

Quantum evolution of an unstable field in a de Sitter-space thermal bath

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We discuss the evolution in time, from an arbitrary initial state, of the low-momentum modes of an unstable field (i.e., the inflation field) which is coupled to a thermal bath in de Sitter space. For convenience, the thermal bath is modeled as a massless scalar field conformally coupled to a background de Sitter space. We give the exact solution for the Feynman-Vernon influence functional which describes the coupling of the thermal bath to the inflation field, as well as an exact solution to a simple evolution problem in which instability of the inflation field is modeled with a negative mass term, in the manner of Guth and Pi. Coupling to a thermal bath leads, in principle, to viscosity and momentum-space diffusion; these effects are not describable with a conventional Hamiltonian. Viscosity and diffusion govern the approach to thermal equilibrium, and compete with de Sitter-space gravitational expansion.

I. INTRODUCTION

The concept of an inflationary early Universe,¹ now modified to embrace a slow-rollover phase transition,^{2,3} seems to be a very successful solution to the problems which plagued the standard big-bang cosmology.⁴ Slow-rolling inflation raises a number of dynamical questions concerning the thermal properties of the inflation field Φ , and we begin to address these problems in this paper. One fundamental problem is that the standard assumption of thermal quasiequilibrium throughout the inflation period requires a certain strength of coupling between the inflation field, which we call Φ , and whatever fields constitute the thermal bath. Yet such couplings are limited by the constraint that inflation not yield too large density fluctuations. Since there are competing requirements on the thermal-bath-inflation field couplings, we must be prepared to examine a range of possibilities for expansion versus thermalization time scales. Moreover, there may be no compelling reason to assume that Φ is initially in thermal equilibrium at a very high temperature.

The unstable Φ potential plus de Sitter-space expansion always tends to drive Φ out of thermal equilibrium, while coupling with the heat bath, which is stable but subject to expansion effects, tends to restore thermal "equilibrium" at a temperature which depends on the degree of expansion. In the semiclassical limit these restoring forces can be described as a combination of viscosity and thermal diffusion, both arising from inelastic scattering of Φ from the bath fields. (The scattering also tends to destroy quantum phase-coherence effects, leading to an approach to classical behavior in Φ .) Beyond the semiclassical limit diffusion and viscosity appear as the real and imaginary parts of a certain function, and are related by a fluctuation-dissipation theorem.

Our purpose here is to give exact solutions to three prototype problems which arise when Φ is coupled to a thermal bath in a de Sitter space. The de Sitter space is treated as a fixed background metric, parametrized by the Hubble constant H ; because of the time dependence of

the background metric, the concept of true thermal equilibrium loses meaning, and a canonical thermal distribution can be at most invoked as an initial condition for, e.g., a density matrix. (In a complementary development, Éboli, Jackiw, and Pi⁵ have discussed out-of-equilibrium quantum thermal dynamics using time-dependent variational principles akin to those used for the functional Schrödinger equation.)

The standard machinery for dealing with out-of-equilibrium quantum thermal dynamics is the Feynman-Vernon^{6,7} influence functional. One couples a system of interest—in our cases, the inflation field—to a thermal reservoir, described by other quantum fields, and then integrates out the reservoir degrees of freedom to arrive at a description of the time evolution of the system density matrix. The influence functional F is a multiplicative contribution to the path integral which would describe the system evolution in the absence of coupling to the reservoir. In general, F is not describable in terms of an effective Hamiltonian; if we write $F = \exp(iS)$, the action S has both real and imaginary parts. Under certain plausible simplifying assumptions^{6,7} F can be calculated exactly. These are (1) the thermal bath consists of free fields (i.e., an infinite set of harmonic oscillators), except for their coupling to the system of interest, (2) the bath-system coupling is weak, and can be treated as a linear perturbation of the bath degrees of freedom, and (3) at some initial time, the bath is in thermal equilibrium, and the density matrix of the combined bath and system is a product of density matrices, one for the bath and one for the system. The system need not be in thermal equilibrium at the initial time. Some examples of influence functionals calculated under these assumptions can be found in Refs. 6 and 7.

In the present case, even though we adopt assumptions (1)–(3), there are complications which make it rather remarkable that we can calculate F exactly, yet that is what we do. The main complication is that both the thermal bath and the system live in a background metric which is time dependent, which in general makes it impossible to

solve even free-field problems explicitly. But the special properties of a de Sitter space,⁸ with its time-independent Hubble constant, allow us to find F by making a canonical transformation of a time-independent free field. For convenience we will model the thermal bath as a massless field, conformally coupled to gravity; the conformal coupling obviates certain infrared problems which arise for a massless nonconformally coupled field in de Sitter space.^{8,9} This is not the most general case which can be handled exactly, but for this simplest case the formulas are already quite complicated, and we leave to the reader the task of working out more complicated special cases. We will, in fact, solve two problems exactly; not only do we find the influence functional F , but also we give the exact time-dependent density matrix for the thermal bath uncoupled to the inflation field. The primary effect of de Sitter space is to replace ordinary time by conformal time [see Eq. (3.4b)].

With the influence functional F in hand, the work has only begun; the next step is to calculate the evolution of the inflation field Φ under the influence of F and of the self-couplings of Φ . This is impossible in general, even if only quadratic terms in the Lagrangian for Φ are saved, because F represents a complicated time-dependent quadratic action with both real and imaginary parts. With certain further simplifications we can solve our third problem exactly, which is to model the evolution of the Φ field in an unstable potential, while coupled to a thermal bath in de Sitter space. One simplification, already discussed without the coupling to a heat bath by Guth and Pi,¹⁰ is to mimic the unstable potential by a wrong-sign mass term. Other simplifications have to do with F itself. As Caldeira and Leggett⁷ discuss, with a suitable distribution of thermal-bath oscillators and in the high-temperature (or semiclassical) limit the effects described in F reduce to simple viscosity and momentum-space diffusion. For time-stationary problems, with stable potentials, the competition between these two effects eventually leads to thermal equilibrium. We also will discuss only the semiclassical limit. However, for us things are more involved, since the “temperature” which follows from the momentum-space diffusion constant is time dependent, and because of de Sitter-space horizon effects which influence the largest-wavelength modes of the thermal bath. Actually, these latter effects are not quantitatively dominant, and we gain much in simplicity by dropping them while losing little that is of interest in principle. As for the time-dependent temperature, it turns out that we can exactly solve the evolution problem of an upside-down oscillator in a thermal bath in de Sitter space for any time dependence of the temperature, as long as we are allowed to approximate the viscosity by a constant. This approximation we will make, in somewhat the same spirit as one goes from a general Friedmann-Robertson-Walker metric to de Sitter space by assuming that the expansion rate $\dot{a}a^{-1} \equiv H$ is constant. In view of the great generality allowed, because of our present ignorance, of the coupling between the thermal bath and the inflation field, we will not offer specific physical justification of the assumption of constant viscosity. The final simplification we will make, which is not essential to

the existence of exact solutions, is to save only one mode of the inflation field, corresponding to an average of this field over the horizon volume or equivalently to saving only the largest-wavelength mode of Φ . In future work we will discuss a very important fourth problem, to which the three problems discussed here are preliminary: the evolution of density fluctuations in the early Universe.¹¹ This will require discussion of short-wavelength modes of Φ as well as the long-wavelength mode.

