Third quantization and the Wheeler-DeWitt equation

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Beginning with a proposal for the normalization of solutions to the Wheeler-DeWitt equation put forth by DeWitt we argue that the Wheeler-DeWitt equation naturally lends itself to a second quantization in analogy to the second quantization of the Klein-Gordon equation. We identify a conserved current, as well as DeWitt's proposal for normalization, as coming from a Lagrangian which is the analog of a second-quantized string theory whose spatial coordinates parametrize the coset manifold SL(3,R)/SO(3). We derive a mode decomposition of the second-quantized Wheeler-DeWitt field in the linearized approximation to quantum gravity, the zero modes of which are given by the total three-volume as well as various anisotropy parameters. We discuss the possibility of adding topological interactions for the linearized theory and find a representation in terms of vertex operators. In a two-dimensional setting we discuss a connection between our formalism and a proposal by Green which may shed light on some of the interpretational problems of string theory.

I. INTRODUCTION

Recently there has been renewed interest in the quantum theory of gravity with emphasis on topology-changing processes.[1-3] All such inquiries are eventually based on the Wheeler-DeWitt equation. In the pioneering work of DeWitt[4] it was noticed that the Wheeler-DeWitt (WDW) equation is a hyperbolic differential equation of signature $(-+++++)$ over the space of three-metrics γ_{ij}, superspace, and $\gamma^{1/2}=\sqrt{\det\gamma}$ was a timelike coordinate. Because of the fact that the WDW equation is of second order in its derivative with respect to this timelike variable, the equation is of the Klein-Gordon type and led DeWitt to a natural definition of the inner product of two of the solutions to this equation (that is, consistent with the invariances of the WDW equation itself) this in turn led to a problem with negative probabilities. Although in certain cases this problem was not insurmountable, the resolution in the ordinary Klein-Gordon equation (which can be interpreted as the WDW equation for one-dimensional gravity coupled to four scalar fields) of the negative-energy and negative-probability problems lead to the reformulation of the theory in a many-body context. Then the inner product

$$(\phi_1,\phi_2)=i\int d^3x(\phi_1^*\partial_0\phi_2-\phi_2^*\partial_0\phi_1) \tag{1.1}$$

becomes reinterpreted as inheriting its invariance from a charge $Q=i\int d^3x(\phi^*\partial_0\phi-\phi\partial_0\phi^*)$ which is associated with a global U(1) symmetry. Also one can define a current density which is conserved, and a Lagrangian from which all this follows. The negative norm becomes reinterpreted as the charge associated with antiparticles. We shall follow the same procedure with the Wheeler-DeWitt equation. The analogy with the Klein-Gordon theory is not perfect, however, and we shall highlight some of the differences. A somewhat different motivation for second quantizing the WDW equation comes from extending what we know about second quantizing particle theory (one-dimensional gravity plus matter) and string theory (two-dimensional gravity plus matter) to the four-dimensional case. (The recent work of Banks[5] follows this line of thought.) We shall touch upon that argument in Appendix C.

This paper is organized as follows. In Sec. II we review the Wheeler-DeWitt equation and the natural inner product which can be formed between its solutions. We shall construct a current density and a Lagrangian from which it follows by considering global U(1) invariance. The Lagrangian is that of a scalar-field theory taking its values on superspace. The indefinite nature of the metric on superspace led to the problem of negative probabilities referred to above. In Sec. III we derive an analogy between second quantizing the WDW equation and second quantizing a string whose spatial coordinates are constrained to move on a coset manifold. In the first case the WDW field Φ maps

$$\Phi:[\mathrm{map}(S^1\times S^1\times S^1\to R\times \mathrm{SL}(3,R)/\mathrm{SO}(3))]\to C \tag{1.2}$$

while when second quantizing a string which moves on a group or coset manifold one needs a field which maps

$$\Phi:[\mathrm{map}(S^1\to R^d\times G/H)]\to C\ . \tag{1.3}$$

We specialize to $S^1\times S^1\times S^1$ spatial topology because it is closest to the S^1 spatial topology used in closed-string theories and a great deal is known about how to second quantize this system and to normalize the states. Also $k=1$ S^3 spatial topology is treated by Hawking in Ref. 1.

In order to understand the physical interpretation of our second-quantized Wheeler-DeWitt field we develop a mode expansion in terms of creation and annihilation operators. In doing so we shall need a complete set of solutions to the WDW equation orthonormal with respect to the above-mentioned inner product. The fields

$\gamma^{1/2}(\mathbf{x})$ and $\gamma_{ij}(\mathbf{x})$ living on the three-manifold naturally separate into a piece which is independent of $\mathbf{x}$, which will be called zero modes, and a piece which depends on higher oscillations of the fields. The terminology is inherited from string theory where one separates the fields $X^{\mu}(\sigma)$ into a piece which is independent of σ, x_0^{μ}, and a piece which depends on the oscillating modes X_n^{μ}. We shall show that the portion of the WDW equation that depends on the zero modes is equivalent to a minisuperspace model with an effective cosmological term generated by the oscillating modes. We discuss in detail the structure of this minisuperspace model as most of the qualitative features of our second-quantized WDW field are contained in the zero modes. In the space $R\times SL(3,R)/SO(3)$ the zero modes of the first factor R are associated with the total three-volume, and those of the second factor $SL(3,R)/SO(3)$ give rise to various anisotropy parameters. The advantage of going to such a formalism is that we no longer have to restrict ourselves to single-universe states. By operating with creation operators on the so-called third-quantized vacuum (the vacuum state of our second-quantized WDW equation) we can construct multiuniverse states or even coherent states of universes discussed by Coleman.[3] In addition it appears that such states are necessary to form a complete Hilbert space. These states have physical applications with regard to the anthropic principle[6] as well as to the cosmological-constant problem.[5-7] Finally in Sec. IV we discuss the possibility that one can include interactions in our second-quantized WDW equation in order to generate certain topology-changing effects.

II. THE HYPERBOLIC NATURE OF THE WHEELER-DeWITT EQUATION

When one tries to extend ideas with which we have intuition in one and two dimensions to gravity in $D\geq 4$ one must realize that one is generalizing from a system which is usually taken to be noninteracting on the world sheet to one in which we certainly have interactions on the world volume. As yet, for gravity the best we can do is to expand the metric about flat space and treat the interactions perturbatively. By adding an infinite number of matter fields to gravity with such content that all matter fields plus gravity can be encoded into one string field, string theory has been remarkably successful in describing the interactions of gravity perturbatively. So when considering gravity plus matter in $D\geq 4$ dimensions we assume that the path integral

$$Z(0)=\int D\phi^a(x)Dg_{\mu\nu}(x)\exp\left[i\int d^4x\sqrt{-g}\,({}^{(4)}R+g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^a)\right] \tag{2.1}$$

has an appropriate generalization with an infinite number of matter fields to a string model which can give (2.1) a well-defined meaning perturbatively. The action has the symmetry $g_{\mu\nu}(x)\rightarrow g_{\mu\nu}(x)+\nabla_\mu\epsilon_\nu+\nabla_\nu\epsilon_\mu$ associated with the reparametrization of the world volume and leads to four constraints for which g_{00} and g_{0_i} act as Lagrange multipliers. They are[4]

$$\begin{aligned} H&=\tfrac{1}{2}\gamma^{1/2}(\gamma_{ik}\gamma_{jl}+\gamma_{il}\gamma_{jk}-\gamma_{ij}\gamma_{kl})\pi^{ij}\pi^{kl}\\ &\quad+(-\gamma^{1/2\,(3)}R)=0\ ,\\ \chi^i&=-2\pi^{ij}{}_{;j}\\ &=-2\pi^{ij}{}_{,j}-\gamma^{il}(2\gamma_{jl,k}-\gamma_{jk,l})\pi^{jk}=0\ , \end{aligned} \tag{2.2}$$

where $\gamma_{ij}=g_{ij}$ the remaining spatial components of the metric and π^{ij} are the canonical momentum conjugate to it. Again $\gamma=\det(\gamma_{ij})$.

The constraints become conditions on the wave function in the usual tradition of quantum gravity:

$$\chi^i\psi=2i\left[\frac{\delta}{\delta\gamma_{ij}}\psi\right]_{;j}=0 \tag{2.3}$$

with dynamics determined by

$$\begin{aligned} H\psi&=\tfrac{1}{2}\gamma^{1/2}(\gamma_{ik}\gamma_{jl}+\gamma_{il}\gamma_{jk}-\gamma_{ij}\gamma_{kl})\frac{\delta}{\delta\gamma_{ij}}\frac{\delta}{\delta\gamma_{kl}}\psi\\ &\quad+(-\gamma^{1/2\,(3)}R)\psi=0\ . \end{aligned} \tag{2.4}$$

The above equation is the Wheeler-DeWitt (WDW) equation. The "metric" for this Klein-Gordon-type equation $G_{ijkl}=\frac{1}{2}\gamma^{1/2}(\gamma_{ik}\gamma_{jl}+\gamma_{il}\gamma_{jk}-\gamma_{ij}\gamma_{kl})$ is indefinite of index $(-+++++)$. This caused concern for DeWitt who realized this fact naturally leads to a problem with negative probabilities.[4] One can change variables to make the indefinite nature of (G^{ijkl}) more explicit. Take as variables $\gamma^{1/2}$ and $\tilde{\gamma}_{ij}$ where $\det(\tilde{\gamma}_{ij})=1$ so that we replace $(\gamma_{ij})\rightarrow(\gamma^{1/2},\tilde{\gamma}_{ij})$ through the relation $\gamma_{ij}=\gamma^{1/2}\tilde{\gamma}_{ij}$. It is straightforward to verify that the canonical momentum conjugate to these variables are given by

$$\pi=\tfrac{4}{3}K=\tfrac{4}{3}(\tfrac{1}{2}\gamma^{-1/2}\pi^{ik}\gamma_{ij})$$

and

$$\tilde{\pi}^{ij}=(\pi^{ij}-\tfrac{1}{3}\gamma^{ij}\pi^k k)\gamma^{1/3}\ . \tag{2.5}$$

In terms of these variables H becomes

$$H=-\tfrac{3}{8}\gamma^{1/2}\pi^2+\gamma^{-1/2}\tilde{\gamma}_{ij}\tilde{\gamma}_{kl}\tilde{\pi}^{ik}\tilde{\pi}^{jl}+(-\gamma^{1/2\,(3)}R) \tag{2.6}$$

and the indefinite nature is manifest. For "particle" theory the analogous expression is

$$H=-(\pi^0)^2+G_{ij}(x)\pi^i\pi^j+m^2\ , \tag{2.7}$$

where we have allowed the particle to move on a curved surface. Thus one may suspect that the proper treatment of the wave functions which solve (2.4) should be similar to the treatment of wave functions associated with particles constrained to move on a surface. The original motivation for this paper was to see in what sense this is true. In our case the inverse of this background metric is given by $G_{ijkl}=\frac{1}{2}\gamma^{-1/2}(\gamma_{ik}\gamma_{jl}+\gamma_{il}\gamma_{jk}-\gamma_{ij}\gamma_{kl})$. H is invariant under SL$(3,R)$ group multiplication and the set $\{\tilde{\gamma}_{ij}\}$ becomes identified with the coset space $M=\mathrm{SL}(3,R)/\mathrm{SO}(3)$ which is five dimensional. The inner product introduced by DeWitt[4],

$$(\psi_b,\psi_a)=Z\int_\Sigma \psi_b^* \prod_{\mathbf{x}} \left[D\Sigma^{ij}G_{ijkl}\frac{\overrightarrow{\delta}}{\delta\gamma_{kl}} - \frac{\overleftarrow{\delta}}{\delta\gamma_{kl}}G_{ijkl}D\Sigma^{ij}\right]\psi_a \tag{2.8}$$

[where $\prod_{\mathbf{x}}D\Sigma^{ij}(\mathbf{x})$ denotes a topological product of the set of five-dimensional hypersurfaces in the superspace], respects the global symmetries of H. A difficulty with the above definition is that negative-frequency components with respect to the timelike variable $\gamma^{1/2}$ can make the above norm negative. If one restics oneself to positive-frequency components then ψ can vanish nowhere in the range $-\infty<\gamma^{1/2}<\infty$. However, one wishes to have the wave function vanish for $\gamma^{1/2}<0$, so that one is forced to admit negative frequencies and must worry about negative probablites. The above inner product can be given a natural interpretation by constructing the Lagrangian density in superspace

$$L=\tfrac{1}{2}[-G]^{1/2}\Phi^*[\Box-(-\gamma^{1/2}\,{}^{(3)}R)]\Phi\ , \tag{2.9}$$

where

$$\Box\Phi=(-G)^{-1/2}\frac{\delta}{\delta\gamma_{ij}}\left[(-G)^{1/2}G_{ijkl}\frac{\delta}{\delta\gamma_{kl}}\Phi\right]$$

is the Laplacian in the presence of our background field and $G=\det(G^{ijkl})$ where G is expressed as a 6×6 matrix. Up to a total divergence in superspace we can express L as

$$L=-\tfrac{1}{2}(-G)^{1/2}G_{ijkl}\frac{\delta}{\delta\gamma_{ij}}\Phi^*\frac{\delta}{\delta\gamma_{kl}}\Phi -\tfrac{1}{2}(-\gamma^{1/2}\,{}^{(3)}R)\Phi^*\Phi\ . \tag{2.10}$$

The Lagrangian has the symmetry $\Phi\to e^{i\theta}\Phi$. Because of this symmetry a conserved current can be derived and is given by

$$J_{ij}=\sqrt{-G}\ G_{ijkl}\left[\Phi^*\frac{\delta}{\delta\gamma_{kl}}\Phi-\Phi\frac{\delta}{\delta\gamma_{kl}}\Phi^*\right]. \tag{2.11}$$

It can readily be verified that $(\delta/\delta\gamma_{ij})J_{ij}=0$ if Φ solves the WDW equation. One can define a conserved charge with respect to a timelike displacement in superspace by forming

$$\begin{aligned}Q&=\int D\Sigma_{ij}G^{ijkl}J_{kl}\\ &=i\int D\Sigma_{ij}\left[\Phi^*\frac{\delta}{\delta\gamma_{ij}}\Phi-\Phi\frac{\delta}{\delta\gamma_{ij}}\Phi^*\right].\end{aligned} \tag{2.12}$$

Now this charge is invariant under global rotations in (γ_{ij}) space just as the corresponding quantity in Klein-Gordon theory is invariant under Lorentz transformations. We can then naturally reinterpret DeWitt's inner product in terms of a conserved charge derived from the Lagrangian L. In terms of the decomposition $(\gamma_{ij})\to(\gamma^{1/2},\tilde{\gamma}_{ij})$ we obtain

$$Q=\int D\tilde{\gamma}_{ij}(2\gamma)^{-1/2}\left[\left[\frac{\delta}{\delta\gamma^{1/2}}\Phi^*\right]\Phi-\Phi^*\left[\frac{\delta}{\delta\gamma^{1/2}}\Phi\right]\right], \tag{2.13}$$

where we have used the fact that $(-G)^{1/2}=[-\det(G^{ijkl})]^{1/2}=(2\gamma)^{-1/2}$ and we have chosen a spacelike surface $\gamma^{1/2}=$const. The WDW equation can be obtained by varying (2.10) with respect to Φ and one obtains the form

$$\begin{aligned}\Box\Phi&=(-G)^{-1/2}\frac{\delta}{\delta\gamma_{ij}}\left[(-G)^{1/2}G_{ijkl}\frac{\delta}{\delta\gamma_{kl}}\Phi\right]\\ &=-(-\gamma^{1/2}\,{}^{(3)}R)\Phi\ .\end{aligned} \tag{2.14}$$

The quantity $(-\gamma^{1/2}\,{}^{(3)}R)$ contributes as an effective $(\text{mass})^2$ term for our Wheeler-DeWitt field. This is analogous to the role played by $[(\partial/\partial\sigma)X^\mu]^2$ in string theory where one imposes the condition $-(\delta/\delta X^\mu)(\delta/\delta X_\mu)\Psi=[(\partial/\partial\sigma)X^\mu]^2\Psi$ associated with timelike reparametrizations of the world surface.

