2\phi^*\phi-\frac{\lambda}{4}(\phi^*\phi)^2 \ , \tag{2}$$

where $D_\mu=\partial_\mu+ieA_\mu$.

In 1973, Nielsen and Olesen observed[8] that this model admitted classical solutions with the long-distance behavior

$$\phi(\mathbf{x},t)\underset{|\mathbf{x}|\to\infty}{\sim}\rho_0 e^{i\arg(\mathbf{x})}, \quad A_i(\mathbf{x},t)\underset{|\mathbf{x}|\to\infty}{\sim}-\frac{1}{e}\partial_i\arg(\mathbf{x}) \ , \tag{3}$$

where $\arg(\mathbf{x})$ is the angle of the vector $\mathbf{x}$ with respect to some arbitrary axis in the (x^1,x^2) plane (e.g., the x^1 axis). Such a solution possesses a nonzero value for the topological charge associated with the identically conserved current $j^\mu=\frac{1}{2}\epsilon^{\mu\alpha\beta}F_{\alpha\beta}$, which is nothing but the magnetic flux along the (x^1,x^2) plane. It was called a "vortex." We are going to introduce now a fully quantized vortex creation operator through an algebra which is a generalization of (1). An important difference from the (1+1)-dimensional case now appears. Observe that the kink operator in (1) was a local field[4] and the algebra (1) is based on the concept of y being at the right or at the left of x. Since in 2+1 dimensions these concepts no longer make sense, we arrive at the conclusion that an algebra which generalizes (1) should involve the local fields $\phi(x)$ and $\mathbf{A}(x)$ and a *nonlocal* vortex operator $\mu(c)$, defined on a certain curve C. The concept of being at the left or at the right is now exchanged by being inside or outside C. (Investigations about local vortex operators were described in Ref. 9.)

In the same spirit of the algebra (1) and keeping in mind the asymptotic behavior (3), we introduce the vortex creation operator through the equal-time commutation rules

$$\mu(\mathbf{x},t;c)\phi(\mathbf{y},t)=\begin{cases} e^{i\arg(\mathbf{y}-\mathbf{x})}\phi(\mathbf{y},t)\mu(\mathbf{x},t;c), & \mathbf{y}-\mathbf{x}\notin T(c), \\ \phi(\mathbf{y},t)\mu(\mathbf{x},t;c), & \mathbf{y}-\mathbf{x}\in T(c) \end{cases} \tag{4a}$$

and

$$\mu(\mathbf{x},t;c)A_i(\mathbf{y},t)=\begin{cases} \left[A_i(\mathbf{y},t)-\frac{1}{e}\partial_i^y\arg(\mathbf{y}-\mathbf{x})\right]\mu(\mathbf{x},t;c), & \mathbf{y}-\mathbf{x}\notin T(c), \\ A_i(\mathbf{y},t)\mu(\mathbf{x},t;c), & \mathbf{y}-\mathbf{x}\in T(c) \ . \end{cases} \tag{4b}$$

In these expressions C is a plane curve contained in the t =const plane. $T(c)$ is the minimal surface bounded by C. $\mathbf{x}$ is a point belonging to $T(c)$ and characterizes the center of the vortex, i.e., the point in relation to which the angle $\arg(\mathbf{x}-\mathbf{y})$ is defined. This angle is measured with respect to an arbitrary direction characterized by a vector $\vec{r}_0$. Without loss of generality we may choose $\vec{r}_0$ pointing in the x^1 direction. The commutation rules above may be generalized to different times, provided the separations between $(\mathbf{y},t)$ and $(\mathbf{x},t)$ and between $(\mathbf{y},t)$ and the points of C are all spacelike (otherwise the concepts of being inside or outside T would become senseless). For $t'>t$, we just change $(\mathbf{y},t)$ for $(\mathbf{y},t')$ in (4). For $t'<t$, we impose $[\mu(\mathbf{x},t;c),\phi(\mathbf{y},t')]=[\mu(\mathbf{x},t;c),A_i(\mathbf{y},t')]=0$ for causality reasons. Observe that $\partial_i \arg(\mathbf{y}-\mathbf{x})$ is the potential of a Dirac string. The above choice of commutation rules implies that the Dirac string goes from $(\mathbf{x},t)$ to $(\mathbf{x},+\infty)$ along the t axis. This condition, hence, expresses the fact that the Dirac string cannot exist before the vortex is created. One may verify that (4) is Lorentz invariant by keeping in mind that the Dirac string as well as the vector $\vec{r}_0$ should also be rotated by the Lorentz transformation.