The exact solution for evolution of the upside-down oscillator will be presented in terms of the Wigner distribution function W rather than in terms of the density matrix. Using W is appropriate for a semiclassical treatment, and makes it physically evident how viscosity and diffusion due to the thermal bath play a role.⁷ As Guth and Pi¹⁰ point out, at late time the evolution of an unstable oscillator becomes describable in classical terms, with quantum-mechanical effects appearing as a distribution of delay times (the times at which the oscillator begins to roll classically). In the Wigner-function case, the distribution of delay times is determined by the initial density matrix. Coupling to a thermal bath in de Sitter space changes this picture somewhat; as long as diffusion is important, the trajectory is not classical, but Brownian. However, the diffusion constant is proportional to the temperature T which is decreasing like e^{-Ht} (until reheating begins). At late inflationary times only viscosity (and decay) may remain to signal the coupling to the thermal bath, in which case the evolution equation for W can be solved in terms of its characteristics, or classical trajectories, but this time with viscous damping. The concept of delay time can still be used, provided that viscosity is accounted for. When diffusion is important, the evolution equation for W is a Fokker-Planck equation with various time-dependent coefficients. One could deal directly with the evolution equation for the reduced density matrix, but it is perhaps more perspicuous to use the Fokker-Planck equation for W , since the effects occurring have an intuitive and classical interpretation. Unfortunately, the formulas in the solution are long and complicated and it is not easy to see what is happening by mere inspection, especially as such a wide range of parameters can play a role.

The reader, seeing the words viscosity, diffusion, Fokker-Planck equation, etc., may be tempted to draw analogs between our paper and other recent works in which the same words appear in an early-Universe context. For example, Starobinsky¹² and Linde¹³ have given a Fokker-Planck equation with damping due to Hubble expansion and diffusion due to infrared quantum fluctuations.⁹ Graziani,¹⁴ following this lead, has discussed the uses of the Wigner distribution in the early Universe. Not only the physics but also the mathematics of our Fokker-Planck problem differs from that of the above references. Other authors¹⁵ have invoked strong viscosity as a possible driver of inflation, by giving rise to an effective negative pressure. This certainly cannot occur for sufficiently weak viscous coupling¹⁶ and we do not advocate this as one of the physical applications of our formalism.

What, then, are the potential physical applications of this work? (We say potential because none will be discussed in detail here; future work will be devoted to applications.) First, one may ask questions related to the preinflationary universe: It is usually assumed that preinflationary dynamics brings the Φ field to a state of thermal equilibrium at the initial time when inflation is about to start, and at this time the effective potential for Φ has a $T^2\Phi^2$ term which stabilizes the inflation field (e.g., Refs. 4 and 10). But the time constants of thermal processes may not allow for establishment of full equilibrium, and the approximation of an effective potential may break down. In any case, one wishes to be able to describe the evolution of an arbitrary initial Φ -field configuration, not just an initial thermal distribution. Second, during inflation viscosity could have an effect on the evolution of density fluctuations, which drive galaxy formations and microwave background fluctuations. Conventional (i.e., ignoring cosmic strings, chaotic inflation, etc.) wisdom,¹¹ based on the assumption of thermal equilibrium of Φ throughout inflation, holds that the Φ self-couplings are very small; e.g., λ in a $\lambda\Phi^4$ coupling is $\sim 10^{-12}$. This is because the fractional density fluctuation $\delta\rho/\rho$ scales like $\lambda^{1/2}$. The couplings of Φ to other fields, such as the thermal bath, are limited—usually like $\lambda^{1/4}$ —so that radiative corrections do not induce a large Φ^4 term. In consequence, the effect of the thermal bath during or before inflation may be too weak to be noticed. This leads to the logical dilemma that there is then no reason to assume that Φ was ever in thermal equilibrium, at least before reheating. On the other hand, it is conceivable (we have no arguments to back up this possibility) that λ is not so small, and that fluctuations are damped to an acceptable level by viscous effects. In either case, it is essential to have in hand a dynamical formalism capable of dealing with out-of-equilibrium coupling of Φ to a thermal bath. Third, the exit from inflation to the reheating era is again one where the assumption of thermal equilibrium is only approximate. One could imagine dealing with this problem, in which both the Hubble constant and the viscosity may change rapidly, by joining together our exact solutions over approximately chosen time intervals and at some point turning the upside-down oscillator into a stable oscillator to yield the potential well in which reheating takes place. Finally, we know that CP -violating effects cannot lead to a baryon asymmetry unless the Universe evolves for some time out of thermal equilibrium, so it will be interesting to discuss baryon generation in the context of our model.

The work is organized as follows: Section II contains a

$$\begin{aligned} \bar{\phi}(\phi, \phi'; t) &= \int \prod dq_k \langle \phi q_k | \rho(t) | \phi' q_k \rangle \\ &= \int \prod dq_k dq_{1k} dq_{2k} d\phi_1 d\phi_2 K(\phi q_k, \phi_1 q_{1k}; t) \langle \phi_1 q_{1k} | \rho(0) | \phi_2 q_{2k} \rangle K^*(\phi' q_k, \phi_2 q_{2k}; t), \end{aligned} \quad (2.4)$$

where K is an evolution operator. If the Hamiltonian \hat{H} is time independent, K can be written as a path integral of the usual type:

$$K(\phi q_k, \phi' q'_k; t) = \int \left[d\phi_1 \prod dq_{1k} \right] e^{iS}, \quad (2.5)$$

brief introduction to the Feynman-Vernon⁶ ideas and sets the notation; Sec. III gives the calculation of the time-dependent density matrix and the influence functional for the thermal bath; Sec. IV solves the evolution problem for an upside-down oscillator; Sec. V discusses the meaning and possible applications of the solutions. An Appendix gives some details of the algebra needed for Sec. IV.

II. THE INFLUENCE FUNCTIONAL PICTURE

The Feynman-Vernon formalism is not an everyday tool of particle physicists and cosmologists, so we give a brief description of it here. (A recent work by Niemi¹⁷ also discusses the incorporation of this formalism into quantum field theory.)

We are interested in the time evolution of a single quantum degree of freedom ϕ (later to be identified with a low-momentum mode of the inflation field Φ). The restriction to a single degree of freedom is unnecessary, and made only for concise notation. The ϕ degree of freedom will be called the system. It is coupled to another quantum system, described by an infinite number of coordinates q_k . These coordinates are the Fourier-space components of one or more quantum fields, free except for their coupling to the system and to a background gravitational field. The q_k degrees of freedom are called the (thermal) bath. We will assume that the bath is in thermal equilibrium at $t=0$, at which time the system-bath density matrix ρ is a direct product of a system density matrix $\rho_S(0)$ and a bath density matrix $\rho_B(0)$. The overall Hamiltonian \hat{H} (we use \hat{H} instead of H , a notation used later for the Hubble constant) is the sum of three terms, as are the Lagrangian and action:

$$\hat{H} = \hat{H}_S(\phi) + \hat{H}_B(q_k) + \hat{H}_I(\phi, q_k). \quad (2.1)$$

Each of these terms may have an explicit time dependence, because of the background gravitational field. The density matrix ρ obeys

$$\frac{\partial \rho}{\partial t} = -i[\hat{H}, \rho], \quad (2.2)$$

$$\rho(0) = \rho_S(0) \otimes \rho_B(0), \quad (2.3a)$$

$$\rho_B(0) = Z_B^{-1} e^{-\beta \hat{H}_B}, \quad (2.3b)$$

where $Z_B = \text{Tr} e^{-\beta \hat{H}_B}$, so $\text{Tr} \rho_B(0) = 1$, and β is the inverse temperature at $t=0$. Of course, $\text{Tr} \rho_S(0) = 1$ also. Note that (2.2) holds even if \hat{H} explicitly depends on t .

We are interested in the coordinate-space representative of the reduced (traced over bath coordinates) density matrix, which we call $\bar{\rho}$:

where the parentheses indicate functional integration, and $S = S_S + S_B + S_I$ is the total action of the system plus bath. In our case, \hat{H} does depend on time, so integrations over coordinates are replaced (up to an overall normali-

zation) by

$$\int \left[d\phi \prod dq_k \right] e^{iS} \rightarrow \int \left[d\phi dp \prod dq_k dp_k \right] e^{iS}, \quad (2.6)$$

where p, p_k are momenta conjugate to the coordinates. Also the action S is written in the form

$$S = \int_0^t dt' \left[\sum p_k \dot{q}_k + p \dot{\phi} - \hat{H} \right] \quad (2.7)$$

and $\dot{q}_k, \dot{\phi}$ are to be expressed in terms of the coordinates, momenta, and time. In practice, all evolution operators needed in this paper will be calculated without actually doing any path integrations, and we will, for simplicity of notation, use the form (2.5) for K in the formal developments of this section.