III. THE MODE EXPANSION FOR THE WHEELER-DeWITT FIELD

In this section we develop a mode expansion for our WDW field. Given the normalization

$$(\phi_m,\phi_n)=i\int D\Sigma_{ij}\left[\phi_m^*\frac{\delta}{\delta\gamma_{ij}}\phi_n-\phi_n^*\frac{\delta}{\delta\gamma_{ij}}\phi_m\right] \tag{3.1}$$

and WDW equation in the form

$$\begin{aligned}\Box\phi_m&=(-G)^{-1/2}\frac{\delta}{\delta\gamma_{ij}}\left[(-G)^{1/2}G_{ijkl}\frac{\delta}{\delta\gamma_{kl}}\phi_m\right]\\ &=-(-\gamma^{1/2\,(3)}R)\phi_m\end{aligned} \tag{3.2}$$

one can expand Φ in a set of modes if one has a complete set of solutions ψ_n to (3.2) which are orthonormal with respect to (3.1). This can easily be done in the case of the particle or the string but the presence of the metric G^{ijkl} as well as the term $(-\gamma^{1/2\,(3)}R)$ which is not quadratic in the γ_{ij} or its derivatives will prevent us from finding such solutions here. Actually both problems do not occur in the linearized (noninteracting) approximation to gravity and in that case we can explicitly construct ψ_n and develop an expansion

$$\Phi(\gamma_{ij})=\sum_n \psi_n(\gamma_{ij})A_n+\sum_n \psi_n^*(\gamma_{ij})A_n^\dagger\ , \tag{3.3}$$

where the index n symbolizes the particular set of quantum numbers associated with that state. For example, in the linearized form of gravity they are occupation numbers for gravitons with certain values of the spacelike momentum. For particle theory in flat space the complete set is given by $\psi_{\mathbf{k}}(x)=(2\omega_{\mathbf{k}}V)^{-1/2}\exp(i\mathbf{k}\cdot\mathbf{x}-i\omega_{\mathbf{k}})$ where $\omega_{\mathbf{k}}=(\mathbf{k}^2+m^2)^{1/2}$ and V the volume in $\mathbf{x}$ space. We need to find such a set for our system. The presence of the background metric G^{ijkl} over superspace provides only a minor obstacle as, for example, one can quantize a particle or a string on a group or coset manifold.[8,9] In this case the relevant manifold has been identified by DeWitt[4] and the coordinates $\tilde{\gamma}_{ij}$ parametrize the coset manifold SL(3,R)/SO(3). Our second-quantized WDW field maps $(\gamma^{1/2},\tilde{\gamma}_{ij})$ into C the complex numbers. That is,

$$\Phi:[\text{map}(S^1\times S^1\times S^1\to R\times \text{SL}(3,R)/\text{SO}(3))]\to C \ . \tag{3.4}$$

This is similar to the problem of second quantizing a string which moves on a group or coset manifold

$$\Phi:[\text{map}(S^1\to R^d\times(G/H))]\to C \tag{3.5}$$

or in the usual notation (3.4) is expressed as $\Phi(\gamma^{1/2}(\mathbf{x}),\tilde{\gamma}_{ij}(\mathbf{x}))$ and (3.5) is expressed as $\Phi(X^0(\sigma),X^i(\sigma),g^a(\sigma))$. So our problem is analogous to second quantizing a string that moves with its spatial coordinates restricted to SL(3,R)/SO(3) a five-dimensional space. The timelike coordinate $\gamma^{1/2}$ is all that is left of the Minkowski-space factor [the R^d in (3.5)] but additional matter fields will add additional spatial coordinates to the superspace. We shall add a single scalar field ϕ for illustrative purposes.

Following Ref. 1 we expand our WDW field as

$$\Phi(\gamma^{1/2},\tilde{\gamma}_{ij},\phi)=\sum_{(n)}\psi^0_{(n)}(\gamma_0^{1/2},\tilde{\gamma}_{ij0},\phi_0)\psi^{\text{osc}}_{(n)}(\gamma^{1/2}(\mathbf{k}),\tilde{\gamma}_{ij}(\mathbf{k}),\phi(\mathbf{k}))A_{(n)} + \sum_{(n)}\psi^{0*}_{(n)}(\gamma_0^{1/2},\tilde{\gamma}_{ij0},\phi_0)\psi^{\text{osc}*}_{(n)}(\gamma^{1/2}(\mathbf{k}),\tilde{\gamma}_{ij}(\mathbf{k}),\phi(\mathbf{k}))A^\dagger_{(n)} \ , \tag{3.6}$$

where we have separated off the zero modes, $\mathbf{x}$-independent portions of $\gamma^{1/2},\tilde{\gamma}_{ij}$, and ϕ. We further wish to isolate the dependence of the wave function which describes the physical components of the graviton as was done in Ref. 1 for the case of $k=1$ S^3 spatial topology. In our case we have chosen the $k=0$ $S^1\times S^1\times S^1$ spatial topology as it is somewhat simpler and more straightforward to generalize to a string model where one has an infinite tower of matter fields. Because of the boundary conditions on the $S^1\times S^1\times S^1$ spatial topology the scalar field (as well as all other fields) can be Fourier expanded as

$$\phi(\mathbf{x})=a^{-3}\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\sum_{n_3=-\infty}^{\infty}\phi(\mathbf{n})e^{i\mathbf{n}\cdot\mathbf{x}/a} \approx \int\frac{d^3k}{(2\pi)^3}\phi(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} \ , \tag{3.7}$$

where we have assumed that the radius a is so much larger than any length scale inherent in $\phi(k)$ that we can approximate the sum by an integral, that is, the allowed values of the momenta become approximately continuous. We develop the following expansions for $\gamma^{1/2},\tilde{\gamma}_{ij}$, and ϕ:

$$\begin{aligned}
\gamma^{1/2}(\mathbf{x})&=\int\frac{d^3k}{(2\pi)^3}\gamma^{1/2}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} \ \text{and}\ \gamma^{1/2}(\mathbf{k})=\int d^3x\ \gamma^{1/2}(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}} \ ,\\
\tilde{\gamma}_{ij}(\mathbf{x})&=\int\frac{d^3k}{(2\pi)^3}\tilde{\gamma}_{ij}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} \ \text{and}\ \gamma^{1/2}(\mathbf{k})=\int d^3x\ \tilde{\gamma}_{ij}(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}} \ ,\\
\phi(\mathbf{x})&=\int\frac{d^3k}{(2\pi)^3}\phi(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} \ \text{and}\ \phi(\mathbf{k})=\int d^3x\ \phi(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}} \ .
\end{aligned} \tag{3.8}$$

The zero modes, $\mathbf{x}$-independent portions of the fields, are defined by $\gamma_0^{1/2}=\gamma^{1/2}(\mathbf{k}=0)$, $\tilde{\gamma}_{ij0}=\tilde{\gamma}_{ij}(\mathbf{k}=0)$, and $\phi_0=\phi(\mathbf{k}=0)$. These portions of the fields do not oscillate spatially. Now clearly $\gamma_0^{1/2}=\int d^3x\,\gamma^{1/2}(\mathbf{x})$ is the total volume of the system. We shall denote it by a^3. The zero modes of $\tilde{\gamma}_{ij}$ are associated with the anisotropy parameters of the three-space. These are conventionally denoted by $\tilde{\gamma}_{ij0}=(e^{2\beta/\sqrt{6}})_{ij}$ with $\text{tr}(\beta)=0$. So our zero-mode metric is given by

$$\gamma_{ij0}=\gamma_0^{1/3}\tilde{\gamma}_{ij0}=a^2(e^{2\beta/\sqrt{6}})_{ij} \ \text{with}\ \text{tr}(\beta)=0 \tag{3.9}$$

and we write the zero-mode portion of the wave function as $\psi^0(\gamma_0^{1/2},\tilde{\gamma}_{ij0})=\psi^0(a,\beta_{ij},\phi_0)$.

Now let us turn to the portion of our solution to the WDW equation associated with the oscillating modes of $\gamma^{1/2},\tilde{\gamma}_{ij}$ and ϕ. As we expect $\psi^{\text{osc}}_{(n)}$ to give us the wave function associated with $n(\mathbf{k})$ gravitons of momenta $\mathbf{k}$, we keep only the portions of $\gamma^{1/2}(\mathbf{k}),\tilde{\gamma}_{ij}(\mathbf{k})$ associated with physical gravitons. Setting $\tilde{\gamma}_{ij}=(e^h)_{ij}$ these will be the transverse traceless components of h_{ij}. We can decompose $h_{ij}(\mathbf{x})$ into transverse and longitudinal parts (it must be traceless from the condition $\det\tilde{\gamma}=1$) as $h_{ij}=h_{ij}^{TT}+(\partial_ih_j+\partial_jh_i)-\frac{2}{3}(\partial_kh_k)\delta_{ij}$ (Refs. 10 and 11). So we will transform the oscillating portion of $\tilde{\gamma}_{ij}\to(h_{ij}^{TT},h_k)$. Now h_{ij}^{TT} has two components—the physical degrees of freedom of the graviton. Keeping only these degrees of freedom in the weak-field approximation, as we are for the moment interested in constructing the wave function in an asymptotic region of superspace, we can find their contribution to H. So we rewrite our fields as

$$\begin{aligned}
\phi(\mathbf{x}) &= \phi + \int \frac{d^3k}{(2\pi)^3} q(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} , \\
\pi_\phi(\mathbf{x}) &= a^{-3}\pi_\phi + \int \frac{d^3k}{(2\pi)^3} p(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} , \\
\gamma^{1/2}(\mathbf{x}) &= a^3 + \int \frac{d^3k}{(2\pi)^3} \gamma^{1/2}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} , \\
\pi(\mathbf{x}) &= a^{-3}\pi_a + \int \frac{d^3k}{(2\pi)^3} \gamma^{1/2}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} , \\
\tilde{\gamma}_{ij}(\mathbf{x}) &= (e^{2\beta/\sqrt{6}})_{ij} + \int \frac{d^3k}{(2\pi)^3} q_{ij}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} , \\
\tilde{\pi}^{ij}(\mathbf{x}) &= a^{-3}\tilde{\pi}^{ij}_0 + \int \frac{d^3k}{(2\pi)^3} p^{ij}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} ,
\end{aligned} \tag{3.10}$$

where we have listed our fields and below them their canonical conjugate momenta. We shall denote the Planck length l_P as l. In terms of these variables the action $S=\int d^4x\sqrt{-g}\,(l^{-2}\,{}^{(4)}R + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2)$ becomes $S=\int dt\,(L_{0G}+L_{0M}+L_{\rm osc}) = S_{0G}+S_{0M}+S_{\rm osc}$, where

$$\begin{aligned}
S_{0G} &= \frac{1}{l^2}\frac{1}{2}\int dt\, ca^3 \left[-\frac{\dot{a}^2}{c^2a^2} + \frac{24}{c^2}\gamma^{ij}_0\gamma^{kl}_0\dot{\gamma}_{ik0}\dot{\gamma}_{jl0} + \frac{k}{a^2}[1-V(\gamma_{ij0})] \right] , \\
S_{0M} &= \tfrac{1}{2}\int dt\, ca^3 \left[\frac{\dot{\phi}^2}{c^2} - m^2\phi^2 \right] , \\
S_{\rm osc} &= \int dt\, c\tfrac{1}{4}\int d^3x\,[h^{TT}_{ik},{}_0 h^{TT}_{ik},{}_0 - h^{TT}_{ik},{}_l h^{TT}_{ik},{}_l + 2(\phi,{}_0\phi,{}_0 - \phi,{}_i\phi,{}_i - m^2\phi^2)] ,
\end{aligned} \tag{3.11}$$

and we have set $c=g_{00}$ which will become the Lagrange multiplier for the constraint $H=0$. The function $V(\gamma_{ij0})$ represents the complicated dependence of ${}^{(3)}R$ on the anisotropy. Note that we have not included a cosmological constant term in S_{0G}. Such a term will be generated by quantum fluctuations in the oscillating fields. The canonical momenta are obtained from the above Lagrangian in the usual way. Further simplification results if one passes over to momentum space for the oscillating modes and defines $a(\mathbf{k})$ and $a^{ij}(\mathbf{k})$ by

$$\begin{aligned}
q(\mathbf{k}) &= \tfrac{1}{2}[a(\mathbf{k})+a^\dagger(-\mathbf{k})](2/\omega_k)^{1/2}a^{3/2} , \\
p(\mathbf{k}) &= \frac{1}{2i}[a(\mathbf{k})-a^\dagger(-\mathbf{k})](2\omega_k)^{1/2}a^{3/2} , \\
q^{ij}(\mathbf{k}) &= \tfrac{1}{2}[a^{ij}(\mathbf{k})+a^{ij\dagger}(-\mathbf{k})](2/\omega_k)^{1/2}a^{3/2} , \\
p^{ij}(\mathbf{k}) &= \frac{1}{2i}[a^{ij}(\mathbf{k})-a^{ij\dagger}(-\mathbf{k})](2\omega_k)^{1/2}a^{3/2} .
\end{aligned} \tag{3.12}$$