The algebra (4) is analogous to the one introduced by 't Hooft for the Wilson loop in (2+1)-dimensional Yang-Mills theory.[10,11] The main difference is that the symmetry in our case is local.

We now generalize for the present case our prescription for the computation of kink correlation functions in 1+1 dimensions. Extending the arguments introduced in Ref. 4 (and Ref. 6), we write, in Euclidean three-dimensional (3D) space,

$$\langle \mu(x;c_1)\mu^*(y;c_2)\rangle = Z^{-1}\int D\phi\, D\phi^* D A_\mu \exp\Big[-\int d^3z \Big[\mathcal{L}+\mathcal{L}(S,T,L)+\int_{S(c_1c_2)} d^2\xi^\mu \Psi_\mu \delta^3(z-\xi) + \int_{x,L}^{y} d\eta^\mu \Phi_\mu(S)\delta^3(z-\eta) + \int_{T_1 U T_2} d^2\xi^\mu \chi_\mu \delta^3(z-\xi)\Big]\Big] . \tag{5}$$

In this expression $S(C_1,C_2)$ is an arbitrary surface such that its boundary is $\partial S=C_1 U C_2$. T_1 and T_2 are the minimal surfaces bounded, respectively, by the plane curves C_1 and C_2. $d^2\xi^\mu$ are the surface elements along S, T_1, and T_2. L is an arbitrary curve connecting x and y and $d\eta^\mu$ is the line element along it. Ψ_μ, $\Phi_\mu(s)$, and χ_μ are functionals of the fields, to be determined (observe that we allow Φ_α to depend on S). $\mathcal{L}(S,T,L)$ is a renormalization counterterm, also to be determined, introduced in order to compensate for the eventual singularities coming from the line and surface terms. Z is the vacuum functional.

As in Refs. 4 and 11 we are going to determine Ψ_μ, Φ_μ, χ_μ, and $\mathcal{L}(S,T,L)$, by imposing surface and path independence on (5). Let us take an arbitrary surface $S'(c_1,c_2)$ such that we also have $\partial S'=C_1 U C_2$ and call ΔV the volume bounded by SUS'. We will assume for simplicity that the surface S' is always exterior to S. Let us call Γ the closed surface made out of S and S' such that $\int_S=\int_{S'}-\oint_\Gamma[\partial(\Delta V)=\Gamma]$. Later on we will consider the most general case.

Let us perform now the following change in the functional integration variables inside ΔV:

$$\phi(z)\to e^{-i\omega(z)\theta(\Delta V)} , \tag{6a}$$

$$A_\mu(z)\to A_\mu(z)+\omega(z)\frac{1}{e}\partial_\mu\theta(\Delta V)+\theta(\Delta V)\frac{1}{e}\partial_\mu\omega(z) \tag{6b}$$

[$\theta(\Delta V)$ is the 3D Heaviside function with support on ΔV].

If we choose $\omega(z)=\alpha(z;x,y)\equiv[\theta(z^3-x^3)\arg(\mathbf{z}-\mathbf{x}) -\theta(z^3-y^3)\arg(\mathbf{z}-\mathbf{y})]$, corresponding to a vortex placed on $x=(\mathbf{x},x^3)$ and an antivortex placed on $y=(\mathbf{y},y^3)$, we see that the last term in (6b) corresponds to two Dirac strings going from $(\mathbf{x},x^3)$ to $(\mathbf{x},+\infty)$ and from $(\mathbf{y},+\infty)$ to $(\mathbf{y},y^3)$, respectively, along the z^3 axis. Of course, this is equivalent by a U(1) transformation to a configuration having a single Dirac string going from z to y along an arbitrary curve. This configuration may be introduced by choosing $\omega(z)=\alpha_L(z;x,y)$ in (6), where α_L is the obvious generalization of α defined by (see the Appendix)

$$F_{\mu\nu}^{\text{Dirac string}}=\frac{1}{e}\int_{x,L}^{y}\epsilon_{\mu\nu\alpha}\delta^3(z-\xi)d\xi^\alpha = \frac{1}{2\pi e}[\partial_\mu,\partial_\nu]\alpha_L(z;x,y) . \tag{7}$$

In this expression, L is an arbitrary curve connecting x and y and $d\xi^\alpha$ is its line element. The U(1) transformation which switches from one to another configuration of Dirac strings is given by (6) with $\omega=\alpha_L-\alpha$. Since L is arbitrary, we are going to choose it coinciding with the L in (5).