We now combine the factors of (2.4) depending on the q_k into a single factor, the influence functional

$$F(\phi, \phi'; t) = \int \prod dq_k dq_{1k} dq_{2k} K_\phi(q_k, q_{1k}; t) \times \langle q_{1k} | \rho_B(0) | q_{2k} \rangle K_{\phi'}^*(q_k, q_{2k}; t) \quad (2.8)$$

with the definition

$$K_\phi(q_k, q_{1k}; t) = \int (dq_{2k}) e^{i[S_B + S_I(\phi)]}. \quad (2.9)$$

Thus K_ϕ is the evolution operator of the bath in the presence of coupling to the system at functional coordinate $\phi(t)$; that is, K_ϕ is a functional of the history $\phi(t)$. With (2.8), we find that (2.4) becomes

$$\bar{\rho}(\phi, \phi'; t) = \int d\phi_1 d\phi_2 (d\phi_3 d\phi_4) e^{iS_S(\phi_3) - iS_S(\phi_4)} \times F(\phi_3, \phi_4; t) \langle \phi_1 | \rho_S(0) | \phi_2 \rangle \quad (2.10)$$

with the coordinates ϕ, ϕ' appearing as boundary conditions on the functional integrals ($\phi_3 = \phi$ at time t , $\phi_3 = \phi_1$ at time 0, etc.). As we will see, F is not in general of the form $\exp[iS(\phi_3, \phi_4)]$ with real S , so that the time evolution of $\bar{\rho}$ is not given by the quantum Liouville equation (2.2).

When the thermal bath is a set of harmonic oscillators coupled linearly to ϕ , Feynman and Vernon have shown that the general form of the influence functional is

$$F(\phi, \phi'; t) = \exp \left[- \int^t dt_1 \int^{t_1} dt_2 [\phi(t_1) - \phi'(t_1)] [\phi(t_2) \alpha(t_1, t_2) - \phi'(t_2) \alpha^*(t_1, t_2)] \right]. \quad (2.11)$$

The real and imaginary parts of α represent, respectively, diffusive and viscous processes. For example, suppose that α is a real function $\alpha_R(t_1, t_2)$. Using (2.11) in (2.10) shows that the latter expression is equivalent to adding a term $\int V\phi dt$ to the action for ϕ and averaging over V , where V is a random Gaussian-distributed potential with zero mean and variance:

$$\langle V(t_1) V(t_2) \rangle = \alpha_R(t_1, t_2). \quad (2.12)$$

These random forces represent thermal diffusion of ϕ in the presence of the bath. The imaginary part of α , called α_I , contributes two terms to $\ln F$:

$$\ln F = -i \int dt_1 \int dt_2 \alpha_I(t_1, t_2) \{ [\phi(t_1)\phi(t_2) - \phi'(t_1)\phi'(t_2)] + [\phi(t_1)\phi'(t_2) - \phi(t_2)\phi'(t_1)] \} + \text{real terms}. \quad (2.13)$$

The first term in square brackets is equivalent to adding an extra quadratic term to the action S_S . The effect of the second square-bracketed term is not immediately obvious, but we will see in Sec. IV that it represents viscous damping, damping which is related to the fluctuations (2.12) by a fluctuation-dissipation theorem.

This reshuffling of functional integrals would be of little use unless F could actually be calculated, which requires the previously mentioned assumptions. First, the action S_B describes a free quantum field, except for the coupling to gravity. The second assumption, which is always made but requires some discussion here, is that the system-bath interaction can be linearized:

$$S_I = \int dt \mathcal{L}_I = \int dt \sum_k g_k q_k \phi. \quad (2.14)$$

The coefficients g_k may be time dependent also. Of course, the concept of an influence functional does not depend on this linearization, but the existence of exact solutions does. For example, suppose that the thermal bath corresponds to a real scalar field $\psi(x)$ and that ϕ is the zero-momentum mode of another field $\Phi(x)$. These

fields interact through a term $g\Phi\psi^2$. For the zero mode ϕ , the corresponding \mathcal{L}_I is

$$\mathcal{L}_I = g\phi \int d^3x \psi^2 \sim g\phi \sum_k \psi_k \psi_{-k}, \quad (2.15)$$

where ψ_k is the Fourier transform of ψ (Ref. 18). A simple procedure is to identify q_k with $\psi_k \psi_{-k}$ and assume, in the spirit of the random-phase approximation, that the q_k dynamics is that of free oscillators. It is not the purpose of this paper to discuss detailed models of the system-bath coupling in de Sitter space, but in Sec. V we will make some general remarks on the possible ranges of couplings in the context of inflation.

Further details of the calculation of F , for a specific realization of the bath (a massless conformally coupled field) will be given in Secs. III and IV.

III. DENSITY MATRIX AND INFLUENCE FUNCTIONAL OF A BATH OF CONFORMAL OSCILLATIONS

In this section we solve two of the three problems set in the Introduction, and in the following section we use

these results in calculating the evolution of an upside-down oscillator coupled to the thermal bath of conformal oscillators. All three problems refer to evolution in a background de Sitter space, whose metric we write

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - e^{2Ht} d\mathbf{x}^2. \quad (3.1)$$

The Hubble constant H is given by $\dot{a}a^{-1}$ in terms of the scale factor, and is not literally unchanging in time, but we will take it to be constant throughout this paper.

Let ψ be a real scalar field, free except for its conformal coupling to gravity. In the de Sitter metric (3.1), the Lagrangian for ψ is

$$\mathcal{L}_B = \int d^3x e^{3Ht} \left[\frac{1}{2} \dot{\psi}^2 - \frac{1}{2} e^{-2Ht} (\nabla\psi)^2 - \frac{1}{2} \xi R \psi^2 \right], \quad (3.2)$$

where R is the curvature scalar ($R = 12H^2$), and $\xi = \frac{1}{6}$ for conformal coupling. By decomposition of ψ into a Fourier sum,

$$\psi = \sum_{\mathbf{k}} [e^{i\mathbf{k}\cdot\mathbf{x}} \psi_{\mathbf{k}}(t) + \text{H.c.}], \quad (3.3)$$

we can write \mathcal{L} as a sum of terms, involving the (real) coefficients of $\cos(\mathbf{k}\cdot\mathbf{x})$ and $\sin(\mathbf{k}\cdot\mathbf{x})$ in (3.3). We will return to this mode sum, but for now let us note that the change of variables

$$\psi(t) = e^{-Ht} \chi(\eta), \quad (3.4a)$$

$$\eta = \int_0^t dt' e^{-Ht'} = H^{-1}(1 - e^{-Ht}), \quad (3.4b)$$

allows us to write the action $S = \int \mathcal{L} dt$ as a flat-space free-field action plus end-point terms:

$$\mathcal{L} = \int d^3x \left[e^{-Ht} \left[\frac{1}{2} (\chi')^2 - \frac{1}{2} (\nabla\chi)^2 \right] - \frac{H}{2} \frac{d}{dt} (\chi^2 e^{Ht}) \right], \quad (3.5)$$

$$\begin{aligned} S_B &= \int_0^t dt' \mathcal{L}(t') \\ &= \int d^3x \int_0^\eta d\eta' \left[\frac{1}{2} (\chi')^2 - \frac{1}{2} (\nabla\chi)^2 \right] \\ &\quad - \frac{H}{2} \int d^3x \psi^2 e^{3Ht'} \Big|_0^t, \end{aligned} \quad (3.6)$$

where $\chi' = dx/d\eta'$. It follows that the classical action S_B , on which all further developments depend, can be found by inspection of the well-known results for a free field.