For these variables the Hamiltonian derived from (3.11) equals

$$H = H_{0G}+H_{0M}+H_{\rm osc} , \tag{3.13}$$

where

$$\begin{aligned}
H_{0G} &= \frac{1}{l^2}c\left[-\pi_a^2\frac{l^4}{2a} + \frac{1}{2a^3}24\tilde{\gamma}_{0ik}\tilde{\gamma}_{0jl}\tilde{\pi}^{ij}_0\tilde{\pi}^{kl}_0 - a\frac{k}{2}[1-V(\beta_{ij})] \right] , \\
H_{0M} &= c\left[\pi_\phi^2\frac{1}{2a^3} + \frac{m^2}{2}\phi^2a^3 \right] , \\
H_{\rm osc} &= ca^3\int\frac{d^3k}{(2\pi)^3}\{ |k| [a^{ij\dagger}(\mathbf{k})a^{ij}(\mathbf{k})+\tfrac{1}{2}] + \sqrt{k^2+m^2}[a^\dagger(\mathbf{k})a(\mathbf{k})+\tfrac{1}{2}]\} .
\end{aligned} \tag{3.14}$$

So in the linearized form we obtain the separation $H=H_{0G}+H_{0M}+H_{\rm osc}$. Wave functions are constrained by $H\psi=0$ [analogous to the $(L_0+\tilde{L}_0-2)\psi=0$ condition in string theory] and the Hamiltonian is naturally separated into a piece involving the zero modes and those involving the oscillators [as $L_0+\tilde{L}_0-2=p^\mu p_\mu+(N+\tilde{N})-2$ in string theory]. The main complication is that the radius a is treated as a quantum variable and couples to every term including the oscillators. This keeps one from being able to separate variables. However, under the condition that the variable a changes only adiabatically[10] the oscillating portion of the wave function retains its form with frequencies evaluated as $\omega_{n/a}=[(n/a)^2+m^2]^{1/2}$ and the WDW equation effectively separates. The WDW equation $H\psi=0$ reduces to the equations

$$a^3\left[-\frac{l^2}{a^4}\pi_a^2 - \frac{k}{l^2a^2}[1-V(\tilde{\gamma}_{ij0})] + 24\tilde{\gamma}_{ij0}\tilde{\pi}^{jk}_0\tilde{\gamma}_{kn0}\tilde{\pi}^{ni}_0 + \frac{1}{a^6}\pi^2_{\phi_{0I}} + m_I^2\phi_{0I}^2 + \frac{1}{3l^2}\Lambda(n) \right]\psi^0(a,\tilde{\gamma}_{ij0},\phi_{0I}) = 0 \tag{3.15a}$$

and

$$a^3\int\frac{d^3k}{(2\pi)^3}\{[a^{ij\dagger}(\mathbf{k})a^{ij}(\mathbf{k})+\tfrac{1}{2}]|k| + [a_I^\dagger(\mathbf{k})a_I(\mathbf{k})+\tfrac{1}{2}]\sqrt{k^2+m_I^2}\}\psi^{\rm osc}_{(n)} = \frac{a^3}{l^2}\Lambda(n)\psi^{\rm osc}_{(n)} . \tag{3.15b}$$

The solution to (3.15b) simply reduces to an infinite set of harmonic oscillators with

$$\Lambda(n) = l^2\int\frac{d^3k}{(2\pi)^3}\left[[n_g(\mathbf{k})+\tfrac{1}{2}]|k| + \sum_I [n_I(\mathbf{k})+\tfrac{1}{2}]\sqrt{k^2+m_I^2} \right] , \tag{3.16}$$

where we have generalized our system of matter fields to a collection of scalar fields labeled by (I) and have defined the number operators $n_g(\mathbf{k})=a^{ij\dagger}(\mathbf{k})a^{ij}(\mathbf{k})$ and $n_I(\mathbf{k})=a_I^\dagger(\mathbf{k})a_I(\mathbf{k})$. We can find explicit expressions for the wave functions $\psi^{\rm osc}_{(n)}$ as follows. Define

$$q_1(\mathbf{k})=\tfrac{1}{2}[q(\mathbf{k})+q^\dagger(\mathbf{k})]a^{-3/2} \quad \text{and} \quad p_1(\mathbf{k})=\tfrac{1}{2}[p(\mathbf{k})+p^\dagger(\mathbf{k})]a^{-3/2} ,$$
$$q_2(\mathbf{k})=\frac{1}{2i}[q(\mathbf{k})-q^\dagger(\mathbf{k})]a^{-3/2} \quad \text{and} \quad p_2(\mathbf{k})=\frac{1}{2i}[p(\mathbf{k})-p^\dagger(\mathbf{k})]a^{-3/2} , \tag{3.17}$$

as well as

$$a_1(\mathbf{k})=(2)^{-1/2}[(\omega_k)^{1/2}q_1(\mathbf{k})+i(\omega_k)^{-1/2}p_1(\mathbf{k})] ,$$
$$a_2(\mathbf{k})=(2)^{-1/2}[(\omega_k)^{1/2}q_2(\mathbf{k})+i(\omega_k)^{-1/2}p_2(\mathbf{k})] . \tag{3.18}$$

In this (1,2) representation the oscillator contribution to H can be written

$$H_{\rm osc}=a^3\int_{k_3>0}\frac{d^3k}{(2\pi)^3}[a_1^\dagger(\mathbf{k})a_1(\mathbf{k})+a_2^\dagger(\mathbf{k})a_2(\mathbf{k})+\tfrac{1}{2}+\tfrac{1}{2}]\omega_k . \tag{3.19}$$

In this representation one restricts oneself to the region of momentum space $k_3>0$ to avoid overcounting[10] as the variables of the phase space (q_1,p_1,q_2,p_2) obey the relations $q_1(-\mathbf{k})=q_1(\mathbf{k})$, $p_1(-\mathbf{k})=p_1(\mathbf{k})$, $q_2(-\mathbf{k})=-q_2(\mathbf{k})$, $p_2(-\mathbf{k})=-p_2(\mathbf{k})$. Now defining $n_1(\mathbf{k})$ and $n_2(\mathbf{k})$ as the eigenvalues associated with $a_1^\dagger(\mathbf{k})a_1(\mathbf{k})$ and $a_2^\dagger(\mathbf{k})a_2(\mathbf{k})$, respectively, one writes the solutions as

$$\psi^{\rm osc}_{(n)}(q_1,q_2)=\prod_{k_3>0}\left[\frac{\omega_k}{2^{n_1(k)}[n_2(k)!]^2}\right]^{1/4}\left[\frac{\omega_k}{2^{n_2(k)}[n_2(k)!]^2}\right]^{1/4}$$
$$\times H_{n_1(k)}(\omega_k^{1/2}q_1(\mathbf{k}))H_{n_2(k)}(\omega_k^{1/2}q_2(\mathbf{k}))\exp\left[-\tfrac{1}{2}a^3\int_{k_3>0}\frac{d^3k}{(2\pi)^3}[q_1(\mathbf{k})q_1(\mathbf{k})\ +q_2(\mathbf{k})q_2(\mathbf{k})]\omega_k\right] , \tag{3.20}$$

the full wave function is given by $\psi^{\rm osc}_{(n_g)}(q_1^{ij},q_2^{ij})\prod_I\psi^{\rm osc}_{(n_I)}(q_1^I,q_2^I)$ and $H_n(\omega^{1/2}q)$ is the nth Hermite polynomial. The value of $\Lambda(n)$ that feeds back into (3.15a) is given by

$$\Lambda(n)=l^2\int_{k_3>0}\frac{d^3k}{(2\pi)^3}\left[[n_{1g}(\mathbf{k})+n_{2g}(\mathbf{k})+\tfrac{1}{2}]\,|\,k\,|+\sum_I[n_{1I}(\mathbf{k})+n_{2I}(\mathbf{k})+\tfrac{1}{2}]\sqrt{k^2+m_I^2}\right] . \tag{3.21}$$

Clearly $\Lambda(0)$ is the quantum-mechanically induced cosmological constant:

$$\Lambda(0)=l^2\int_{k_3>0}\frac{d^3k}{(2\pi)^3}\left[2\,|\,k\,|+\sum_I\sqrt{k^2+m_I^2}\right]=l^2\tfrac{1}{2}\int\frac{d^3k}{(2\pi)^3}\left[2\,|\,k\,|+\sum_I\sqrt{k^2+m_I^2}\right] . \tag{3.22}$$

Although this expression diverges, the corresponding quantity in string theory is finite and well defined.[13]

The states we have constructed above do not obey the constraint $\chi^i\psi=0$ in (2.3) however. We shall have to make a projection from the above states to physical states which satisfy this constraint. Again decomposing $\tilde{\gamma}_{ij}=h_{ij}^{TT}+\partial_ih_j+\partial_jh_i-\tfrac{2}{3}\delta_{ij}(\partial_kh_k)$ as well as $\tilde{\pi}_{ij}=\tilde{\pi}_{ij}^{TT}+\partial_i\pi_j+\partial_j\pi_i-\tfrac{2}{3}\delta_{ij}(\partial_k\pi_k)$ the integrated form of the χ^i constraint becomes

$$\int d^3x\,\chi^i=\int d^3x[-2\pi^{ij}{}_{,j}-\gamma^{il}(2\gamma_{jl,k}-\gamma_{jk,l})\pi^{jk}]\approx\int d^3x[-2(\Delta\pi^i+\pi^k{}_{,ki})-h_{rs,i}^{TT}\pi^{TTrs}] . \tag{3.23}$$

The integrand of the first of these operators can be readily verified to be canonically conjugate to $h_i(\mathbf{x})$ (Refs. 10 and 11) and is conventionally denoted by $p^i(\mathbf{x})=-2(\Delta\pi^i+\pi^k{}_{,ki})(\mathbf{x})$. Introducing the Fourier decomposition $p^i(\mathbf{x})=a^3\int[d^3k/(2\pi)^3]p^i(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}$ we obtain the expression

$$\int d^3x\,\chi^i\approx a^3p^i(\mathbf{k}=0)-a^3\int\frac{d^3k}{(2\pi)^3}k^ia_{rs}^\dagger(\mathbf{k})a_{rs}(\mathbf{k})-a^3\int\frac{d^3k}{(2\pi)^3}k^ia_I^\dagger(\mathbf{k})a_I(\mathbf{k}) , \tag{3.24}$$

where we have included the contribution from our matter fields. Quantum mechanically then the condition $\int d^3x\,\chi^i=0$ becomes the following condition on the wave function $\psi^{\rm osc}_{(n)}$:

$$\left[a^3\frac{\partial}{\partial h_{i0}}-a^3\int\frac{d^3k}{(2\pi)^3}k^ia_{rs}^\dagger(\mathbf{k})a_{rs}(\mathbf{k})-a^3\int\frac{d^3k}{(2\pi)^3}k^ia_I^\dagger(\mathbf{k})a_I(\mathbf{k})\right]\psi^{\rm osc}_{(n)}(q^{ij},q^I,h_{i0})=0 . \tag{3.25}$$

So we see that the wave functions above will, in general, develop a dependence on the zero mode h_{i0} given by

$$\psi^{\rm osc}_{(n)}(q^{ij},q^I,h_{i0})=\exp(ih_{i0}P^i_{\rm osc})\psi^{\rm osc}_{(n)}(q^{ij},q^I,0)\ , \tag{3.26}$$

where $P^i_{\rm osc}$ is defined by

$$P^i_{\rm osc}=\int d^3x\, h^{TT}_{rs,i}\pi^{TTrs}+\int d^3x\,\phi_{I,i}\pi^I=a^3\int\frac{d^3k}{(2\pi)^3}k^i a^\dagger_{rs}(\mathbf{k})a_{rs}(\mathbf{k})+a^3\int\frac{d^3k}{(2\pi)^3}k^i a^\dagger_I(\mathbf{k})a_I(\mathbf{k})\ . \tag{3.27}$$

This represents the oscillator contribution to the spatial momentum of the Universe. In order to form "physical" states which do not have spurious h_{i0} dependence one constructs

$$\psi_{\rm phys}(q^{ij},q^I)=\int d^3h_{i0}\psi(q^{ij},q^I,h_{i0})=\int d^3h_{i0}\exp(ih_{i0}P^i_{\rm osc})\psi(q^{ij},q^I)\ . \tag{3.28}$$

Our expansion for the WDW field must be modified to include this constraint and is given by

$$\begin{aligned}\Phi(\gamma^{1/2},\tilde\gamma_{ij})&=\sum_{(n)}\psi^0_{(n)}(a,\gamma_{ij0},\phi^I_0)\psi^{\rm osc}_{(n)\rm phys}(q^{ij},q^I)A(n)+\text{c.c.}\\&=\sum_{(n)}\psi^0_{(n)}(a,\gamma_{ij0},\phi^I_0)\delta^3(P^i_{\rm osc})\psi^{\rm osc}_{(n)}(q^{ij},q^I)A(n)+\text{c.c.}\\&=\sum_{(n)}\psi^0_{(n)}(a,\gamma_{ij0},\phi^I_0)\delta^3\left[a^3\int\frac{d^3k}{(2\pi)^3}k^i[n_g(\mathbf{k})+n_{\phi I}(\mathbf{k})]\right]\psi^{\rm osc}_{(n)}(q^{ij},q^I)A(n)+\text{c.c.}\ ,\end{aligned} \tag{3.29}$$

so that we must restrict our expansion to include only wave functions which describe universes of zero spatial momentum. In terms of the (1,2) representation used to find the explicit form of the wave function the operator $P^i_{\rm osc}$ is written

$$P^i_{\rm osc}=a^3\int_{k_3>0}\frac{d^3k}{(2\pi)^3}\frac{1}{2i}[a_2(\mathbf{k})a^\dagger_1(\mathbf{k})-a_1(\mathbf{k})a^\dagger_2(\mathbf{k})]k^i\ . \tag{3.30}$$