The only terms in (5) which are affected by the change of variable (6) (with $\omega=\alpha_L$) are $F_{\mu\nu}$ and possibly Ψ_μ, Φ_μ, and χ_μ. We assume $\mathcal{L}(S,T,C)$ to be invariant under it. An explicit computation taking in account the properties of the derivative of the Heaviside function shows that under (6) $F_{\mu\nu}\to F_{\mu\nu}+\tilde{F}_{\mu\nu}$, with

$$\tilde{F}_{\mu\nu}=-\frac{\alpha_L}{e}\Big[\partial_\mu\oint_{\Gamma=S'-S}d^2\xi_\nu\delta^3(z-\xi)-(\mu\leftrightarrow\nu)\Big] + \frac{1}{e}\theta(\Delta V)[\partial_\mu,\partial_\nu]\alpha_L . \tag{8}$$

After the transformation (6), we may write the exponent of the integrand in (5) $[E(S)]$ as

$$E(S)=\int d^3z\left[\mathcal{L}_E+\tfrac{1}{2}F_{\mu\nu}\tilde{F}_{\mu\nu}+\tfrac{1}{4}(\tilde{F}_{\mu\nu})^2+\left[\int_{S'}d^2\xi^\mu-\oint_\Gamma d^2\xi^\mu\right]\Psi_\mu\delta^3(z-\xi)\right.$$
$$+\int_S d^2\xi^\mu\delta\Psi_\mu\delta^3(z-\xi)+\int_{x,L}^{y}d\eta^\mu[\Phi_\mu(s)+\delta\Phi_\mu(s)]\delta^3(z-\eta)$$
$$\left.+\int_{T_1UT_2}d^2\xi^\mu(\chi_\mu+\delta\chi_\mu)\delta^3(z-\xi)+\mathcal{L}(S,T,L)\right]. \tag{9}$$

In this expression, $\delta\Psi_\mu$, $\delta\Phi_\mu$, and $\delta\chi_\mu$ represent the possible variations of the respective functionals under the transformation (6).

Introducing (8) in (9) and imposing the cancellation of terms containing a single closed surface integral, we get

$$\Psi^\mu=\frac{\alpha_L}{e}(z;x,y)F^{\mu\nu}\partial_\nu\ , \tag{10}$$

from which we immediately find

$$\delta\Psi^\mu=\frac{\alpha_L}{e}(z;x,y)\tilde{F}_{\mu\nu}\partial_\nu\ . \tag{11}$$

Let us call V_S the volume bounded by $S(c_1,c_2)UT_1UT_2$. Then, we may write $\theta(\Delta V)=\theta(V_{S'})-\theta(V_S)$ (remember we are assuming S' is always exterior to S). Inserting this form of $\theta(\Delta V)$ in the second term in (8), which by its turn was introduced in (9) one may see, making use of (7) that surface invariance forces us to choose

$$\Phi^\alpha(S)=\frac{\pi}{e}\theta(V_S)F_{\mu\nu}\epsilon^{\mu\nu\alpha}\ , \tag{12}$$

whereupon

$$\delta\Phi^\alpha(S)=\frac{\pi}{e}\theta(V_S)\tilde{F}_{\mu\nu}\epsilon^{\mu\nu\alpha}\ . \tag{13}$$

Introducing (8), (10), and (12) in (9), we get

$$E(S)=\int d^3z\left[\mathcal{L}_E+\int_{S'(c_1,c_2)}d^2\xi_\mu\Psi_\mu\delta^3(z-\xi)+\int_{x,L}^{y}d\eta^\mu\Phi_\mu(S')\delta^3(z-\eta)+\tfrac{1}{4}(\tilde{F}_{\mu\nu})^2\right.$$
$$+\int_{S(c_1,c_2)}d^2\xi^\mu\delta\Psi_\mu\delta^3(z-\xi)+\int_{x,L}^{y}d\eta^\mu\delta\Phi_\mu(S)\delta^3(z-\eta)$$
$$\left.+\int_{T_1UT_2}d^2\xi^\mu(\chi_\mu+\delta\chi_\mu)\delta^3(z-\xi)+\mathcal{L}(S,T,L)\right]. \tag{14}$$