It is now convenient to decompose ψ into its Fourier modes which, for a given wave vector \mathbf{k} , evolve as would a harmonic oscillator of frequency $k \equiv |\mathbf{k}|$. Let q be the coordinate of such a mode; we suppress the labels, such as \mathbf{k} , which identify the particular mode in question. The Lagrangian for this mode is

$$\mathcal{L} = e^{3Ht} \left(\frac{1}{2} \dot{q}^2 - \frac{1}{2} k^2 q^2 e^{-2Ht} - H^2 q^2 \right). \quad (3.7)$$

The canonical momentum p is

$$p = \dot{q} e^{3Ht} \quad (3.8)$$

and the Hamiltonian \hat{H} is

$$\hat{H} = \frac{1}{2} e^{-3Ht} p^2 + \frac{1}{2} k^2 e^{Ht} q^2 + H^2 q^2 e^{3Ht}. \quad (3.9)$$

Corresponding to (3.4) we change variables to $Q(\eta)$:

$$q(t) = e^{-Ht} Q(\eta) \quad (3.10)$$

in terms of which the action is

$$S_B = \int_0^\eta d\eta' \left[\frac{1}{2} (Q')^2 - \frac{1}{2} k^2 Q^2 \right] - \frac{H}{2} e^{3H\eta'} q^2 \Big|_0^\eta. \quad (3.11)$$

In particular, the classical action $S_c(q_f, q_i; t)$ for the path beginning at q_i at $t'=0$ and ending at q_f at $t'=t$ is

$$\begin{aligned} S_c(q_f, q_i; t) &= \frac{k}{2} \cot(k\eta) (e^{2Ht} q_f^2 + q_i^2) - \frac{k q_f q_i}{\sin(k\eta)} e^{Ht} \\ &\quad - \frac{H}{2} (e^{3Ht} q_f^2 - q_i^2). \end{aligned} \quad (3.12)$$

We will use S_c to construct the evolution operator:

$$K(q_f, q_i; t) = \langle q_f | T \exp \left[-i \int_0^t dt' \hat{H}(t') \right] | q_i \rangle. \quad (3.13)$$

This can be written as a path integral, which differs from the conventional form $\int (dq) e^{iS}$ because of the time dependence of the kinetic terms, as discussed in Sec. II. Nonetheless, K can be expressed in terms of the classical action S_c as

$$K(q_f, q_i; t) = N(t) \exp[iS_c(q_f, q_i; t)], \quad (3.14)$$

which is to be expected for a quadratic Lagrangian. The function $N(t)$ is determined by comparing the equation of motion for K with $p = -i\partial/\partial q_f$ in \hat{H} ,

$$i \frac{\partial K}{\partial t} = \hat{H} K \quad (3.15)$$

and the Hamilton-Jacobi equation satisfied by S_c of (3.12):

$$-\frac{\partial S_c}{\partial t} = \hat{H} \left[p = \frac{\partial S_c}{\partial q_f}, q = q_f; t \right]. \quad (3.16)$$

The correctly normalized result is

$$\begin{aligned} N &= \left[\frac{i}{2\pi t} \right]^{1/2} \exp \left[\int_0^t dt' \left[-\frac{k}{2} \cot(k\eta') e^{2Ht'} \right. \right. \\ &\quad \left. \left. + \frac{1}{2t'} + \frac{H}{2} e^{3Ht'} \right] \right], \end{aligned} \quad (3.17)$$

where $\eta' = \eta(t')$.

A. The density matrix

Now we can construct the time evolution of the density matrix ρ_B , obeying the equation

$$\frac{\partial \rho_B}{\partial t} + i[\hat{H}, \rho_B] = 0. \quad (3.18)$$

The solution for the matrix elements of ρ_B is

$$\langle q | \rho_B | q' \rangle = \int dq_1 dq_2 K(q, q_1; t) \langle q_1 | \rho(0) | q_2 \rangle \times K^*(q', q_2; t) \quad (3.19)$$

in terms of the initial density matrix $\rho_B(0)$. We choose the latter to represent a thermal bath at temperature $T \equiv \beta^{-1}$:

$$\left[\frac{k}{\pi} \tanh \frac{1}{2} \beta k \right]^{-1/2} \langle q | \rho_B(t) | q' \rangle = G_1^{1/2} \exp \left[-G_1 \left[\frac{k}{2} (q^2 + q'^2) \coth \beta k - \frac{kqq'}{\sinh(\beta k)} \right] + iG_2 (q^2 - q'^2) \right], \quad (3.22)$$

where G_1, G_2 are functions of t , given by

$$G_1 = \Delta e^{2Ht}, \quad (3.23)$$

$$G_2 = \frac{k}{2} \cot(\eta k) e^{2Ht} (1 - \Delta) - \frac{H}{2} (e^{3Ht} + \Delta e^{2Ht}), \quad (3.24)$$

$$\Delta = \left[1 + \frac{2H}{k} \sin(\eta k) \cos(\eta k) + \frac{H^2}{k^2} \sin^2(\eta k) \right]^{-1}. \quad (3.25)$$

Of course at $H=0$ where $G_1=1, G_2=0$ we recover the result that ρ_B for a harmonic oscillator in flat space is constant in time. In the present case ρ_B is time dependent but $\text{Tr} \rho_B = 1$.

B. The influence functional

Here, as discussed in Sec. II, we couple the conformal oscillators linearly to a coordinate $\phi(t)$, with coupling

$$S_{c\phi}(q_f, q_i; t; \phi) = S_c(q_f, q_i; t) + \int_0^t dt' g \phi(t') e^{-Ht'} \left[q_f e^{Ht'} \frac{\sin(k\eta')}{\sin(k\eta)} + q_i \frac{\sin[k(\eta - \eta')]}{\sin(k\eta)} \right] + \frac{1}{2} \int_0^t dt_1 \int_0^t dt_2 g \phi(t_1) G(t_1, t_2; t) g \phi(t_2) \quad (3.28)$$

and the Green's function G is given by

$$G = \frac{e^{-H(t_1+t_2)}}{k \sin(k\eta)} \{ \theta(t_1 - t_2) \sin[k(\eta_1 - \eta)] \sin(k\eta_2) + (t_1 \leftrightarrow t_2) \}. \quad (3.29)$$

In (3.27) and (3.28), the functions S_c and $N(t)$ are given in (3.12) and (3.17). As before, the evolution operator obeys (3.15), with \hat{H}_ϕ in place of \hat{H} .

The evolution operator is calculated by doing the threefold integration in (2.8), using (3.20) and (3.28). The result is

$$F(\phi, \phi'; t) = \exp \left[- \left[\int_0^\eta d\eta_1 \int_0^{\eta_1} d\eta_2 [\phi(\eta_1) - \phi'(\eta_1)] [\alpha(\eta_1, \eta_2) \phi(\eta_2) - \alpha^*(\eta_1, \eta_2) \phi'(\eta_2)] \right] \right], \quad (3.30)$$

where

$$\alpha(\eta_1, \eta_2) = \sum_k \frac{|g_k|^2}{2k} \left[2 \coth \frac{1}{2} \beta k \left[\cos k(\eta_1 - \eta_2) + \frac{H}{k} \sin k(\eta_1 + \eta_2) + \frac{H^2}{k^2} \sin k \eta_1 \sin k \eta_2 \right] - i \sin k(\eta_1 - \eta_2) \right]. \quad (3.31)$$

$$\langle q_1 | \rho_B(0) | q_2 \rangle = \left[\frac{k}{\pi} \tanh \frac{1}{2} \beta k \right]^{1/2} \times \exp \left[- \frac{k}{2} \coth \beta k (q_1^2 + q_2^2) + \frac{kq_1 q_2}{\sinh(\beta k)} \right]. \quad (3.20)$$