By making the substitution

$$a'_1(\mathbf{k})=(2)^{-1/2}[a_1(\mathbf{k})+ia_2(\mathbf{k})],\quad a'_2(\mathbf{k})=(2)^{-1/2}[ia_1(\mathbf{k})+a_2(\mathbf{k})]\ , \tag{3.31}$$

which can be shown to be a canonical transformation in (q_1,p_1,q_2,p_2) phase space $H_{\rm osc}$ and $P^i_{\rm osc}$ take the form

$$\begin{aligned}H_{\rm osc}&=a^3\int_{k_3>0}\frac{d^3k}{(2\pi)^3}[a'^\dagger_1(\mathbf{k})a'_1(\mathbf{k})+a'^\dagger_2(\mathbf{k})a'_2(\mathbf{k})+\tfrac12+\tfrac12]\omega_k\ ,\\P^i_{\rm osc}&=a^3\int_{k_3>0}\frac{d^3k}{(2\pi)^3}\frac12[a'_1(\mathbf{k})a'^\dagger_1(\mathbf{k})-a'_2(\mathbf{k})a'^\dagger_2(\mathbf{k})]k^i\ ,\end{aligned} \tag{3.32}$$

and the expansion of our second quantized WDW field in the (1,2) representation is given by

$$\begin{aligned}\Phi(\gamma^{1/2},\tilde\gamma_{ij})=&\sum_{(n_1,n_2)}\psi^0_{(n_1,n_2)}(a,\gamma_{ij0},\phi^I_0)\delta^3\left[a^3\int_{k_3>0}\frac{d^3k}{(2\pi)^3}\frac12 k^i[n_1(\mathbf{k})-n_2(\mathbf{k})]\right]\\&\times\psi^{\rm osc}_{(n_1)}(q^{ij}_1,q^I_1)\psi^{\rm osc}_{(n_2)}(q^{ij}_2,q^I_2)A(n_1,n_2)+\text{c.c.}\ ,\end{aligned} \tag{3.33}$$

where $\psi^{\rm osc}_{(n_1)}(q^{ij}_1,q^I_1)$ are given in (3.20). So physical oscillator states are characterized by $(n_1,n_2)=[n_1(\mathbf{k}),n_2(\mathbf{k})\,|\,\mathbf{k}\in R^3]$ the total occupation numbers for given value of spatial momentum. They contribute to the quantity $\Lambda(n)$ in the zero-mode equation (3.15a) through

$$\Lambda(n)=l^2\int_{k_3>0}\frac{d^3k}{(2\pi)^3}\omega_k[n_1(\mathbf{k})+n_2(\mathbf{k})] \tag{3.34}$$

but are subject to the constraint

$$l^2\int_{k_3>0}\frac{d^3k}{(2\pi)^3}k^i[n_1(\mathbf{k})-n_2(\mathbf{k})]=0\ . \tag{3.35}$$

The constraint $\chi^i\psi=0$ is the analogue of condition

$$\int d\sigma(X^{-\prime}p^++X^{i\prime}p^i)\psi(p^+,x^+,X^i(\sigma))=0 \tag{3.36}$$

which is the second constraint of string theory $\int d\sigma X^{\mu\prime}(\sigma)P_\mu(\sigma)$ written in the light-cone gauge $(\partial/\partial\sigma)X^+(\sigma)=0$. For our system we have chosen the gauge $(\partial/\partial x^i)\gamma_{ij}=\Delta h_i+\frac13\partial_i(\partial\cdot h)=0$ in order to obtain the physical components of the graviton. We have three times the gauge conditions as we have three times the residual gauge invariance. Physical states are formed in string theory by[13]

$$\psi_{\text{phys}}(p^+,x^+,X^i)=\int d\sigma_0 \exp\left[i\sigma_0\frac{1}{p_+}\int_0^{2\pi|p^+|}d\sigma\, X^1(\sigma)'\Pi^i(\sigma)\right]\psi(p^+,x^+,X^i) \tag{3.37}$$

which reduces to the projection operator $\int d\sigma_0 \exp[i\sigma_0(L_0-\tilde{L}_0)]$ on physical states. This amounts to requiring equal numbers of left and right movers. Physical states have the property that they are invariant under translations in σ. That is,

$$\psi_{\text{phys}}(X^i(\sigma))=\psi_{\text{phys}}(X^i(\sigma+\sigma_0)) . \tag{3.38}$$

In our case we have

$$\psi_{\text{phys}}(h_{ij}^{TT}(x^k))=\psi_{\text{phys}}(h_{ij}^{TT}(x^k+x_0^k)) . \tag{3.39}$$

The condition clearly implies that the state be annihilated by the operator which generates spatial translations, that is, the momentum operator P^i_{osc} and the associated absence of the h_i mode from the wave function.

Now let us return to Eq. (3.15a) for the zero modes:

$$a^3\left[-\frac{l^2}{a^4}\pi_a^2-\frac{k}{l^2a^2}[1-V(\tilde{\gamma}_{ij0})]+\frac{l^2}{a^6}24\tilde{\gamma}_{ij0}\tilde{\pi}_0^{jk}\tilde{\gamma}_{kn0}\tilde{\pi}_0^{ni}+\frac{1}{a^6}\pi^2_{\phi_{0I}}+m_I^2\phi_{0I}^2+\frac{1}{3l^2}\Lambda(n)\right]\psi^0(a,\tilde{\gamma}_{ij0},\phi_{0I})=0 . \tag{3.40}$$

Upon substitution $\pi_a=-i\partial/\partial a$, $\tilde{\pi}_0^{ij}=-i\partial/\partial\gamma_{ij0}$, and $\pi_{\phi_{0I}}=-i\partial/\partial\phi_{0I}$ we obtain a partial differential equation governing the minisuperspace consisting of our zero modes. We wish to obtain an expansion of our WDW field in asymptotic regions of superspace (indeed we do not think such an expansion is possible for small values of $\gamma^{1/2}$) which we expect to represent large values of the radius a and small values of the anisotropy β_{ij} where the zero-mode metric $\gamma_{ij0}=a^2(e^{2\beta/\sqrt{6}})_{ij}$ approaches that of three-dimensional flat space. In these regions $a\to\infty$ and $\beta_{ij}\to 0$, Eq. (3.15a) takes the simplified form

$$\left[l^2\frac{\partial^2}{\partial a^2}-\frac{l^2}{a^2}\frac{\partial}{\partial\beta_{ij}}\frac{\partial}{\partial\beta_{ij}}-\frac{1}{a^2}\frac{\partial}{\partial\phi_{0I}}\frac{\partial}{\partial\phi_{0I}}+m_I^2a^4\phi_{0I}\phi_{0I}+\frac{a^4}{3l^2}\Lambda(n)-k\frac{a^2}{l^2}\right]\psi^0_{(n)}(a,\beta_{ij},\phi_{0I})=0 . \tag{3.41}$$

Again if $k\neq 0$ one has to expand the oscillating modes in spherical harmonics.[1] In the following we shall make the simplifying assumptions $m_I^2=0$ and $k=0$. Then we set

$$\psi^0_{(n)}(a,\beta_{ij},\phi_{0I})=\psi^0_{(n)}(a)\exp(i\tilde{p}^{ij}\beta_{ij}+ip_\phi^I\phi_{0I})$$

and (3.41) becomes

$$\left[l^2\frac{\partial^2}{\partial a^2}+\frac{l^2}{a^2}\tilde{p}^{ij}\tilde{p}^{ij}+\frac{1}{a^2}p_\phi^Ip_\phi^I+\frac{a^4}{l^2}\frac{\Lambda(n)}{3}\right]\psi^0_{(n)}(a,\beta_{ij},\phi_{0I})=0 \tag{3.42}$$

or

$$\left[9l^2\frac{\partial^2}{\partial(a^3)^2}+\frac{l^2}{a^6}\tilde{p}^{ij}\tilde{p}^{ij}+\frac{1}{a^6}p_\phi^Ip_\phi^I+\frac{1}{l^2}\frac{\Lambda(n)}{3}\right]\psi^0_{(n)}(a^3,\beta_{ij},\phi_{0I})=0 . \tag{3.43}$$

Now define

$$p_0(a^3)=\left[\frac{1}{9a^6}\tilde{p}^{ij}\tilde{p}^{ij}+\frac{1}{9l^2a^6}p_\phi^Ip_\phi^I+\frac{1}{l^4}\frac{\Lambda(n)}{3\times 9}\right]^{1/2} \tag{3.44}$$

or

$$p_0(a^3)=\left[\frac{1}{9a^6}\tilde{p}^{ij}\tilde{p}^{ij}+\frac{1}{9l^2a^6}p_\phi^Ip_\phi^I+\frac{1}{l^4}\frac{\Lambda(0)}{3\times 9}+\frac{\rho(n)}{l^29a^4}\right]^{1/2} , \tag{3.45}$$

where we have found it convenient to define $\rho(n)=[\Lambda(n)-\Lambda_0]a^4l^{-2}$. This is because this quantity is dimensionless and can often be taken to be independent of a. For example, if $n(\mathbf{k})=(e^{|k|a}-1)^{-1}$ then

$$\rho(n)=a^4\int\frac{d^3k}{(2\pi)^3}|k|n(\mathbf{k}) =\int\frac{d^3m}{(2\pi)^3}|m|(e^{|m|}-1)^{-1}=\frac{\pi^2}{30} . \tag{3.46}$$

A distribution which is useful when the radius a is proportional to the inverse temperature.[14]

Equation (3.43) can be solved using the WKB approximation[15] and one obtains

$$\psi^0_{(n)}(a^3)\approx\psi^0_{(n)}(a_0^3)\frac{[p_0(a_0^3)]^{1/2}}{[p_0(a^3)]^{1/2}}\exp\left[\pm i\int_{a_0^3}^{a^3}da^3p_0(a^3)\right] . \tag{3.47}$$

For large values of a^3 the cosmological constant term dominates and we find

$$\psi^0_{(n)}(a^3)\approx\psi^0_{(n)}(a_0^3)\exp\left[\pm i(a^3-a_0^3)\left[\frac{\Lambda(0)}{3\times 9l^4}\right]^{1/2}\right] . \tag{3.48}$$

Equation (3.47) represents our version of particle theory's

$$\psi(t)=[c^2p^2+(mc^2)^2]^{-1/4}\exp[\pm it\sqrt{c^2p^2+(mc^2)^2}] \tag{3.49}$$

and (3.48) the nonrelativistic version $\psi(t) =m^{-1/2}\exp(\pm itmc^2)$. So we have the relation

$$\{m\to[\Lambda(0)/(3\times 9)]^{1/2}\}\ (t\to a^3)\ (c^2\to l^{-2})\ . \tag{3.50}$$

Note that there is no real time in the WDW equation. It is only that $-\partial^2/\partial(a^3)^2$ comes into the WDW equation with the opposite sign from all other second derivatives in much the same way that $-\partial^2/\partial t^2$ comes into the wave equation for the free particle with the opposite sign. The above is simply a correspondence between two equations of the Klein-Gordon type (that is, they are both second-order hyperbolic differential equations). Note that the term $\Lambda(0)$ behaves as an effective (mass)^2 term in the WDW equation restricted to the zero modes.

We will attempt to use DeWitt's inner product to normalize the WKB wave functions we found above. On the subspace (a^3,β_{ij},ϕ_I) of $(\gamma^{1/2},\tilde{\gamma}_{ij},\phi_I)$ this inner product takes a particularly simple form

$$(\psi,\psi)=-i\int d\phi_I d\beta_{ij}\left[\psi^*\frac{\partial}{\partial(a^3)}\psi-\psi\frac{\partial}{\partial(a^3)}\psi^*\right]\ . \tag{3.51}$$

Applying this definition to (3.47) yields

$$(\psi^0_{(n)},\psi^0_{(n)})=\pm V_\phi V_\beta 2\ , \tag{3.52}$$

where we have cut off our integral over the spacelike variables (ϕ,β_{ij}). This is conventional for equations of this type. We hope that when one computes physical quantities (such as the probability a given Universe will split off into two Universes with various properties) the dependence on this arbitrary cut off will cancel. This certainly happens when one computes the decay rate of a particle into two others. The normalized zero-mode wave functions are

$$\psi^0_{(n)}(a^3,\beta_{ij},\phi_{0I})\approx\frac{1}{[2V_\phi V_\beta p_0(a^3)]^{1/2}}\exp\left[i\left[\tilde{p}^{\,ij}\beta_{ij}+p^I\phi_{0I}\pm\int_{a_0^3}^{a^3}da^3p_0(a^3)\right]\right]\ . \tag{3.53}$$

And our final mode expansion of our WDW field is given by (3.33) with the above zero-mode wave function. The crucial thing to notice about the above formula is the introduction of free parameters $\tilde{p}^{\,ij}$ and p^I on which the solution can depend besides the (n) which are the parameters which describe the oscillator modes. The creation operators $A^\dagger(n)$ must have the ability to create universes with these parameters. So we must write $A((n),\tilde{p}^{\,ij},p^I)$ from now on. Like the wormhole parameters discussed by Hawking,[1] Coleman,[3] and others the four-dimensional observer is powerless to measure the parameters $\tilde{p}^{\,ij}$ and p^I. They are canonical momenta in superspace and have no relation to four-dimensional stress energy, the relevant quantity from the point of view of the four-dimensional observer.

IV. THE INCLUSION OF INTERACTIONS FOR THE SECOND-QUANTIZED WDW FIELD

In particle theory the Klein-Gordon equation is naturally extended to include interactions. One can consider interacting terms with completely different Klein-Gordon fields describing particles with different mass, charge, etc. We have seen that the analogue of (mass)^2 for our problem is the vacuum energy $\Lambda(0)$ or for an excited state of the universe $\Lambda(n)$. Actually in string theory one allows for transitions between different two-dimensional quantum field theories with different vacuum energies and boundary conditions on these fields. For example, a Neveu-Schwarz state can decay into two Ramond states. Indeed the two-dimensional vacuum energy defines the (mass)^2 for the ground state. So that we expect that when one includes interactions in our second-quantized WDW formalism it will become possible to dynamically jump from a Universe with one set of boundary conditions and vacuum energy to another thus altering the physical laws describing the Universe. It should be possible to compute the probability of such a transition within this formalism. To motivate such considerations we note that string theory can be viewed as two-dimensional gravity plus matter via the Polyakov path integral[16]

$$Z(0)=\int Dg_{\alpha\beta}(\sigma^\alpha)DX^\mu(\sigma^\alpha)\exp\left[i\int d^2\sigma\sqrt{-g}\ g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu\right]\ . \tag{4.1}$$

To include interactions in x_0^μ space (the space of zero modes associated with the scalar field X^μ) the above integral is modified by considering

$$G(x_1^\mu,x_2^\mu,\dots,x_n^\mu)=\int Dg_{\alpha\beta}(\sigma^\alpha)DX^\mu(\sigma^\alpha)\exp\left[i\int d^2\sigma\sqrt{-g}\ g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu\right] \times\prod_{i=1}^n d^2\sigma_i\sqrt{-g(\sigma_i)}\delta^D[X^\mu(\sigma_i)-x_i^\mu]\ . \tag{4.2}$$

Note that this has nothing to do with interactions on the world sheet as the two-dimensional fields on the world sheet are indeed free. It simply means that we sum over all two-dimensional scalar fields $X^\mu(\sigma^\alpha)$ which take the values x_i^μ at some point in two-dimensional spacetime. One can further make a Fourier transform with respect to x_i^μ to obtain

$$A(p_1^\mu,p_2^\mu,\ldots,p_n^\mu)=\int \prod_{i=1}^n d^D x_i e^{ip_i^\mu x_i^\mu} G(x_1^\mu,x_2^\mu,\ldots,x_n^\mu)$$

$$=\int Dg_{\alpha\beta}(\sigma^\alpha)DX^\mu(\sigma^\alpha)\exp\left[i\int d^2\sigma\sqrt{-g}\, g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu\right]\prod_{i=1}^n d^2\sigma_i e^{ip_i^\mu X^\mu}(\sigma_i)$$

$$=\int Dg_{\alpha\beta}(\sigma^\alpha)\sqrt{-g(\sigma_i)}\exp\left[-\sum_{ij} p_i^\mu p_j^\mu K(\sigma_i,\sigma_j)\right], \tag{4.3}$$

where $K(\sigma,\sigma')$ is the Green's function on the two-dimensional manifold satisfying

$$\partial_\alpha[\sqrt{-g}\, g^{\alpha\beta}\partial_\beta K(\sigma,\sigma')]=\delta^2(\sigma-\sigma') . \tag{4.4}$$