Let us introduce now (the minus one term is introduced for later convenience; it may be eliminated by a gauge transformation)

$$\tilde{A}_\mu(S)=-\frac{\alpha_L}{e}\int_{S(c_1,c_2)}d^2\xi_\mu\delta^3(z-\xi)$$
$$+\frac{1}{e}[\theta(V_S)-1]\partial_\mu\alpha_L(z;x,y)\ , \tag{15}$$

from which we find that $\tilde{F}_{\mu\nu}(S)=\partial_\mu\tilde{A}_\nu(S)-\partial_\nu\tilde{A}_\mu(S)$ is given by

$$\tilde{F}_{\mu\nu}(S)=-\frac{\alpha_L}{e}\left[\partial_\mu\int_{S(c_1,c_2)}d^2\xi_\nu\delta^3(z-\xi)-(\mu\leftrightarrow\nu)\right]$$
$$+\frac{1}{e}[\theta(V_S)-1][\partial_\mu,\partial_\nu]\alpha_L+T_{\mu\nu}\ , \tag{16a}$$

where

$$T_{\mu\nu}=\frac{1}{e}\partial_\mu\alpha_L\int_{T_1UT_2}d^2\xi_\nu\delta^3(z-\xi)-(\mu\leftrightarrow\nu)\ . \tag{16b}$$

It follows immediately that we may write (8) as $\tilde{F}_{\mu\nu}(S)=\tilde{F}_{\mu\nu}(S')-\tilde{F}_{\mu\nu}(S)$. Inserting (11) and (13) in (14) and taking into account this expression for $\tilde{F}_{\mu\nu}$, we find that the sum of the fourth, fifth, and sixth terms in (14) may be written as

$$t_4+t_5+t_6=-\tfrac{1}{4}[\tilde{F}_{\mu\nu}(S)]^2+\tfrac{1}{4}[\tilde{F}_{\mu\nu}(S')]^2-\tfrac{1}{2}\tilde{F}_{\mu\nu}T^{\mu\nu}\ . \tag{17}$$

Choosing

$$\mathcal{L}(S,T,L)=\tfrac{1}{4}[\tilde{F}_{\mu\nu}(S)]^2\ , \tag{18}$$

we get

$$E(S)=E(S')+\int d^3z\left[-\tfrac{1}{2}\tilde{F}_{\mu\nu}T^{\mu\nu}\right.$$
$$\left.+\int_{T_1UT_2}d^2\xi_\mu\delta\chi^\mu\delta^3(z-\xi)\right]. \tag{19}$$

Taking (16b) into account and choosing

$$\delta\chi^\mu=-\frac{1}{e}\partial_\nu\alpha_L\tilde{F}_{\mu\nu}\ , \tag{20}$$

whereupon

$$\chi^\mu=-\frac{1}{e}\partial_\nu\alpha_L(z;x,y)F^{\mu\nu}\ , \tag{21}$$

we cancel the remaining terms and obtain $E(S)=E(S')$, that is, surface invariance. L independence may be seen by making the transformation (6) with $\omega=\alpha_{L'}-\alpha_L$, which takes L in L' (observe that α_L coincides with $\alpha_{L'}$ on T_1 and T_2). The surfaces T_1 and T_2 are fixed.

Inserting (10), (12), (18), and (21) in the original expression (5) one may write that awkward expression in the nice compact form

$$\langle \mu(x;c_1)\mu^*(y,c_2)\rangle = Z^{-1}\int D\phi\, D\phi^* D A_\mu \times \exp\left[-\int d^3z\{\mathcal{L}[F_{\mu\nu}\to F_{\mu\nu}+\tilde{F}_{\mu\nu}(S)]\}\right], \tag{22a}$$

with $\tilde{F}_{\mu\nu}(S)$ given by (16). It is clear now that surface invariance is just a consequence that, under (6) $\tilde{A}_\mu(S)\to \tilde{A}_\mu(S')$ and $\tilde{F}_{\mu\nu}(S)\to\tilde{F}_{\mu\nu}(S')$.