The normalization is such that

$$\text{Tr} \rho_B(0) \equiv \int dq \langle q | \rho_B(0) | q \rangle = 1, \quad (3.21)$$

a normalization which in fact holds at all times. To calculate the time dependence of ρ_B , use (3.11), (3.13), (3.17), and (3.20) in (3.19) and find

constants g_k . As with the density matrix we suppress the subscripts k . We write our result for any one of the oscillators, forming the results for all oscillators by multiplying together the single-oscillator formulas. For every k we have the action

$$S_\phi \equiv S_B + S_I(\phi) = S_B + \int_0^t dt' g q \phi, \quad (3.26)$$

where S_B is given in (3.11). The influence functional is found by calculating the evolution operator K_ϕ of (2.9) and then performing the integrations in (2.8). It is straightforward to calculate K_ϕ , just as we did for K of (3.14), modifying S_c to account for the source term, linear in q of (3.26). We find

$$K_\phi(q_f, q_i; t) = N(t) \exp(iS_{c\phi}), \quad (3.27)$$

where

We have assumed (see Sec. IV) that g_k is time independent, which is not the most general case. The general case is easily recovered by using (3.28), where g can be dependent on time; it simply amounts to replacing g^2 by $g(\eta_1)g(\eta_2)$. Note that α has both a real and an imaginary part, as we have already discussed in Sec. II. If one sets the explicit H 's in (3.31) to zero, the result is the usual influence functional^{6,7} for flat space, but with conformal time replacing ordinary time. In this case, which we will pursue in Sec. IV, the real and imaginary parts of α are related by a conventional fluctuation-dissipation theorem. The H -dependent terms are unimportant if $k \gg H$. Note that the imaginary, or dissipative, part of α in (3.31) is independent of temperature and depends on H only through the conformal time. The real, or fluctuating, terms in α can be interpreted as in (2.12) via a fluctuating Gaussian-distributed potential whose statistics are not time stationary for k small compared to H . In the next section we will use the approximate α with no explicit H dependence to investigate the evolution of the Wigner distribution function.

IV. EVOLUTION OF AN UNSTABLE OSCILLATOR

In this section we discuss the only problem we know how to solve exactly, involving the (approximate) influence functional in de Sitter space. It is a variant of the Guth-Pi¹⁰ problem of the time evolution of an upside-down oscillator. To get a soluble problem we must make some approximations in the general expression (3.31) for the influence functional, which amounts to dealing with a viscosity coefficient which is constant in time. This may or may not be physically realistic, but we can at least make a quantitative judgment of the influence of viscosity and thermal diffusion on the evolution of an unstable oscillator in de Sitter space, for any given time interval during which the viscosity changes little.

We will describe explicitly only one mode of the inflation field $\Phi(x)$. The mode in question is a long-wavelength mode, with little or no spatial variation over the volume included in a causal horizon; we can consider it as the spatial average of the field Φ . As such, it depends only on time, so we will introduce (as in Sec. II) a single coordinate $\phi(t)$ to describe the inflation field. It would, in fact, be possible to add to our treatment finite-momentum modes, but only at the expense of a great deal of complexity which is not, at the moment, called for.

We begin with a discussion of the dynamics of ϕ when it is uncoupled to the heat bath. The approximate Lagrangian in de Sitter space is

$$\mathcal{L}_S = \frac{1}{2} e^{3Ht} (\dot{\phi}^2 + \omega^2 \phi^2) \quad (4.1)$$

with corresponding canonical momentum and Hamiltonian \hat{H} :

$$p = e^{3Ht} \dot{\phi}, \quad \hat{H} = \frac{1}{2} e^{-3Ht} p^2 - \frac{1}{2} e^{3Ht} \omega^2 \phi^2. \quad (4.2)$$

The solutions to the equations of motion are linear combinations of the exponentials

$$\exp \left[-\frac{3Ht}{2} \right] e^{\pm \lambda t}, \quad \lambda^2 = \frac{9}{4} H^2 + \omega^2. \quad (4.3)$$

(If we considered modes with finite momentum the corresponding solutions would be Hankel functions,⁸ which would lead to a soluble but messy problem.) Equation (4.3) suggests the change of variables

$$\phi = z \exp \left[-\frac{3Ht}{2} \right] \quad (4.4)$$

in terms of which the Lagrangian becomes

$$\mathcal{L}_S = \frac{1}{2} \dot{z}^2 + \frac{1}{2} \lambda^2 z^2 - \frac{3H}{4} \frac{d}{dt} z^2, \quad (4.5)$$

which is a harmonic oscillator plus a total time derivative, just as before. So we can calculate all quantum-mechanical properties of this system, e.g., the evolution function, in terms of the classical action S_{cl} . This turns out to be, for a system beginning at $\phi = \phi_i$ at zero time and $\phi = \phi_f$ at time t ,

$$S_{cl} = \frac{\lambda}{2} \coth \lambda t (\phi_f^2 + \phi_i^2) - \frac{\lambda \phi_i \phi_f}{\sinh \lambda t} - \frac{3H}{4} (\phi_f^2 - \phi_i^2). \quad (4.6)$$

One could try to calculate the evolution operator occurring in (2.10) for the system in contact with the heat bath by using (4.6) and (3.30), then doing the Gaussian functional integrals over ϕ .

Unfortunately, it is impossible to do this analytically because of the complicated time dependence in the integrand of F . We will approximate F so that (2.10) can be exactly calculated by doing some Gaussian functional integrations. However, we will not, in fact, proceed this way; instead (following Caldeira and Leggett⁷) we will deal with the problem through the Wigner distribution function in the semiclassical limit (see also Ref. 14). This way of proceeding has certain conceptual advantages which can be used in more complicated situations, through the use of physically motivated approximations. To begin with, we discuss an approximation to the influence functional F which allows for an exact treatment of the unstable oscillator.

The first step in the approximation is to drop the explicit H -dependent terms in α of (3.31), leaving only terms which implicitly depend on H through the conformal time η . The next, and crucial, step is to replace the t - and k -dependent terms $|g_k|^2$ in (3.31) by a constant γ , which we will introduce via

$$|g_k|^2 = 4\pi\gamma. \quad (4.7)$$

Note that γ has dimensions of mass to the first power, since the coordinate ϕ , being the zero-momentum mode¹⁸ of a scalar field of mass dimension 1, has mass dimension $-\frac{1}{2}$. This makes the action $\int \mathcal{L} dt$ based on (4.1) dimensionless. The final approximation we will make is to consider only the high-temperature, or semiclassical regime in which we replace $\coth \frac{1}{2} \beta k$ by $2Tk^{-1}$, where $T = \beta^{-1}$ is the initial ($t=0$) temperature of the heat bath. Given these three approximations, it is straightforward to calculate the influence functional, with a result quite similar to that of Caldeira and Leggett:⁷

$$F(\phi, \phi'; t) = \exp \left[-2\gamma T \int_0^t dt' e^{-Ht'} [\phi(t') - \phi'(t')]^2 - i\gamma \int_0^t dt' (\dot{\phi} + \dot{\phi}')(\phi - \phi') \right], \quad (4.8)$$

where the overdots indicate derivatives with respect to ordinary time, not conformal time. The appearance of $e^{-Ht'}$ in the first term of (4.8) indicates the time dependence of the temperature; recall that T is the *initial* temperature of the heat bath.