Now we shall attempt to apply this same procedure to four-dimensional gravity plus matter. In this case the interactions on the world volume are not negligible; however, the effect we are interested in has nothing to do with interactions on the world volume within a given universe, but rather the interaction only takes place when two universes interface. So we are led to consider

$$Z_4(0)=\int Dg_{\mu\nu}(x)D\phi_I(x)\exp\left[i\int d^4x\sqrt{-g}\,({}^{(4)}R+g^{\mu\nu}\partial_\mu\phi_I\partial_\nu\phi_I)\right] \tag{4.5}$$

and its modifications to include interactions in the space of zero modes of ϕ_{0_I} and $g_{0_{\mu\nu}}$ given by

$$G(g_{01}^{\mu\nu},g_{02}^{\mu\nu},\ldots;\phi_{01}^I,\phi_{02}^I,\ldots)=\int Dg_{\mu\nu}(x)D\phi_I(x)\exp\left[i\int d^4x\sqrt{-g}\,({}^{(4)}R+g^{\mu\nu}\partial_\mu\phi^I\partial_\nu\phi^I)\right]$$

$$\times\prod_{i=1}^{(n)} d^4x_i\sqrt{-g(x_i)}\delta(g_{\mu\nu}(x_i)-g_{0\mu\nu i})\delta(\phi^I(x_i)-\phi_{0i}^I) . \tag{4.6}$$

For simplicity we shall assume we have only one scalar field $\phi(x)$. Concentrating on a single scalar portion of the insertions we have

$$\prod_{i=1}^n d^4x_i\sqrt{-g(x_i)}\delta(\phi(x_i)-\phi_i)=\prod_{i=1}^n d^4x_i\sqrt{-g(x_i)}\int\frac{dk_{\phi i}}{2\pi}e^{ik_{\phi i}(\phi(x_i)-\phi_i)}$$

$$=\prod_{i=1}^n\int\frac{dk_{\phi i}}{2\pi}e^{-ik_{\phi i}\phi_i}\prod_{i=1}^n d^4x_i\sqrt{-g(x_i)}V(k_{\phi i},x_i) , \tag{4.7}$$

where $V(k_\phi,x)=\exp[ik_\phi\phi(x)]$ the analogue of a vertex operator in string theory. Similar manipulations yield vetex operator representations for the other insertions:

$$V(k_\phi,x^\mu)=e^{ik_\phi\phi(x)},\quad V(k_{ij},x^\mu)=e^{ik_{ij}\tilde{\gamma}^{ij}(x)},\quad V(k_a,x^\mu)=e^{ik_a a(x)} . \tag{4.8}$$

Furthermore we can express (4.6) in terms of its Fourier transform as

$$G(\phi_{01},\phi_{02},\ldots,\phi_{0_n})=\int\prod_{i=1}^n\int\frac{dk_{\phi i}}{2\pi}e^{-ik_{\phi i}\phi_i}A(k_{\phi 1},k_{\phi 2},\ldots,k_{\phi n}) , \tag{4.9}$$

so that we constrain the path integral so as to integrate over the field variables which take the designated values $g_{0\mu\nu i}$ and ϕ_{0i} somewhere in spacetime.

Using our mode decomposition we can give a more tractable representation of the above path integral. From Sec. III we have obtained the decomposition

$$L=L_{0G}+L_{0M}+L_{\rm osc} . \tag{4.10}$$

Thus we write the path integral as

$$Z_4(0)=\int Da\, D\beta_{ij}D\phi_{0I}Dq^{ij}(\mathbf{k})Dq^i(\mathbf{k})$$

$$\times\exp\left[i\int dt\,\tfrac{1}{2}(l^{-2}\{-a\dot a^2+ak[1-V(\beta)]+a^3\dot\beta_{ij}\dot\beta_{ij}\}+a^3(\dot\phi_I^2-m^2\phi_I^2))\right]$$

$$\times\exp\left[i\int dt\,\tfrac{1}{2}a^3l^{-2}\int_{k_3>0}\frac{d^3k}{(2\pi)^3}[\dot q_\alpha^{ij}(k)\dot q_\alpha^{ij}(k)-|k|^2q_\alpha^{ij}(k)q_\alpha^{ij}(k)]\right]$$

$$\times\exp\left[i\int dt\,\tfrac{1}{2}a^3\int_{k_3>0}\frac{d^3k}{(2\pi)^3}[\dot q_\alpha^I(k)\dot q_\alpha^I(k)-\omega_k^2q_\alpha^I(k)q_\alpha^I(k)]\right], \tag{4.11}$$

where α takes the values 1 or 2 in the (1,2) representation. Although this integral is a vast simplification as it ignores interactions on the world volume the effect we are looking for should still be present in this limit. The insertion of factors

$$\prod_{i=1}^{n} d^4x_i(ca^3)\exp\left[ik_\phi^i\left[\phi+\int\frac{d^3k}{(2\pi)^3}q(k)e^{i\mathbf{k}\cdot\mathbf{x}_i}\right]\right] \tag{4.12}$$

generate the interactions in this simple model and can actually be implemented by the insertion of vertex operators in the usual manner. Or one can simply note that the above interactions are completely analogous to the forced oscillator problem with

$$\begin{aligned} L_{\rm osc}&=\tfrac{1}{2}\int\frac{d^3k}{(2\pi)^3}[\dot q(k,t)\dot q(-k,t)-\omega_k^2 q(k,t)q(-k,t)+\dot q^{ij}(k,t)\dot q^{ij}(-k,t)-|k|^2q^{ij}(k,t)q^{ij}(-k,t)]\ ,\\ L_{\rm int}&=\int\frac{d^3k}{(2\pi)^3}j(k,t)q(k,t)\ , \end{aligned} \tag{4.13}$$

where

$$j(k,t)=\sum_{i=1}^{N}p_{\phi i}\delta(t-t_i)e^{i\mathbf{k}\cdot\mathbf{x}_i}\ . \tag{4.14}$$

The problem of finding the correlation function in such a model has been completely solved by Feynman and Hibbs.[17] We shall use the result

$$\begin{aligned} G_{00}&=\exp\left[i\tfrac{1}{2}\int\frac{d^3k}{(2\pi)^3}\int dt\, j(k,t)D_{1F}(t-t';\omega_k)j(-k,t')dt'\right]\ ,\\ G_{m_kn_k}&=G_{00}\sum_{r=0}^{\min(m_k,n_k)}\frac{\sqrt{m_k!}\sqrt{n_k!}}{(m_k-r)!(n_k-r)!r!}(i\beta_k)^{n_k-r}(i\beta_k^*)^{n_k-r}\ , \end{aligned} \tag{4.15}$$

where $\beta_k=(2\omega_k)^{-1/2}\int j(k,t)e^{-i\omega_k t}dt$ and m_k is the number of particles with momentum $\mathbf{k}$. $G_{m_kn_k}$ represents the probability amplitude for the system of oscillators to evolve from a state of occupation number m_k to one of n_k driven by the force term $L_{\rm int}$. A Euclideanized version of the above formula has been developed by Coleman in Ref. 3. A state of the Universe can be represented by $(a^\dagger)^{n(k)}/\sqrt{n(k)!}\,|0\rangle$. This state does not obey the χ^i constraint (3.25) however. States which obey that are given in the (1,2) representation as $|n_1(k_1)n_1(-k_1)\rangle$ where

$$|n_1(k_1)n_2(k_2)\rangle=\frac{1}{\sqrt{n_1(k_1)!}}(a_1(k_1))^{\dagger n(k_1)}\frac{1}{\sqrt{n_2(k_2)!}}(a_2(k_2))^\dagger n(k_2)\,|0,0\rangle\ . \tag{4.16}$$

One can ask for the probability that the Universe can make a transition from the state $|n_1(k)n_1(-k)\rangle$ to $|n_2(k)n_2(-k)\rangle$ through the interactions introduced above. The oscillator portion of the transition amplitude is given by

$$G_{00}\sum_{r=0}^{\min(n_1,n_2)}\frac{\sqrt{n_1!}\sqrt{n_2!}}{(n_1-r)!(n_2-r)!r!}(i\beta_k)^{n-r}(i\beta_k^*)^{n-r}\ ,$$

where

$$\beta_k=(2\omega_k)^{-1/2}p_\phi e^{ik\cdot x-i\omega_k t}\ . \tag{4.17}$$

The above transition amplitude can be written as

$$\langle f\,|\,V(p_\phi,x,t)\,|\,i\rangle=\langle f\,|\,e^{itH_{\rm osc}+ix\cdot P_{\rm osc}}V(p_\phi,0,0)e^{-itH_{\rm osc}-ix\cdot P_{\rm osc}}\,|\,i\rangle\ . \tag{4.18}$$

As the initial and final states should be physical states they should be annihilated by $P^i_{\rm osc}$ and when one includes the zero-mode portion of the vertex operator one finds that one can set x_2 equal to zero. Also if the initial and final states are eigenstates of $H_{\rm osc}$ the t_2 dependence will be in an overall phase.

If one uses the original $(h_{ij}^{TT}(\mathbf{x}),\phi(\mathbf{x}))$ representation

$$\begin{aligned} Z_4(0)=\int &Da(t)D\beta_{ij}(t)D\phi_0(t)Dh_{ij}^{TT}(\mathbf{x},t)D\phi(\mathbf{x},t)e^{iS_{0G}+iS_{0M}}\exp\left[\tfrac{1}{4}i\int dt\,d^3x(h^{TT}_{ij,0}h^{TT}_{ij,0}-h^{TT}_{ij,k}h^{TT}_{ij,k})\right]\\ &\times\exp\left[\tfrac{1}{2}i\int dt\,d^3x(\phi_{,0}\phi_{,0}-\phi_{,k}\phi_{,k})\right]\ . \end{aligned} \tag{4.19}$$

The insertions (4.12) can be written in the form

$$\exp\left[i\int dt\, d^3x\,[J(\mathbf{x},t)\phi(\mathbf{x},t)+J^{ij}(\mathbf{x},t)h_{ij}^{TT}(\mathbf{x},t)]\right] \tag{4.20}$$

with

$$J(x)=\sum_{i=1}^{n} k_{\phi i}\delta^4(x-x_i),\quad J^{rs}(x)=\sum_{i=1}^{n} k_i^{rs}\delta^4(x-x_i)\ . \tag{4.21}$$

In this form the path integral can be evaluated by standard techniques:

$$Z_4(J)=\int Da(t)D\beta_{ij}(t)D\phi_0(t)Dh_{ij}^{TT}(\mathbf{x},t)D\phi(\mathbf{x},t)e^{iS_{0G}+iS_{0M}}\exp\left[\frac{1}{4}i\int dt\, d^3x(h_{ij,0}^{TT}h_{ij,0}^{TT}-h_{ij,k}^{TT}h_{ij,k}^{TT})\right]$$

$$\times\exp\left[\tfrac{1}{2}i\int dt\, d^3x[\phi_{,0}\phi_{,0}-\phi_{,k}\phi_{,k}+J(\mathbf{x},t)\phi(\mathbf{x},t)+J^{ij}(\mathbf{x},t)h_{ij}^{TT}(\mathbf{x},t)]\right]\ . \tag{4.22}$$

The k_ϕ, k^{ij} dependence of the result is given by

$$Z_4(J)=\exp\left[-\tfrac{1}{2}i\int d^4x\, d^4y\, D_F(x-y)J^I(x)J^I(y)-\tfrac{1}{2}i\int d^4x\, d^4y\, D_F^{ij,kl}(x-y)J^{ij}(x)J^{kl}(y)\right], \tag{4.23}$$

where the Feynman Green's function $D_F(x-y)$ is given by

$$D_F(x-y)=\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}}{k^2-m^2+i\epsilon},\quad D_F^{ij,lm}(x-y)=\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}}{k^2-m^2+i\epsilon}P^{ij,lm}(\mathbf{k})\ , \tag{4.24}$$

where

$$P^{ij,lm}(\mathbf{k})=\sum_{\lambda=1}^{2}\epsilon^{ij}(\mathbf{k},\lambda)\epsilon^{lm}(\mathbf{k},\lambda)$$

$$=\frac{1}{2}\left[\left[\delta^{il}-\frac{k^ik^l}{\mathbf{k}\cdot\mathbf{k}}\right]\left[\delta^{jm}-\frac{k^jk^m}{\mathbf{k}\cdot\mathbf{k}}\right]+\left[\delta^{im}-\frac{k^ik^m}{\mathbf{k}\cdot\mathbf{k}}\right]\left[\delta^{jl}-\frac{k^jk^l}{\mathbf{k}\cdot\mathbf{k}}\right]-\left[\delta^{ij}-\frac{k^ik^j}{\mathbf{k}\cdot\mathbf{k}}\right]\left[\delta^{lm}-\frac{k^lk^m}{\mathbf{k}\cdot\mathbf{k}}\right]\right]\ . \tag{4.25}$$

The Feynman propagator can also be expressed as

$$D_F(x-y)=i\theta(x_0-y_0)\int\frac{d^3k}{(2\pi)^3}\frac{1}{(2\omega_k)^{1/2}}e^{i\omega_k(x_0-y_0)+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}+i\theta(y_0-x_0)\int\frac{d^3k}{(2\pi)^3}\frac{1}{(2\omega_k)^{1/2}}e^{i\omega_k(y_0-x_0)+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$$

$$=-\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}D_{1F}(x_0-y_0;\omega_k)\ , \tag{4.26}$$

so that we obtain agreement with (4.15). When the scalar field is massless we have

$$D_F(x-y)=-\frac{1}{4\pi}\delta((x-y)^2)+i\frac{1}{4\pi^2}(x-y)^{-2}\ . \tag{4.27}$$

For comparison we record the Feynman propagator for the two- and one-dimensional scalar fields given by

$$\lim_{m\to 0}D_{F^2}(x-y)=\frac{i}{4\pi}\ln(m^2)+\frac{i}{4\pi}\ln[(x-y)^2]+\left[-\frac{1}{4}+\frac{i}{2\pi}(\gamma-\ln 2)\right],$$

$$D_{F1}(t-t';m)=\frac{1}{2im}[\theta(t-t')e^{-im(t-t')}+\theta(t'-t)e^{im(t-t')}]\ , \tag{4.28}$$

and for small values of m we have

$$\lim_{m\to 0}D_{F1}(t-t';m)=\frac{1}{2im}-\tfrac{1}{2}|t-t'|+\cdots\ . \tag{4.29}$$