Shifting the A_μ variable of functional integration as $A_\mu\to A_\mu-\tilde{A}_\mu(S)$, we get the equivalent expression

$$\langle \mu(x;c_1)\mu^*(y,c_2)\rangle = Z^{-1}\int D\phi\, D\phi^* D A_\mu \times \exp\left[-\int d^3z[\mathcal{L}(D_\mu\phi\to\tilde{D}_\mu\phi)]\right], \tag{22b}$$

where $\tilde{D}_\mu=\partial_\mu+ie[A_\mu-\tilde{A}_\mu(S)]$. Expressions (22) are our final result for the vortex two-point function. Mixed correlation functions may be obtained by just introducing ϕ and A_μ fields in (22) in the usual way. Surface invariance for these functions is attained up to multiplicative factors $e^{\pm i\alpha_L(z;x,y)}$ for ϕ fields and additive factors $(1/e)\partial_\mu\alpha_L(z;x,y)$, for A_μ fields. These ambiguities are just a manifestation of the dual algebra (4) in the functional integral. By analytic continuation back to the Minkowski region, the different possibilities would correspond to the various operator orderings in the correlation function.[4] Upper vortex correlation functions would be obtained by just introducing additional external fields $\tilde{A}_\mu(S)$.

In our derivation, we assumed S' to be always exterior to S. In the most general case where S' is sometimes exterior and sometimes interior to S, we must subdivide ΔV in regions ΔV_i such that in each of them S' is either exterior or interior to S. We should then perform the transformation (6) in each region ΔV_i with $\theta(\Delta V_i)$ exchanged by $(-1)^P\theta(\Delta V_i)$, where $P=0$ (1) for S' exterior (interior) to S, along with the corresponding obvious modifications in the subsequent formulas. The final result (22) remains unchanged and is completely general.

From (22a) and (22b), dropping the renormalization counterterms and analytically continuing to Minkowski space, we may extract two equivalent realizations for the vortex operator $\mu(x;c)$. We first use surface invariance and choose $S(c_1,c_2)$ in (22) as $S(c_1,c_2)=(\mathbb{R}^2_x-T_1)U(\mathbb{R}^2_y-T_2)$, where $\mathbb{R}^2_x$ is the plane at $x^3=$const. We see, then, that V_S in (15) is the infinite slice between $\mathbb{R}^2_x$ and $\mathbb{R}^2_y$, $\mathbb{R}^2_y$, implying that that the second term in (15) vanishes. With the choices above, we immediately see, from (22a) and (22b), that the vortex operator is given by (Minkowski space)

$$\mu(x;c)=\exp\left[-i\int d^3z\, F^{\mu\nu}\partial_\mu\tilde{A}_\nu(S_x)\right]=\exp\left[-\frac{i}{e}\int_{\mathbb{R}^2_x-T_x} d^2z\,\arg(\mathbf{z}-\mathbf{x})\partial_i F^{i0}(\mathbf{z},t)\right] \tag{23a}$$

or

$$\mu(x;c)=\exp\left[i\int d^3z\, j^\mu\tilde{A}_\nu(S_x)\right]=\exp\left[\int_{\mathbb{R}^2_x-T_x} d^2z[\phi^*(\mathbf{z},t)\pi^*(\mathbf{z},t)-\pi(\mathbf{z},t)\phi(\mathbf{z},t)]\arg(\mathbf{z}-\mathbf{x})\right]. \tag{23b}$$

In the expressions above $j_\mu=ie[\phi^*D_\mu\phi-(D_\mu\phi)^*\phi]$, $\tilde{A}_\mu(S_x)=-(1/e)\arg(\mathbf{z}-\mathbf{x})\int_{\mathbb{R}^2_x-T_x}\delta^3(z-\xi)d^2\xi_\mu[\tilde{A}_\mu(S)=\tilde{A}_\mu(S_x)-\tilde{A}_\mu(S_y)]$, and $\pi=(D_0\phi)^*=\dot{\phi}^*-ieA^0\phi$ is the momentum canonically conjugate to ϕ. That (23a) and (23b) are equivalent may be seen by the use of $\partial_\mu F^{\mu\nu}=j^\nu$ plus integration by parts.

Using the expansion for the operator product $e^{A}Be^{-A}$ it is straightforward to verify that $\mu(x;c)$ as given by (23) satisfies the dual algebra (4), from which we started.