At this point, the equation of motion for the reduced density matrix can be found following Caldeira and Leggett. Because of the influence functional, the reduced density matrix $\bar{\rho}$, defined in (2.4) as the trace over the reservoir coordinates of the full density matrix, obeys an equation such as (2.2) with some additions:

$$\frac{\partial \bar{\rho}}{\partial t} = -i[\hat{H}(\phi) - \hat{H}(\phi')] \bar{\rho} - \gamma(\phi - \phi') \left[\frac{\partial \bar{\rho}}{\partial \phi} - \frac{\partial \bar{\rho}}{\partial \phi'} \right] - 2\gamma \hat{T}(\phi - \phi')^2 \bar{\rho}. \quad (4.9)$$

Here,

$$\hat{H}(\phi) = -\frac{1}{2} e^{-3Ht} \left[\frac{\partial}{\partial \phi} \right]^2 - \frac{1}{2} e^{3Ht} \omega^2 \phi^2$$

is the coordinate-space version of the quantum Hamiltonian (4.2) and

$$\hat{T} = T e^{-Ht} \quad (4.10)$$

is the time-dependent temperature. We wish to convert this into a dynamic equation for the Wigner distribution function, defined by

$$W(p, \phi, t) = \frac{1}{2\pi} \int dy e^{ipy} \bar{\rho}(\phi - \frac{1}{2}y, \phi + \frac{1}{2}y; t). \quad (4.11)$$

Substitution of (4.9) into (4.11) yields the Fokker-Planck equation for W :

$$\begin{aligned} \frac{\partial W}{\partial t} = & -pe^{-3Ht} \frac{\partial W}{\partial \phi} - \omega^2 \phi e^{3Ht} \frac{\partial W}{\partial p} + 2\gamma \frac{\partial}{\partial p} (pW) \\ & + D \frac{\partial^2 W}{\partial p^2}, \end{aligned} \quad (4.12)$$

where

$$D = 2\gamma \hat{T}. \quad (4.13)$$

This equation is exact as it stands for a quadratic potential, but for a general potential $V(\phi)$ in $\hat{H}(\phi)$ there is an infinite series of terms explicitly dependent on \hbar . More generally, the second term on the right-hand side (RHS) of (4.12) is of the form

$$V'(\phi) \frac{\partial W}{\partial p} \quad (4.14)$$

and the leading correction in powers of \hbar is of order $\hbar^2 V''''(\phi) \partial^3 W / \partial p^3$. The techniques we use to solve (4.12) are valid semiclassically, and $O(\hbar^2)$ and higher corrections can be systematically included for more general $V(\phi)$, but we will not discuss that here. Recall also that we have used the high-temperature (classical) expansion of $\coth \frac{1}{2}\beta k$ in evaluating the influence functional, which

would not be justified if higher-order terms in \hbar are to be saved.

It is convenient to solve (4.12) by exploiting its formal similarity to a Euclidean-space evolution function equation, such as (3.15) with the factor of i deleted. That is, we can define a pseudo-Hamiltonian \mathcal{H} and corresponding action S , solving the Hamilton-Jacobi equation, such that W is of the form

$$W(p, \phi, t) = \int dp' d\phi' K(p\phi, p'\phi'; t) W_0(p', \phi') \quad (4.15)$$

with

$$K = Q(t) e^{-S}, \quad K(t=0) = \delta(p-p') \delta(\phi-\phi'), \quad (4.16)$$

and $W_0(p, \phi)$ is the initial Wigner distribution function at $t=0$ [i.e., the Fourier transform of the initial density matrix $\bar{\rho}(t=0)$]. Of course, K itself satisfies the equation for W in (4.12). In constructing the pseudo-Hamiltonian, we think of both p and ϕ as coordinates, for each of which we introduce a corresponding canonical momentum. So we use the notation

$$y_1 = p, \quad y_2 = \phi \quad (4.17)$$

with corresponding momenta p_1, p_2 . The required pseudo-Hamiltonian is

$$\mathcal{H} = Dp_1^2 - (2\gamma y_1 - \omega^2 e^{3Ht} y_2) p_1 + y_1 e^{-3Ht}. \quad (4.18)$$

We note that if $S(y_i, y_i'; t)$ is a classical action based on \mathcal{H} , satisfying the Hamilton-Jacobi equation

$$-\frac{\partial S}{\partial t} = \mathcal{H} \left[p_i = \frac{\partial S}{\partial y_i}, y_i; t \right] \quad (4.19)$$

and Q obeys

$$\frac{\dot{Q}}{Q} = 2\gamma - D \frac{\partial^2 S}{\partial y_1^2}, \quad (4.20)$$

then K (and consequently W) obeys the Fokker-Planck equation (4.12). Although \mathcal{H} is degenerate, in having no p_2^2 term, this will give us no trouble.

To find the classical action S , we need the classical orbits satisfying Hamilton's equations. These equations are

$$\dot{y}_1 = \frac{\partial \mathcal{H}}{\partial p_1} = 2Dp_1 - 2\gamma y_1 + \omega^2 e^{3Ht} y_2, \quad (4.21a)$$

$$\dot{y}_2 = \frac{\partial \mathcal{H}}{\partial p_2} = y_1 e^{-3Ht}, \quad (4.21b)$$

$$\dot{p}_1 = -\frac{\partial \mathcal{H}}{\partial y_1} = 2\gamma p_1 - p_2 e^{-3Ht}, \quad (4.21c)$$

$$\dot{p}_2 = -\frac{\partial \mathcal{H}}{\partial y_2} = -\omega^2 e^{3Ht} p_1. \quad (4.21d)$$

Note that the p equations do not depend on the y_i , and can be solved without reference to them. If D were equal to zero, the y equations would be independent of the p_i .

In fact, in the $D=0$ limit (4.21a) and (4.21b) are simply the Hamiltonian equations for the original Hamiltonian $\hat{H}(\phi)$ in (4.2), except that the equation for \dot{p} has an additional term $-2\gamma p$. In this limit (4.21a) and (4.21b) are the characteristic equations of (4.12) and can be interpreted as classical trajectories in the presence of viscosity. Knowledge of these characteristics leads to a complete solution of Eqs. (4.21) for $D \neq 0$, but the concept of a classical trajectory is replaced by that of a Brownian trajectory resulting from momentum-space diffusion.

We proceed by eliminating y_1 from (4.21a) and (4.21b) which leads to

$$\ddot{y}_2 + (2\gamma + 3H)\dot{y}_2 - \omega^2 y_2 = 2Dp_1 e^{-3Ht}. \quad (4.22)$$

Similarly, elimination of p_1 in (4.21c) and (4.21d) gives

$$\ddot{p}_2 - (2\gamma + 3H)\dot{p}_2 - \omega^2 p_2 = 0. \quad (4.23)$$

At $D=0$ the solutions are linear combinations of $e^{-\Gamma_{\pm}t}$ for y_2 , and $e^{+\Gamma_{\pm}t}$ for p_2 , where

$$\Gamma_{\pm} = \gamma + \frac{3}{2}H \pm [(\gamma + \frac{3}{2}H)^2 + \omega^2]^{1/2}. \quad (4.24)$$

From these and the Green's function for (4.22), the necessary classical solutions are found. For example, y_2 has the form

$$y_2(t') = Y_2(t') + \int dt'' 2D(t'') p_1(t'') e^{2\gamma t''} G(t', t''; t), \quad (4.25)$$

where the homogeneous ($D=0$) solution is

$$Y_2 = \phi \frac{u(t')}{u(t)} + \phi' \frac{u(t'-t)}{u(-t)} \quad (4.26)$$

with

$$\begin{aligned} u(t') &= e^{-\Gamma_+ t'} - e^{-\Gamma_- t'}, \\ u(-t) &= -e^{(\Gamma_+ + \Gamma_-)t} u(t), \end{aligned} \quad (4.27)$$

and

$$\begin{aligned} G(t', t''; t) &= (\Gamma_+ - \Gamma_-)^{-1} u^{-1}(-t) \\ &\times [u(t')u(t''-t)\theta(t''-t') + (t' \leftrightarrow t'')] \end{aligned} \quad (4.28)$$

is the Green's functions vanishing at t' or $t''=0, t$. The function y_2 solves (4.22) with t' as the independent variable, subject to

$$y_2(t'=0) = \phi', \quad y_2(t'=t) = \phi. \quad (4.29)$$

The pseudomomentum p_1 is of the form

$$p_1 = A(t) e^{(\Gamma_+ - 3H)t'} + B(t) e^{(\Gamma_- - 3H)t'}, \quad (4.30)$$

where A, B are found by evaluating (4.25) and imposing the boundary conditions following from (4.21b) and (4.17):

$$\left. \frac{\partial}{\partial t'} y_2 \right|_{t'=0} = p', \quad e^{3Ht} \left. \frac{\partial}{\partial t'} y_2 \right|_{t'=t} = p. \quad (4.31)$$