Note that for $D\geq 3$ the propagator for a massless scalar is infrared finite, that is, it has a well-defined limit as $m\to 0$. It is known that energy-momentum conservation of strings or particles is related to this infrared divergence in $D=2$ and $D=1$ dimensions[13,18] and the consequential inability to generate vacuum expectation values for the one- and two-dimensional scalar fields, the zero modes of which represent position operators. One may well wonder if this infrared divergence is necessary for the existence of energy-momentum conservation. The formulation of a membrane theory of elementary particles based on any object than a string or a particle is in a somewhat more precarious position as the $D\geq 3$ Green's functions which appear in the membranes' analogue of the Veneziano amplitude are infrared finite. As $D=4$ in the case we are considering we do not expect anything as momentum conservation in the scattering amplitude for our WDW field. Actually as the Lagrangian on the world volume is interacting we would not expect canonical momenta to be conserved anyhow. This does not concern us as we are only dealing with an "analogue" of a three-dimensional membrane and not a theory of elementary particles. So the transition amplitude is proportional to

$$\prod_{i=1}^{n}\int d^4x_i a^3(t_i)\exp\left[\sum_{i,j} ik_\phi^i k_\phi^j\left[-\frac{1}{4\pi}\delta((x_i-x_j)^2)+i\frac{1}{4\pi^2}(x_i-x_j)^{-2}\right]\right]. \tag{4.30}$$

We now consider the case of three-universe scattering, $n=3$. This can be represented at least in the above approximations by

$$\langle n_3|\langle p_{3a},p_{3ij},p_{3\phi}|\exp\left[i\left[p_{2a}\hat{a}+\tilde{p}_{2ij}\hat{\tilde{\gamma}}_0^{ij}+\tilde{p}_{2\phi}\hat{\phi}_0+\tilde{p}_{2ij}\int\frac{d^3k}{(2\pi)^3}q^{ij}(k)+\tilde{p}_{2\phi}\int\frac{d^3k}{(2\pi)^3}q(k)\right]\right]|p_{1a},p_{1ij},p_{1\phi}\rangle|n_1\rangle . \tag{4.31}$$

We assume $|n_1\rangle=a^\dagger(\mathbf{k}_1)a^\dagger(-k_1)|0\rangle$ a two-particle state which satisfies the constraint $P^i_{\text{osc}}|n_1\rangle=0$, and take as the final oscillator state $\langle n_3|=\langle 0|$. In the (1,2) representation we really have two separate oscillators as $q(k)=[q_1(k)+iq_2(k)]a^{3/2}$ and consequently two values for β in (4.13) given by $\beta_{1k}=(2\omega_k)^{-1/2}p_{2\phi}$ and $\beta_{2k}=i(2\omega_k)^{-1/2}p_{2\phi}$ then we obtain, from the application of (4.15),

$$\langle 1,1|e^{ip_{2\phi}[q_1(k)+iq_2(k)]\hat{a}^{3/2}}|0,0\rangle=i\beta_{1k}i\beta_{2k}G_{00}=-iG_{00}(2\omega_k)^{-1}p_{2\phi}^2 . \tag{4.32}$$

So we have

$$\begin{aligned}A(1,2,3)&=\langle p_{3a},p_{3ij},p_{3\phi}|e^{ip_{2a}\hat{a}+i\tilde{p}_{2ij}\hat{\tilde{\gamma}}^{ij}+ip_{2\phi}\hat{\phi}}[-iG_{00}(2\omega_k)^{-1}p_{2\phi}^2]|p_{1a},p_{1ij},p_{1\phi}\rangle\\&=-iG_{00}p_{2\phi}^2\delta(p_{3\phi}-p_{2\phi}-p_{1\phi})\delta(p_{3ij}-p_{2ij}-p_{1ij})\langle p_{3a}|e^{ip_{2a}\hat{a}}(2\omega_k)^{-1}|p_{1a}\rangle\\&=-iG_{00}p_{2\phi}^2\delta(p_{3\phi}-p_{2\phi}-p_{1\phi})\delta(p_{3ij}-p_{2ij}-p_{1ij})V(p_3,p_2,p_1) .\end{aligned} \tag{4.33}$$

Formula (4.33) gives the scattering amplitude for three-momentum eigenstates. The solutions to the WDW equation even in asymptotic regions of superspace are not eigenstates of the canonical momenta. However, as

$$|\psi\rangle=\int dp\,|p\rangle\langle p|\psi\rangle=\int dp\,|p\rangle\langle p|a\rangle\langle a|\psi\rangle \tag{4.34}$$

we can always represent our solution as a superposition of such states. Given our solution (3.47) we obtain

$$|\psi\rangle=\int dp\,|p\rangle\int da\,e^{ipa}\psi(a)=\int dp\,|p\rangle\int da\,e^{ipa}[2V_\phi V_{ij}p_0(a^3)]^{-1/2}\exp\left[\pm i\int d(a^3)p_0(a^3)\right] . \tag{4.35}$$

Defining

$$\begin{aligned}V(a_1,a_2,a_3)&=\int dp_1dp_2dp_3e^{ip_1a_1-ip_2a_2-ip_3a_3}V(p_1,p_2,p_3)\\&=\int dp_1dp_2dp_3\langle p_1|a_1\rangle\langle a_2|p_2\rangle\langle a_3|p_3\rangle\langle p_3|e^{ip_2\hat{a}}(2\omega_k)^{-1}|p_1\rangle\\&=\int dp_2\langle a_3|e^{ip_2\hat{a}}(2\omega_k)^{-1}|a_1\rangle e^{-ip_2a_2}=\delta(a_3-a_1)\delta(a_2-a_1)(2\omega_k)^{-1} .\end{aligned} \tag{4.36}$$

Our amplitude should be proportional to

$$\int da_1da_2da_3\psi_1^0(a_1)\psi_2^0(a_2)\psi_3^0(a_3)V(a_1,a_2,a_3)=\int_{a_0}^{\infty}da\,\psi_1^0(a)\psi_2^0(a)\psi_3^0(a)(2\omega_k)^{-1} . \tag{4.37}$$

So we have finally

$$A(1,2,3)=-iG_{00}p_{2\phi}^2\delta(p_{3\phi}-p_{2\phi}-p_{1\phi})\delta(p_{3ij}-p_{2ij}-p_{1ij})\int_{a_0}^{\infty}da\,\psi_1^0(a)\psi_2^0(a)\psi_3^0(a)(2\omega_k)^{-1} , \tag{4.38}$$

where

$$\psi_3^0(a)=\left[l^{-4}\frac{\Lambda(0)}{3}a^4+l^{-2}\rho(n,a)+p_{3\phi}p_{3\phi}l^{-2}a^{-2}+p_{3ij}p_{3ij}a^{-2}\right]^{-1/4}$$
$$\times\exp\left[+i\int_{a_0}^{a}da'\left[l^{-4}\frac{\Lambda(0)}{3}a^4+l^{-2}\rho(n,a')+p_{3\phi}p_{3\phi}l^{-2}a'^{-2}+p_{3ij}p_{3ij}a'^{-2}\right]^{1/2}\right],$$
$$\psi_2^0(a)=\left[l^{-4}\frac{\Lambda(0)}{3}a^4+l^{-2}\rho(n,a)+p_{2\phi}p_{2\phi}l^{-2}a^{-2}+p_{2ij}p_{2ij}a^{-2}\right]^{-1/4}$$
$$\times\exp\left[-i\int_{a_0}^{a}da'\left[l^{-4}\frac{\Lambda(0)}{3}a^4+l^{-2}\rho(n,a')+p_{2\phi}p_{2\phi}l^{-2}a'^{-2}+p_{2ij}p_{2ij}a'^{-2}\right]^{1/2}\right], \tag{4.39}$$
$$\psi_1^0(a)=\left[l^{-4}\frac{\Lambda(0)}{3}a^4+l^{-2}\rho(n,a)+p_{1\phi}p_{1\phi}l^{-2}a^{-2}+p_{1ij}p_{1ij}a^{-2}\right]^{-1/4}$$
$$\times\exp\left[-i\int_{a_0}^{a}da'\left[l^{-4}\frac{\Lambda(0)}{3}a^4+l^{-2}\rho(n,a')+p_{1\phi}p_{1\phi}l^{-2}a'^{-2}+p_{1ij}p_{1ij}a'^{-2}\right]^{1/2}\right].$$

In Appendix B we compute this overlap integral (4.38) and discuss how one computes the probability that one universe will decay into two others for the special case when $\rho(n,a)=0$, that is, when the universes are in the oscillator ground state corresponding to the absence of particles. The main distinction between such a computation and standard derivation for the decay rate of a particle into two others is that in the latter case (4.38) would be replaced by

$$\int_{-\infty}^{\infty}dt\,(2VE_3)^{-1/2}e^{itE_3}(2VE_1)^{-1/2}e^{-itE_1}(2VE_2)^{-1/2}e^{-itE_2}=(2E_3V2E_1V2E_2)^{-1/2}(2\pi)\delta(E_3-E_2-E_1)\,, \tag{4.40}$$

where $E_i=\sqrt{p_i^2+m_i^2}$ and V is a spatial cutoff. Upon squaring the above, one obtains

$$(2E_3V2E_1V2E_2V)^{-1}T(2\pi)\delta(E_3-E_2-E_1)\,, \tag{4.41}$$

where we have introduced a temporal cutoff T. In our case instead of (4.40) we have

$$\int_{x_0}^{\infty}dx[2p_{03}(x)V2p_{02}(x)V2p_{01}(x)V]^{-1/2}\exp\left[i\int_{x_0}^{x}dx'[p_{03}(x')-p_{02}(x')-p_{01}(x')]\right], \tag{4.42}$$

where we have defined $x=l^{-2}a^3$, $V=V_\phi V_\beta$, and

$$p_{0i}(x)=\left[\frac{\Lambda^{(i)}(0)}{3\times 9}+l^{-2/3}\rho(n^{(i)},x)\frac{x^{-4/3}}{9}+p_\phi^{(i)}p_\phi^{(i)}x^{-2}\frac{l^{-2}}{9}+p_{jk}^{(i)}p_{jk}^{(i)}\frac{x^{-2}}{9}\right]^{1/2}. \tag{4.43}$$

Equation (4.42) differs from the familiar particle case in that x is restricted to the positive and in the interval $[x_0,\infty]$. The result of the integration is not a delta function and there is no analogue of energy conservation. Consequently the square of (4.42) does not introduce any time cutoff and one ends up computing probabilities rather than transition rates. The volume cutoff introduced in (3.52) cancels out when one calculates this probability. This is explicitly shown in Appendix B.

Another case which is under investigation involves the decay of universes which belong to different Hilbert spaces. Just as one can allow the transition in string theory between different sectors specified by different choices for the two-dimensional boundary conditions so we should be able to make transitions in our four-dimensional setting between field configurations with different boundary conditions. The universes with different boundary conditions have different values of the cosmological constant as well as different physical properties. There is some hope of obtaining a workable form of the anthropic principle with such a model.[6] The only predictions that can be made deal with the most probable final-state configurations. Such behavior has an analogue in particle theory. After many subsequent decays a very heavy particle will decay into a final-state configuration through a cascade process. The most probable outcome is that one is left with particles of the smallest mass[2] available, massless particles if they are present. We hope to find that the final-state configurations do not contain any universes with nonzero values of the cosmological constant as the most probable configurations.[7] It is possible that this is the only quantity that the final-state universes have in common. So we would be left with many final-state universes with a myriad of physical properties all with cosmological constant zero. A happy state of affairs to invoke the anthropic principle.

V. CONCLUSION AND DISCUSSION

We have argued that the WDW equation naturally leads itself to a second quantization in analogy with particle and string theories which can be considered as one- and two-dimensional gravities coupled to matter. We have identified a conserved current, as well as DeWitt's proposal for the normalization of solutions to the WDW equation as coming from a Lagrangian which is the analogue of a second-quantized string theory moving on the coset manifold GL(3,R)/SO(3). Finally we have attempted to construct a mode expansion of the WDW second-quantized field and isolate the zero-mode degrees of freedom.

For a given theory (gravity plus matter) one can construct a second-quantized WDW field associated to it. In much the same way as one can associate a second-quantized field to each elementary particle. Just as allowing interactions among particles can give rise to processes in which one elementary particle decays into others, the presence of an interacting term between different WDW fields associated with the different sectors should allow one to study the decay of a vacuum associated with a given theory into vacuum states described by completely different theories. This can be of great practical value to superstring theories where each vacuum looks like a different string theory with apparently no natural way to pass from one vacua to another. The two-dimensional gravity one uses in string theory naturally includes processes in which different sectors, associated with the choice of different boundary conditions on the two-dimensional fields living on the world surface, can decay into one another. The many different four-dimensional string theories mentioned above simply arise from choosing different boundary conditions on the massless ten-dimensional fields. For example, choosing periodic boundary conditions on some components of the gravitational field gives rise to the string theory associated with a toroidal compactification. It is clear that a large class if not all the different string models can be constructed in this way. It should be possible in this formalism to compute the probability for the process of one vacuum decaying into another. In the two-dimensional case one can conveniently obtain the spectra of the twisted sectors by operating on the untwisted sector with global diffeomorphisms (modular transformations). A higher-dimensional version of this process might make it unnecessary to explicitly add a different second-quantized WDW field for each choice of vacuum, as is customarily done in particle theory (one associates to each separate particle a different field) as there should exist a mapping from one sector to another.