Let us remark here that the algebra used to determine the vortex operator just depends on the asymptotic behavior of the classical vortex field. The full vortex solution contains a smooth function $f(|x|)$, such that $f(0)=0$ and $f(\infty)=1$, that is, a smeared out (around c) 2D Heaviside function with support in R^2-T (Ref. 8). Since we want to work with unsmeared fields, in the spirit of local QFT, we use a Heaviside function in (4) which means that only the asymptotic behavior (3) is important. It is interesting to remark that $\mu(\mathbf{x},t;c)$ creates a vortex in a definite point of spacetime, $(\mathbf{x},t)$, the curve c being related to its extension. As a consequence, in order to obtain vortex eigenstates of the Hamiltonian, one would have to consider the Fourier transform of $\mu(\mathbf{x},t;c)|0\rangle$.

One of the greatest virtues of the method introduced above is that the evaluation of vortex correlation functions reduces to a standard computation of QFT in the presence of an external field. We are presently applying this method to the Abelian Higgs model[12] in order to obtain explicit expressions for the vortex two-point function and for the quantum vortex mass. We are now also extending the method for the case of magnetic monopoles in the Georgi-Glashow model in 3+1 dimensions.[13]

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APPENDIX

Let us demonstrate here Eq. (7) and explain what $\alpha_L(z;x,y)$ is. Let us start by considering the plane $z^3=0$ in 3D Euclidean space. The Cauchy-Riemann equation relating the real and imaginary parts of $\ln z(z=|z|e^{i\arg(z)})$,

$$\epsilon^{ij}\partial_j \arg(\mathbf{z}-\mathbf{x})=\partial^i \ln|\mathbf{z}-\mathbf{x}| \ , \quad (A1)$$

where **z** and **x** are vectors in the $z^3=0$ plane, implies that

$$\epsilon^{ij}\partial_i\partial_j \arg(\mathbf{z}-\mathbf{x})\equiv[\partial_1,\partial_2]\arg(\mathbf{z}-\mathbf{x}) =\partial^2\ln|\mathbf{z}-\mathbf{x}| \ . \quad (A2)$$

The derivatives are taken with respect to **z**. Since the 2D Green's function satisfying $-\partial^2 D(\mathbf{z})=\delta^2(\mathbf{z})$, is $D(\mathbf{z})=-(1/2\pi)\ln|\mathbf{z}|$, it follows that

$$[\partial_1,\partial_2]\arg(\mathbf{z}-\mathbf{x})=2\pi\delta^2(\mathbf{z}-\mathbf{x}) \quad (A3a)$$

or

$$[\partial_i,\partial_j]\arg(\mathbf{z}-\mathbf{x})=2\pi\epsilon^{ij}\delta^2(\mathbf{z}-\mathbf{x}) \ . \quad (A3b)$$

Going now to 3D Euclidean space and choosing L as a straight line parallel to the z^3 axis, connecting $x=(\mathbf{x},-\infty)$ and $y=(\mathbf{x},+\infty)$, we may write the last expression as

$$[\partial_\mu,\partial_\nu]\arg(\mathbf{z}-\mathbf{x})=2\pi\epsilon_{\mu\nu 3}\int_{-\infty}^{\infty}d\xi^3\delta(\xi^3-z^3)\delta^2(\mathbf{z}-\mathbf{x}) \ . \quad (A4)$$

Restricting x and y to finite times: $x_0=(\mathbf{x},x^3)$ and $y_0=(\mathbf{x},y^3)$, we immediately get

$$[\partial_\mu,\partial_\nu]\alpha_{L_0}(z;x_0,y_0)=2\pi\epsilon_{\mu\nu\alpha}\int_{x_0,L_0}^{y_0}d\xi^\alpha\delta^3(z-\xi) \quad (A5a)$$

with

$$\alpha_{L_0}(z;x_0,y_0)-\theta(z^3-x^3)\arg(\mathbf{z}-\mathbf{x}) -\theta(z^3-y^3)\arg(\mathbf{z}-\mathbf{x}) \ . \quad (A5b)$$

In the above expressions L_0 is the segment of the z^3 axis connecting x_0 and y_0. The generalization of $\alpha_{L_0}(z;x_0,y_0)$ for arbitrary x, y, and L may be obtained straightforwardly as

$$\alpha_L(z;x,y)=[\theta(z^3-x^3)-\theta(2^3-y^3)] \times\arg[\mathbf{z}-\mathbf{x}_L(z^3)] \ ,$$

where $x_L(z^3)$ are the points of L. Observe that $\alpha_L(z^3-x^3)=\arg(\mathbf{z}-\mathbf{x})$ and $\alpha_L(z^3-y^3)=\arg(\mathbf{z}-\mathbf{y})$. This and (A5) establish Eq. (7).

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