[In the simple case at hand, where D depends exponentially on t'' via (4.10) and (4.13), an alternative to using

(4.25) is to write y_2 as the sum of four exponentials, with coefficients to be determined by solving four simultaneous linear equations. However, if D has a more complicated time dependence, (4.25) must be used.] All we need out of these extensive calculations are the coefficients A, B in (4.30), since the classical action S is given by

$$S = \int_0^t dt' \left[\sum p_i \dot{y}_i - \mathcal{H} \right] = \int_0^t dt' D p_1^2, \quad (4.32)$$

where the last equality uses (4.21a) and (4.21b). In terms of A and B , S is given by

$$\begin{aligned} S = 2\gamma T &\left[\frac{A^2}{2\Gamma_+ - 7H} (e^{(2\Gamma_+ - 7H)t} - 1) \right. \\ &+ \frac{2AB}{\Gamma_+ + \Gamma_- - 7H} (e^{(\Gamma_+ + \Gamma_- - 7H)t} - 1) \\ &\left. + \frac{B^2}{2\Gamma_- - 7H} (e^{(2\Gamma_- - 7H)t} - 1) \right]. \end{aligned} \quad (4.33)$$

Recall that T stands for the time-independent initial temperature. The coefficients A, B are given by the expressions

$$A = \frac{(\gamma - 2H)(2\Gamma_+ - 7H)}{\gamma T} \mathcal{A}_1, \quad (4.34a)$$

$$B = \frac{(\gamma - 2H)(2\Gamma_- - 7H)}{\gamma T} \mathcal{A}_2, \quad (4.34b)$$

where

$$\mathcal{A}_i = (Q + R\mathcal{W}^{-1})_{ij}^{-1} \mathcal{P}_j \quad (4.35)$$

and the vector \mathcal{P} is

$$\begin{aligned} \mathcal{P}_1 = p' - \frac{1}{u(t)} &[\phi(\Gamma_- - \Gamma_+) \\ &+ \phi'(\Gamma_+ e^{-\Gamma_- t} - \Gamma_- e^{-\Gamma_+ t})], \end{aligned} \quad (4.36a)$$

$$\begin{aligned} \mathcal{P}_2 = p - \frac{1}{u(t)} &[\phi(\Gamma_- e^{(3H - \Gamma_-)t} - \Gamma_+ e^{(3H - \Gamma_+)t}) \\ &+ \phi'(\Gamma_+ - \Gamma_-) e^{(3H - \Gamma_+ - \Gamma_-)t}]. \end{aligned} \quad (4.36b)$$

The matrices Q and R are

$$Q = \begin{pmatrix} \Gamma_+ - 7H & \Gamma_- - 7H \\ (\Gamma_+ - 7H)e^{(\Gamma_+ - 4H)t} & (\Gamma_- - 7H)e^{(\Gamma_- - 4H)t} \end{pmatrix}, \quad (4.37)$$

$$R = -u(-t) \begin{pmatrix} \Gamma_+ & \Gamma_- \\ \Gamma_+ e^{(3H - \Gamma_+)t} & \Gamma_- e^{(3H - \Gamma_-)t} \end{pmatrix}. \quad (4.38)$$

Finally, the matrix \mathcal{W} is given in the appendix, Eqs. (A2) and (A3). This appendix gives some details of the derivation of these expressions.

It only remains to solve the differential equation (4.20) for the normalization factor $Q(t)$. This is trivial in principle but messy in practice, and we do not write the lengthy and unilluminating formula here. In the con-

cluding section we discuss the interpretation and potential applications of these results.

V. DISCUSSION

Even though we have not had to deal with anything more difficult than a harmonic oscillator, the final formulas for the evolution of an upside-down oscillator are very complicated and hard to see through. Serious applications will depend on extensive numerical work. Still, it is useful to make some general comments.

Our results extend the simple upside-down oscillator¹⁰ in three different ways, governed by three parameters: (1) inclusion of a de Sitter-space background, with parameter H (Refs. 8 and 10); (2) inclusion of viscosity, governed by γ (3) inclusion of diffusion, governed by $D=2\gamma\hat{T}$. One or more of these effects may be negligible, depending on whether one is dealing with preinflation, inflation, or postinflation reheating.

The simplest case is $H=\gamma=D=0$, which is the Guth-Pi case. These authors discussed their results in terms of a delay time, showing that the upside-down quantum oscillator behaved classically at large times, but that the initial wave packet led to a distribution of classical trajectories parametrized by the time at which the classical oscillator started to roll. The Wigner-distribution approach we use leads through (4.21) to the following solution to (4.12) at $H=\gamma=D=0$:

$$W(p, \phi; t) = W_0(p', \phi'), \quad (5.1)$$

$$p' = p \cosh \omega t - \omega \phi \sinh \omega t, \quad (5.2a)$$

$$\omega \phi' = \omega \phi \cosh \omega t - p \sinh \omega t. \quad (5.2b)$$

Here W_0 is the $t=0$ Wigner function, and p', ϕ' are the $t=0$ values of p and ϕ ; they can be found from (4.36) by setting $\mathcal{P}_j=0$. From (5.2) one finds

$$(p \pm \omega \phi) = e^{\pm \omega t} (p' \pm \omega \phi'), \quad (5.3)$$

which shows that the classical result $p = \omega \phi$ holds, up to exponentially small terms at large t if the initial values p', ϕ' are bounded. It is now convenient to think of W_0 as a function of the variables $a^\pm \equiv p' \pm \omega \phi'$. Obviously the distribution of W_0 in a^+ can be thought of as a distribution of delay times in (5.3) by writing $p + \omega \phi = e^{\omega t} a^+ \equiv e^{\omega(t-t_0)}$. At the same time, as long as W_0 decreases rapidly for large a^- , $p - \omega \phi$ is forced toward the classical value of zero.

Similar remarks hold when H and/or γ are not zero but D is zero; the case $H \neq 0$ and $\gamma = 0$ has been discussed by Guth and Pi, for example, while $H = 0$ and $\gamma, D \neq 0$ has been studied by Caldeira and Leggett⁷ for a right-side-up oscillator; the upside-down oscillator at $D = H = 0$ is easily recovered from their results. In these cases the concepts of a classical trajectory [in the sense of a unique relation between p and ϕ , as determined by the $D=0$ solutions of (4.21a) and (4.21b)] and of a distribution of delay time are valid. We will return later to cases when both H and γ are nonzero.

Next, consider what happens when D , the thermal momentum-space diffusion constant, is not zero. The

simplest case is $H=0$, so the temperature is constant, and then the time-independent solution to (4.12) is

$$W(p, \phi) = \text{const} \times \exp \left[-\frac{1}{T} \left[\frac{1}{2} p^2 + V(\phi) \right] \right], \quad (5.4)$$

where $V(\phi)$ is the classical potential ($V = -\frac{1}{2} \omega^2 \phi^2$, in our case). As expected, W is Maxwellian. This time-independent solution is valid when $\gamma t \gg 1$, so that viscosity and diffusion have had plenty of time to act. Of course, this Maxwellian distribution makes sense only if the energy $\frac{1}{2} p^2 + V(\phi)$ is positive, which it is not for the unstable oscillator. In general, an unstable system will not approach any time-independent equilibrium.