We prefer to call the process of second quantizing the WDW equation third quantization (this terminology is also used in the work of Giddings and Strominger[2]). In first quantization one constructs a Schrödinger wave function of the spatial coordinates $\mathbf{x}$. The Schrödinger equation describes its evolution in time

$$\left[i\frac{\partial}{\partial t}+H(\mathbf{x},-i\nabla)\right]\Psi(\mathbf{x},t)=0\ . \tag{5.1}$$

This is inconsistent with the global symmetries of special relativity, however, so one writes a new equation: the Klein-Gordon equation

$$\left[-\frac{\partial^2}{\partial t^2}+\nabla^2-m^2\right]\phi(\mathbf{x},t)=0\ . \tag{5.2}$$

This equation leads to problems with negative probabilities. To remedy the situation one treats the $\phi(\mathbf{x})$ as a dynamical variable and writes

$$\left[i\frac{\partial}{\partial t}+\int d^3x\left[-\frac{\delta^2}{\delta\phi(\mathbf{x})\delta\phi(\mathbf{x})}+m^2\phi(\mathbf{x})\phi(\mathbf{x})\right]\right]\Psi(\phi(\mathbf{x}),t)=\left[i\frac{\partial}{\partial t}+H\left[\phi(\mathbf{x}),-i\frac{\delta}{\delta\phi(\mathbf{x})}\right]\right]\Psi(\phi(\mathbf{x}),t)=0\ . \tag{5.3}$$

Unfortunately this form of the equation does not possess reparametrization invariance of the world volume. Also it assumes that there is a global way to define time, whereas general relativity tells us the best we can hope for is a function $t(\mathbf{x})$ as clocks tick at different rates depending on the magnitude of the gravitational field at $\mathbf{x}$. This is remedied by going to the WDW equation:

$$\left[\int d^3x\left[\tfrac{1}{2}\gamma^{1/2}(\gamma_{ik}\gamma_{jl}+\gamma_{il}\gamma_{jk}-\gamma_{ij}\gamma_{kl})\frac{\delta}{\delta\gamma_{ij}}\frac{\delta}{\delta\gamma_{kl}}+(-\gamma^{1/2}\,{}^{(3)}R)\right]\right.$$
$$\left.+\int d^3x\left(\tfrac{1}{2}\right)\left[\gamma^{-1/2}\frac{\delta}{\delta\phi}\frac{\delta}{\delta\phi}-\gamma^{1/2}\gamma^{ij}\partial_i\phi\partial_j\phi-\gamma^{1/2}m^2\phi\phi\right]\right]\Phi(\gamma_{ij},\phi)=0\ . \tag{5.4}$$

This equation also possesses global symmetries. Attempting to normalize states while respecting these symmetries naturally leads to a problem with negative probabilities. One way to avoid them is to treat $\Phi(\tilde{\gamma}_{ij},\phi)$ as a dynamical variable. Define a wave function $\Psi(\Phi(\tilde{\gamma}_{ij},\phi),\gamma^{1/2})$ and evolve in $\gamma^{1/2}$ through the equation

$$\left[\frac{i\delta}{\delta\gamma^{1/2}}+H(\Phi(\tilde{\gamma}_{ij},\phi),\Pi(\tilde{\gamma}_{ij},\phi),\gamma^{1/2})\right]\times\Psi(\Phi(\tilde{\gamma}_{ij},\phi),\gamma^{1/2})=0\ , \tag{5.5}$$

where

$$H(\Phi(\tilde{\gamma}_{ij},\phi),\Pi(\tilde{\gamma}_{ij},\phi),\gamma^{1/2})=\int D\tilde{\gamma}_{ij}D\phi\left[-\tfrac{8}{3}\gamma^{-1/2}\Pi^2+\gamma^{-1/2}\tilde{\gamma}_{ik}\tilde{\gamma}_{ik}\frac{\delta\Phi}{\delta\tilde{\gamma}_{ij}}\frac{\delta\Phi}{\delta\tilde{\gamma}_{ij}}+\gamma^{-1/2}\frac{\delta\Phi}{\delta\phi}\frac{\delta\Phi}{\delta\phi}+\gamma^{1/2}(\tilde{\gamma}^{ij}\partial_i\phi\partial\phi_j+m^2l^2\phi^2-{}^{(3)}R)\Phi\Phi\right] \tag{5.6}$$

and $\Pi=-\gamma^{1/2}\frac{3}{8}(\delta/\delta\gamma^{1/2})\Phi$. This would then represent a third quantization as the wave function associated with the second-quantized field becomes a dynamical variable.

Another aspect of this theory might help with one of the interpretational problems of string theory. If one takes the point of view that string theory is two-dimensional gravity plus matter, with ten-dimensional spacetime emerging as an effective concept, as for example, has been emphasized by Friedan. Then what properties of the two-dimensional world become promoted to properties of the ten-dimensional world built out of the zero modes of the two-dimensional fields. We can rephrase this argument by saying that ten-dimensional spacetime arises from second quantizing the WDW equation of two-dimensional gravity and D scalar fields as well as their supersymmetric counterparts. Or in the above language ten-dimensional spacetime arises from third quantizing the two-dimensional field theory.

A prime example of a property of the two-dimensional field theory having a dramatic effect on the ensuing third-quantized theory is the emergence of translational invariance in D dimensions. If the two-dimensional field theory were interacting on the world sheet then the integral

$$G(x_1^\mu,x_2^\mu,\ldots,x_n^\mu)=\int Dg_{\alpha\beta}(\sigma^\alpha)DX^\mu(\sigma^\alpha)\exp\left[i\int d^2\sigma\sqrt{-g}\,[g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu+V(X)]\right]\times\prod_{i=1}^{n}d^2\sigma_i\sqrt{-g(\sigma_i)}\,\delta^D(X^\mu(\sigma_i)-x_i^\mu) \tag{5.7}$$

would not be invariant under the translation of the zero modes x_i^μ and this would lead to the loss of energy-momentum conservation in the D-dimensional theory. That is if we include an arbitrary interaction term $V(X)$ in the two-dimensional field theory only for $V=\text{const}$ do we obtain translation invariance in the D-dimensional theory built out of the zero modes x_0^μ of our two-dimensional scalar fields.

Even more bizarre is the emergence of general coordinate invariance in the third quantized (ten-dimensional theory). This arises because one of the possible two-dimensional universes that can be created has $\Lambda(n)=0$. From the ten-dimensional point of view this is seen as a massless particle. The WDW field associated with that two-dimensional universe (particle) the gravitational field. We know that only when such a two-dimensional universe is present can the ten-dimensional world be generally coordinate invariant. But as we have seen each two-dimensional theory leads directly upon third quantization to a Lagrangian describing the propagation and interaction of a WDW field which is a function of the zero modes of the two-dimensional fields. This Lagrangian may not have conservation with respect to the canonical momentum of the zero modes and even if it does would not lead to ten-dimensional spacetime unless the two-dimensional theory contained universes with $\Lambda(n)=0$. Whether such properties are present can be determined by inspecting the two-dimensional field theory. With only one such theory it is difficult to see how spacetime properties could emerge naturally, the typical two-dimensional field theory would not give rise to anything like Minkowski spacetime upon third quantization, nor do they typically contain states with $\Lambda(n)=0$ which could be interpreted as massless particles. Fortunately there exists a formulation of string theory which makes use of an infinite tower of two-dimensional field theories.

In defining $Z_4(0)$ in Eq. (2.1) we have assumed that the path integral can be generalized to include an infinite number of fields in such a way they can be obtained from the mode expansion of a string field as

$$\Psi(g_{ij}(\mathbf{x}),\phi_I(\mathbf{x}),\ldots)\to\Psi(\Phi(x^+,x^-,X^i(\sigma)))\to\Psi(\Phi(x^+,p^+,x^i,\alpha_n^i))\ . \tag{5.8}$$

Such expressions can be given a more tangible representation by expanding in the oscillator basis. For example, for the 26-dimensional bosonic string we have[19]

$$\Phi_{26}(x^+,p^+,x^i,\alpha_n^i)=\phi(x)\,|\,0\rangle+g_{ij}(x)\alpha_{-1}^i\tilde{\alpha}_{-1}^j\,|\,0\rangle+\cdots\ . \tag{5.9}$$

The α oscillators are in turn defined by the expansion

$$X^\mu(\sigma^\alpha)=x_0^\mu+p_0^\mu\sigma^0+i(\alpha'/2)^{1/2}\sum_{n\neq0}\frac{1}{n}(\alpha_n^\mu e^{-2i(\sigma^0-\sigma^1)}+\tilde{\alpha}_n^\mu e^{-2i(\sigma^0+\sigma^1)})\ , \tag{5.10}$$

so that (5.8) represents the WDW field associated with the string field. One can proceed as above to eliminate the ultraviolet divergence inherent in the path integral. A similar procedure has been followed by Green to eliminate the only divergence occurring in the free two-dimensional theory, the leading contribution to the two-dimensional cosmological constant.[20,21] In doing so he passed from a two-dimensional gravity to a two-dimensional string theory. Then the Feynman diagrams of the two-dimensional theory become world sheets and

$$\Psi(\Phi(x^+,x^-,X^i(\sigma)))\to\Psi(\Phi_{10}(\Phi_2(\phi^+,\phi^-,\phi^-(\sigma)))\ . \tag{5.11}$$

The mode expansion for this two-dimensional string theory is written

$$\Phi_2(\sigma_0^\alpha,\beta_n^\alpha,\rho_n^I)=W(\sigma)\,|\,0\rangle+X^I(\sigma)\rho_{-1}^I\,|\,0\rangle+Y^{IJ}(\sigma)\rho_{-1}^I\rho_{-1}^J\,|\,0\rangle+\cdots\ , \tag{5.12}$$

where ρ and β oscillators are defined by[20,21]

$$\sigma^1(\eta)=\sigma_0^1+(m\alpha'/r)2\eta^0+nr2\eta^1+i(\alpha'/2)^{1/2}\sum_{n\neq 0}\frac{1}{l}\beta_l^1\exp[-2i(\eta^0-\eta^1)]$$
$$+i(\alpha'/2)^{1/2}\sum_{n\neq 0}\frac{1}{l}\tilde{\beta}_l^1\exp[-2i(\eta^0+\eta^1)]\ ,$$
$$\rho^I(\eta^0-\eta^1)=\rho_0^I+p_0^I(\eta^0-\eta^1)+i(\alpha'/2)^{1/2}\sum_{n\neq 0}\frac{1}{n}(\rho_n^I e^{-2i(\eta^0-\eta^1)})\ . \qquad (5.13)$$

The coordinates $\rho^I(\eta^0-\eta^1)$ $(I=1,\ldots,24)$ are left-moving coordinates which define the torus associated with the Leech lattice. Green also suggested the only way to remove the ultraviolet divergences altogether is to repeat the process indefinitely giving rise to the wave function

$$\Psi(\Phi_{10}(\Phi_2^1(\Phi_2^2(\Phi_2^3(\cdots)))))\ . \qquad (5.14)$$

We have shown that it is possible to extend the process on more levels to the left obtaining

$$\Psi(\Phi_{(5+n_m,1)}(\Phi_{10}(\Phi_2^1(\Phi_2^2(\Phi_2^3(\cdots))))))\ , \qquad (5.15)$$

where n_m denotes the number of matter fields.

Of course superspace is infinite dimensional as is loop space. We simply count the zero modes associated with the massless fields present in each case and list that number as a subscript on the WDW field. For example, in string theory one starts with two-dimensional gravity plus D scalar fields, if one of the scalar fields has the wrong sign for its kinetic energy then the WDW equation associated with this two-dimensional gravity theory has a metric associated with its superspace that is of indefinite signature. Note in two-dimensional gravity this indefinite nature does not arise from the two-dimensional metric this is why one needs the scalars. When one tries to normalize solutions to this two-dimensional WDW equation one runs into the problem with negative probabilities in exactly the same place as with the Klein-Gordon equation. Indeed the zero mode portion of the two-dimensional WDW equation is the Klein-Gordon equation. Each Φ_2^i describes a different two-dimensional field theory in its $\alpha'\to 0$ limit with their own sets of values $\Lambda^{(i)}(n)$. Also, above each two-dimensional theory one can construct another $\Phi_{n_i}^{(i+1)}$ by second quantizing the WDW equation associated with the two-dimensional theory where n_i are the number of fields such that $\Lambda^{(i)}(n)=0$. For $n_i=10$ we would have ten-dimensional theory. Unless this theory had a spin-two graviton in its spectrum there would be no general coordinate invariance and no WDW equation with which to further second quantize. With the infinite tower of two-dimensional theories each its own two-dimensional field content there should come a place along the infinite chain where the two-dimensional universes such that $\Lambda(n)=0$ exist and with them general coordinate invariance at the next level.

In order to continue the process we would need the analogue of general coordinate invariance in the minisuperspace constructed out of the zero modes of the four-dimensional fields. We have not found the analogue of energy-momentum conservation associated with the canonical momentum. Superspace has a background metric $G^{ijkl}(\gamma_{ij})$ on it so we should not expect such conservation. Yet the fact that there exist universes with $\Lambda_4(n)=0$ [take our own Universe with $n=0$, the cosmological constant $\Lambda(0)$ vanishes in our Universe] suggests that perhaps there could exist a reparametrization invariance in the space of zero modes in which case there would not be a Schrödinger equation for our second-quantized WDW field but another WDW equation. The second quantization of this would, in principle, yield

$$\Psi(\Phi_?(\Phi_{(5+n_m,1)}(\Phi_{10}(\Phi_2^1(\Phi_2^2(\Phi_2^3(\cdots)))))))\ . \qquad (5.16)$$

Whether the process can continue depends upon whether it is possible to find universes such that the WDW field which creates them leads to a reparametrization invariance in the configuration space of the second-quantized WDW field theory. One way to avoid having to ever write down a Schrödinger equation is to continue the process indefinitely to the left, one way of never encountering ultraviolet divergences is to continue the process indefinitely to the right. Although the above discussion is highly speculative it seems to provide a framework for a modern formulation of the original bootstrap hypothesis in which the original object in a statistical ensemble is represented as a statistical ensemble of other objects only now we do not assume that second set of objects are identical to the first.

Note added in proof. The Klein-Gordon inner product of the solutions to the WDW equation is also advocated in the work of Alexander Vilenkin.[22]

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This work was inspired by a talk from Sydney Coleman who alluded to certain analogies with string perturbation theory as being useful in the study of quantum gravity. In effect what we have tried to do is generalize the treatment of light-cone string field theory in the context of four- rather than two-dimensional quantum gravity. Recently Tom Banks has proposed second quantizing the WDW equation in the context of the cosmological constant problem along the lines of a four-dimensional version string-field theory which he refers to as universal field theory. Also Steve Giddings and Andrew Strominger have proposed going to a third quantization in order generate all possible four-dimensional topologies in the same way string-field theory generates all two-dimensional topologies. These works partially overlap the treatment given here. Finally we wish to acknowledge useful conversations with Mark Rubin, Carlos Ordoñez, and H. C. Ren. This work was supported in part under Department of Energy Contract No. DE-AC02-87ER40325.