After a certain amount of work, one can see from the explicit formulas of Sec. IV that the classical action S behaves like $D^{-1} F(\mathcal{P}_j)$ for small D , so that at small D the Wigner function has support only at the zeros of F . These zeros turn out to be $\mathcal{P}_j=0$ which, as we have already mentioned, yield the $D=0$ characteristics of Eq. (4.12). Thus, as expected, the solutions for small D join onto the characteristic classical trajectories. But for finite D the Wigner function is determined by Brownian trajectories, and it is not clear how to implement the Guth-Pi concept of a delay time. We can gain some insight into the dynamics when H, γ , and D all play a role by replacing the diffusion term of (4.12) by Langevin forces. This amounts to replacing the term $2Dp_1$ in (4.21a), (4.22), and (4.25) by $\epsilon(t)$, a random force with zero mean and (nonstationary) variance

$$\langle \epsilon(t) \epsilon(t') \rangle = 4\gamma T e^{-Ht} \delta(t-t'). \quad (5.5)$$

A solution for ϕ is thus

$$\begin{aligned} \phi(t) = (\Gamma_- - \Gamma_+)^{-1} & \left[(p' + \Gamma_- \phi') e^{-\Gamma_+ t} \right. \\ & - (p' + \Gamma_+ \phi') e^{-\Gamma_- t} \\ & \left. + \int dt' \epsilon(t') e^{-3Ht'} \tilde{G}(t, t') \right], \quad (5.6) \end{aligned}$$

$$\tilde{G}(t, t') = (e^{\Gamma_-(t-t')} - e^{\Gamma_+(t-t')}) \theta(t-t'), \quad (5.7)$$

which reduces to (5.2) at $H=D=0$. The corresponding momentum can be calculated using (4.21b). From these formulas various moments of ϕ and p can be calculated, which will indicate how diffusion modifies the classical characteristic solutions. We give one example: The dominant terms in $\langle \phi^2 \rangle$ at large t are given by

$$\begin{aligned} \langle \phi^2(t) \rangle \rightarrow (\Gamma_- - \Gamma_+)^{-2} e^{-2\Gamma_- t} & \left[(p' + \Gamma_+ \phi')^2 \right. \\ & \left. + \frac{4\gamma T}{7H - 2\Gamma_-} \right] \quad (5.8) \end{aligned}$$

[recall that, from (4.24), Γ_- is negative]. The exponential growth of the second moment is due to the unstable potential plus expansion, but is quantitatively modified by viscosity, which occurs in Γ_- , and by diffusion. Note that the $t=0$ temperature appears in (5.8).

Let us now ask: under what physical circumstances can H, γ , and D all be important parameters? First, con-

sider the initial conditions for inflation. We will suppose that in any case the thermal bath—consisting of all other fields except for the inflation field—has strong enough self-couplings so that it can be, and is, in thermal equilibrium prior to inflation. Usually it is also assumed that the inflation field Φ is similarly in thermal equilibrium, with a $+T^2\Phi^2$ term in the effective potential stabilizing the slow-rolling potential, and driving Φ near to zero. If this thermal equilibrium has come about because of interactions with the bath, then it is evident that H and γ must be of the same order. Note that this is not the same thing as saying that the rate of decay of the Φ field is comparable to H ; viscosity comes from collisions with the thermal bath, not from decay. It may, in fact, be possible to construct scenarios where $H \sim \gamma$ in the early stages of inflation. Suppose that the slow-rolling potential is quartic (e.g., Ref. 19) instead of quadratic as we have so far used in this paper:

$$V(\Phi) = -\frac{1}{4}\lambda(\Phi^4 - \Phi_0^4). \quad (5.9)$$

Then at small time the Hubble constant H is

$$H = \left[\frac{2\pi\lambda}{3} \right]^{1/2} \frac{\Phi_0^2}{M_p}, \quad (5.10)$$

where $\Phi_0 \simeq M_p$, the Planck mass, at early times. Thus $H = O(\lambda^{1/2}M_p)$ at early times. Next suppose that the Φ -bath coupling is quadratic in the bath field ψ , as in (2.12):

$$\mathcal{L}_{\text{int}} = g\Phi\psi^2. \quad (5.11)$$

The ψ field induces a Φ^4 coupling, of strength $\sim g^4$. This must be of order λ , so $g \sim \lambda^{1/4}$. From (4.7), the viscosity coefficient $\gamma \sim g^2$, so $\gamma \sim \lambda^{1/2}$ times a mass. At the earliest times, one expects the mass to be determined by the Planck mass, that is, $\gamma \sim M_p \lambda^{1/2} \sim H$. Finally, in the reheating phase^{20,21} one expects both γ - and Φ -decay rates to be comparable to or even much larger than H , which decreases as Φ approaches its minimum. When decay is important a term $-\Gamma W$ should be added to the RHS of (4.12), the evolution equation for the Wigner function. During reheating the temperature is changing rapidly, but (4.12) can still be solved if H is not too rapidly changing, with the help of the Green's function (4.25). In this phase a right-side up oscillator potential is appropriate.

In summary, we have found exact solutions to some idealized but still physically interesting problems of the evolution of unstable fields coupled to a thermal bath in de Sitter space. Applications of these solutions to the preinflationary universe, to density fluctuations, and to reheating are now under investigation.

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APPENDIX

We give here a few details of the derivation of (4.35) to (4.39). Equation (4.26) turns out to be

$$y_2 = Y_2 + X_i \mathcal{W}_{ij} \alpha_j + \alpha_i Z_j, \quad (A1)$$

where repeated indices are summed over. The matrix \mathcal{W}_{ij} and the vector α_i ($i, j = 1, 2$) depend only on t , and \mathcal{W}_{ij} is given by

$$\mathcal{W}_{ij}(t) = \begin{bmatrix} W_{-+} & W_{--} \\ -W_{++} & -W_{+-} \end{bmatrix}, \quad (A2)$$

where

$$W_{AB}(t) = e^{\Gamma_A t} - e^{(\Gamma_B - \Gamma_A)t} \quad (A, B = \pm). \quad (A3)$$

The vector α_i is related to the integration constants A, B of (4.30) by

$$\mathcal{W}_{ij}(t)\alpha_j = \mathcal{A}_i, \quad (A4)$$

$$\mathcal{A}_1 = \frac{\gamma T A}{(\gamma - 2H)(2\Gamma_+ - 7H)}, \quad (A5)$$

$$\mathcal{A}_2 = \frac{\gamma T B}{(\gamma - 2H)(2\Gamma_- - 7H)},$$

as one finds by using the following definitions for X_i, Z_i and substituting in either (4.26) or the equation of motion (4.23) for Y_2 :

$$X_i = \begin{bmatrix} e^{(\Gamma_+ - 7H)t'} \\ e^{(\Gamma_- - 7H)t'} \end{bmatrix}, \quad Z_i = u(-t) \begin{bmatrix} e^{-\Gamma_+ t'} \\ e^{-\Gamma_- t'} \end{bmatrix}, \quad (A6)$$

with $u(t)$ given in (4.27). The solution to (A4),

$$\alpha_i = \mathcal{W}_{ij}^{-1}(t)\mathcal{A}_j \quad (A7)$$

can be written in terms of $\mathcal{W}(-t)$ using

$$\mathcal{W}^{-1}(t) = -[4 \sinh^2 \frac{1}{2}(\Gamma_+ - \Gamma_-)t]^{-1} \mathcal{W}(-t). \quad (A8)$$

Thus (A1) is

$$y_2 = Y_2 + Z_i \mathcal{W}_{ij}^{-1} \mathcal{A}_j + X_i \mathcal{A}_i. \quad (A9)$$

Now the \mathcal{A}_i (or equivalently the α_i) are determined by imposing the boundary conditions (4.31). This gives

$$p' - \frac{\partial Y_2}{\partial t'} \Big|_{t'=0} = \mathcal{W}_{ij}^{-1}(t)\mathcal{A}_j(t) \frac{\partial Z_i}{\partial t'} + \mathcal{A}_i(t) \frac{\partial X_i}{\partial t'} \Big|_{t'=0}, \quad (A10)$$

$$p - e^{3Ht} \frac{\partial Y_2}{\partial t'} \Big|_{t'=t} = e^{3Ht} \left[\mathcal{W}_{ij}^{-1}(t)\mathcal{A}_j(t) \frac{\partial Z_i}{\partial t'} + \mathcal{A}_i(t) \frac{\partial X_i}{\partial t'} \right] \Big|_{t'=t}. \quad (A11)$$

These are easily converted into the form given in the text.

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