APPENDIX A: OPERATOR ORDERING FOR DeWITT's NORMALIZATION

In a minisuperspace model one usually allows for an arbitrary choice of operator ordering[23] by writing

$$\frac{1}{2}\left[l^{-2}a^{-p}\frac{\partial}{\partial a}\left[a^{p}\frac{\partial}{\partial a}\Psi^0(a)\right]+\Lambda a^4 l^{-2}\Psi^0(a)\right]=0\ , \tag{A1}$$

where the parameter ρ is included in order to take into account possible operator ordering problems. We have set the matter contributions and $k=0$ for the purpose of illustration. Λ is the cosmological constant. This equation represents the zero-mode portion of

$$(-G)^{-1/2}\frac{\delta}{\delta\gamma_{ij}}\left[(-G)^{1/2}G_{ijkl}\frac{\delta}{\delta\gamma_{kl}}\Phi\right]=(-\gamma^{1/2\,(3)}R)\Phi\ , \tag{A2}$$

where one writes $\Phi=\Psi^0_{(n)}(a)\Psi_{(n)}(q^{ij}(\mathbf{k}))$ as discussed in Sec. III. We express (A1) as

$$g^{1/2}(a)\frac{\partial}{\partial a}[g^{1/2}(a)]g^{-1}(a)\frac{\partial}{\partial a}\Psi^0_{(n)}(a)=\tilde{V}(a)\Psi^0_{(n)}(a)\ , \tag{A3}$$

where $g=a^{-2\rho}$ allows for the operator ordering ambiguity[22] and $\tilde{V}(a)=-g^{-1}\Lambda a^4 l^{-4}$. One can write a Lagrangian from which (A3) follows as a field equation. It is

$$L=(g)^{1/2}g^{-1}\frac{\partial\Psi^*}{\partial a}\frac{\partial\Psi}{\partial a}-g^{1/2}\tilde{V}(a)\Psi^*\Psi\ . \tag{A4}$$

The conserved current associated to the symmetry $\Psi\to e^{i\phi}\Psi$ is

$$J=i(g)^{1/2}g^{-1}\left[\frac{\partial\Psi^*}{\partial a}\Psi-\frac{\partial\Psi}{\partial a}\Psi^*\right]\ . \tag{A5}$$

For the choice $\rho=-2$ so that $g(a)=a^4$ we obtain

$$-a^{-2}\frac{\partial}{\partial a}\left[a^2 a^{-4}\frac{\partial\Psi}{\partial a}\right]=-\Lambda l^{-4}\Psi \tag{A6}$$

which can be written in the convenient form

$$-9\frac{\partial}{\partial(a^3)}\frac{\partial}{\partial(a^3)}\Psi+\Lambda l^{-4}\Psi=0 \tag{A7}$$

and the current J becomes

$$J=3i\left[\frac{\partial\Psi^*}{\partial(a^3)}\Psi-\frac{\partial\Psi}{\partial(a^3)}\Psi^*\right]\ . \tag{A8}$$

Now using our WKB solution for large a in (3.48) we take

$$\Psi(a)=(\Lambda/3)^{-1/4}\exp[\pm ia^3(\Lambda/3)^{1/2}l^{-2}] \tag{A9}$$

and obtain $J=\pm 6$. Note that J is independent of a as it should be. For general $g(a)$ we should choose

$$\Psi(a)=a^{-2}[g(a)]^{1/2}(\Lambda/3)^{-1/4}\times\exp[\pm ia^3(\Lambda/3)^{1/2}l^{-2}] \tag{A10}$$

in order to obtain a current independent of a. Only for the choice $g(a)=a^4$ do we obtain a simple plane-wave current.

APPENDIX B: CUBIC INTERACTION FROM A THREE-UNIVERSE OVERLAP

In this appendix we wish to demonstrate that the superspace volume cutoffs V_ϕ and V_β introduced in (3.52) in order to obtain normalized solutions to the WDW equation cancel out when one computes the probability for the Universe to undergo a given process. This has a direct analogue in particle theory where the spacetime volume cutoffs disappear when one computes transition rates. Rewriting (4.38) as

$$A(1,2,3)=g\delta(p_{3\phi}-p_{2\phi}-p_{1\phi})\delta(p_{3ij}-p_{2ij}-p_{1ij})M\ , \tag{B1}$$

where

$$M=\int_{x_0}^{\infty}dx\,[p_{03}(x)p_{02}(x)p_{01}(x)]^{-1/2}\exp\left[i\int_{x_0}^{x}dx'p_{03}(x')-i\int_{x_0}^{x}dx'p_{02}(x')-i\int_{x_0}^{x}dx'p_{01}(x')\right] \tag{B2}$$

and we have defined

$$p_{0(i)}(x)=[m_i^2+\mathbf{p}_i\cdot\mathbf{p}_i l^2/x^2+l^{-2/3}\rho_i(n,x)x^{-4/3}]^{1/2} \tag{B3}$$

as well as $\mathbf{p}=[(l^{-2}/3)p_\phi,(l^{-1}/3)p_{ij}]$, $m_i^2=\Lambda_i(0)(3\times 9)^{-1}$, and $x=(1/l^2)a^3$. We wish to consider the situation where $\psi_i(x)=[p_{0i}(x)]^{-1/2}\exp[i\int_{x_0}^{x}dx'p_{0i}(x')]$ defines a state representing a universe with different values of the cosmological constant Λ_i. Furthermore, we shall assume they are vacuum states so that $\rho(n,a)=0$ in which case we have $p_{0(i)}(x)=(m_i^2+\mathbf{p}_i\cdot\mathbf{p}_i l^2/x^2)^{1/2}$.

If we take $\mathbf{p}_3=0$ and set $m_3=m$ we obtain

$$\begin{aligned}M=\int_{x_0}^{x}dx\,&e^{im(x-x_0)}m^{-1/2}(m_1^2+p_1^2l^2/x^2)^{-1/4}e^{-i(m_1^2x^2+p_1^2l^2)^{1/2}+i(m_1^2x_0^2+p_1^2l^2)^{1/2}}\\&\times(|\mathbf{p}_1|l+\sqrt{m_1^2x^2+p_1^2l^2})^{i|\mathbf{p}_1|l}(|\mathbf{p}_1|l+\sqrt{m_1^2x_0^2+p_1^2l^2})^{-i|\mathbf{p}_1|l}\left[\frac{x}{x_0}\right]^{-i|\mathbf{p}_1|l}\\&\times(m_2^2+p_2^2l^2/x^2)^{-1/4}e^{-i(m_2^2x^2+p_2^2l^2)^{1/2}+i(m_2^2x_0^2+p_2^2l^2)^{1/2}}\\&\times(|\mathbf{p}_2|l+\sqrt{m_2^2x^2+p_2^2l^2})^{i|\mathbf{p}_2|l}(|\mathbf{p}_2|l+\sqrt{m_2^2x_0^2+p_2^2l^2})^{-i|\mathbf{p}_2|l}\left[\frac{x}{x_0}\right]^{-i|\mathbf{p}_2|l}\ .\end{aligned} \tag{B4}$$

Then the probability that the Universe will undergo such a transition is given by

$$\text{Prob}=|M|^2[(2V)^{-1/2}(2V)^{-1/2}(2V)^{-1/2}]^2 V\delta(\mathbf{p}_1+\mathbf{p}_2)V\frac{d^6p_1 V}{(2\pi)^6}\frac{d^6p_2}{(2\pi)^6}$$
$$=2^{-3}|M|^2\delta(\mathbf{p}_1+\mathbf{p}_2)\frac{d^6p_1}{(2\pi)^6}\frac{d^6p_2}{(2\pi)^6}\ . \qquad \text{(B5)}$$

The superspace volume which we used to normalize our states cancels out when one computes probabilities. Also in the special case when $m_1=m_2=0$ we can explicitly evaluate M:

$$M=\int_{x_0}^{\infty}dx\,(2m)^{-1/2}e^{im(x-x_0)}(2|\mathbf{p}_1|2|\mathbf{p}_2|)^{-1/2}\left[\frac{x}{l}\right]^{1/2}\left[\frac{x}{x_0}\right]^{-i|\mathbf{p}_1|l}\left[\frac{x}{l}\right]^{1/2}\left[\frac{x}{x_0}\right]^{-i|\mathbf{p}_2|l}$$
$$=(2m2|\mathbf{p}_1|2|\mathbf{p}_2|)^{-1/2}e^{-imx_0+i(l|\mathbf{p}_1|+l|\mathbf{p}_2|)\ln(x_0)}l^{-1}\int_{x_0}^{\infty}dx\,x^{1-i|\mathbf{p}_1|l-i|\mathbf{p}_2|l}e^{-imx}\ . \qquad \text{(B6)}$$

The x_0 dependence is contained within a phase and the lower limit of the integral. For x_0 small one can use $\int_0^\infty dx\,x^z e^{ikx}=e^{i(\pi/2)(1+z)}k^{-1-z}\Gamma(1+z)$ to evaluate the integral. With $z=1-i|\mathbf{p}_1|-i|\mathbf{p}_2|$ we obtain

$$M=l^{-1}(m|\mathbf{p}_1||\mathbf{p}_2|)^{-1/2}e^{-imx_0+il\ln x_0(|\mathbf{p}_1|+|\mathbf{p}_2|)l}e^{i(\pi/2)[2-i(|\mathbf{p}_1|+|\mathbf{p}_2|)l]}m^{[-2+i(|\mathbf{p}_1|+|\mathbf{p}_2|)l]}\Gamma(2-i|\mathbf{p}_1|l-i|\mathbf{p}_2|l)\ . \qquad \text{(B7)}$$

The norm squared of the matrix element is given by

$$|M|^2=\tfrac{1}{8}l^{-2}m^{-5}|\mathbf{p}_1|^{-1}|\mathbf{p}_2|^{-1}(\pi l(|\mathbf{p}_1|+|\mathbf{p}_2|)\{\sinh[\pi l(|\mathbf{p}_1|+|\mathbf{p}_2|)]\})^{-1}\prod_{s=1}^{2}[s^2+l^2(|\mathbf{p}_1|+|\mathbf{p}_2|)^2]\ . \qquad \text{(B8)}$$

For small values of p_1^2 and p_2^2 we obtain

$$|M|^2\approx\tfrac{1}{2}l^{-2}m^{-5}|\mathbf{p}_1|^{-1}|\mathbf{p}_2|^{-1}\ . \qquad \text{(B9)}$$

Now computing the probability of the transition from (B5) the integration over $\mathbf{p}_2$ can be done trivially because of the momentum delta function and we are left with

$$\text{Prob}=\tfrac{1}{8}{}^2l^{-2}m^{-5}g^2\int\frac{d^6p}{(2\pi)^6}\frac{1}{|p|^2}\frac{2\pi l|p|}{\sinh(2\pi l|p|)}4(1+5l^2|p|^2+4l^4|p|^4)\ . \qquad \text{(B10)}$$

The above integral has a simple generalization to the case where $(d-1)$ four-dimensional fields contribute spatially to the WDW equation. For example, if we had five fields associated with the anisotropy and $(d-6)$ scalar fields we would have

$$\text{Prob}=\tfrac{1}{8}{}^2l^{-2-(d-3)}m^{-5}g_d^2\Omega_{d-1}\frac{1}{(2\pi)^{d-1}}\int_0^\infty dn\,n^{d-2}\frac{1}{n^2}\frac{2\pi n}{\sinh(2\pi n)}4(1+5n^2+4n^4)$$
$$=\tfrac{1}{8}{}^2l^{-2-(d-3)}m^{-5}g_d^2\Omega_{d-1}\frac{8}{(2\pi)^{d-1}}\left[\frac{\zeta(d-2)}{(2\pi)^{d-3}}(d-3)!(1-2^{-(d-2)})+5\frac{\zeta(d)}{(2\pi)^{d-1}}(d-1)!(1-2^{-d})\right.$$
$$\left.+4\frac{\zeta(d+2)}{(2\pi)^{d+1}}(d+1)!(1-2^{-(d+2)})\right]\ , \qquad \text{(B11)}$$

where Ω_{d-1} is the volume of the $(d-1)$-dimensional unit sphere $2\pi^{d/2}[\Gamma(d/2)]^{-1}$. The coupling constant g_d is dimensionally of the form $g_d=l^{(d-6)/2}g_0$ where g_0 is dimensionless. So we have finally

$$\text{Prob}=\tfrac{1}{8}{}^2l^{-5}m^{-5}g_0^2\Omega_{d-1}\frac{8}{(2\pi)^{d-1}}\left[\frac{\zeta(d-2)}{(2\pi)^{d-3}}(d-3)!(1-2^{-(d-2)})\right.$$
$$\left.+5\frac{\zeta(d)}{(2\pi)^{d-1}}(d-1)!(1-2^{-d})+4\frac{\zeta(d+2)}{(2\pi)^{d+1}}(d+1)!(1-2^{-(d+2)})\right]\ . \qquad \text{(B12)}$$

Typically $m \sim l^{-1}$ so the probability for such an emission is appreciable. The dimensionless coupling g_0 can, in principle, be computed if one knows the mechanism which gives rise to the different values of the cosmological constant in each of the three universes. For example, in the absence of interactions they arise from different choices of boundary conditions on the four-dimensional fields. To illustrate this consider the expectation value of a number operator $a^\dagger a = (1/2m\omega)P^2 + \frac{1}{2}m\omega X^2 - \frac{1}{2}$ in the ground state of another oscillator with different frequency ω'. It is straightforward to verify that $\langle 0' | a^\dagger a | 0' \rangle = (\omega - \omega')^2/4\omega\omega'$. This is seen to be related to the dimensionless coupling describing the emission of two universes with allowed frequencies $\omega'(\mathbf{k})$ from a universe with allowed frequencies $\omega(\mathbf{k})$ containing one particle. We shall return to these matters elsewhere.

APPENDIX C: SECOND QUANTIZATION AS THE THIRD QUANTIZATION OF TWO-DIMENSIONAL GRAVITY

In this appendix we wish to illustrate the process of third quantization discussed in the conclusion in the case of two-dimensional gravity coupled to matter. In doing so we shall adopt the point of view that the second quantization of a string theory is equivalent to third quantizing the two-dimensional gravity describing the world sheet. First-quantized string theory can be regarded as two-dimensional gravity coupled to matter in the form of D-scalar fields $\phi_I(\sigma^\alpha)$. It is really already a second-quantized theory in the form of a two-dimensional field theory. The two-dimensional observer can compute the quantity which we refer to as $\Lambda(n)$ for various oscillations of these matter fields. One might even find that for certain configurations of oscillations the quantity vanishes. However all the two-dimensional experiments he performs involves the exchange of two-dimensional stress energy of two-dimensional momenta and energy,

$$T_{\alpha\beta} = \partial_\alpha \phi_I \partial_\beta \phi_I - \tfrac{1}{2} g_{\alpha\beta}(g^{\gamma\sigma}\partial_\gamma \phi_I \partial_\sigma \phi_i) , \tag{C1}$$

not the exchange of canonical momenta $\pi_1 = \dot{\phi}_I$. Such two-dimensional physical quantities will always have the index I saturated. By studying the two-dimensional WDW equation as well as the two-dimensional version of the X constraint

$$\left[-\frac{\delta}{\delta\phi_I}\frac{\delta}{\delta\phi_I} + \left[\frac{\partial}{\partial\sigma}\phi_I\right]\left[\frac{\partial}{\partial\sigma}\phi_I\right]\right]\psi(\phi_I(\sigma)) = 0 ,$$
$$\left[\frac{\partial}{\partial\sigma}\phi_I\right]\frac{\delta}{\delta\phi_I}\psi(\phi_I(\sigma)) = 0 , \tag{C2}$$

and their solutions in the light-cone gauge (see, for example, Ref. 13)

$$\psi(\phi^-,\phi^+,\phi_i(\sigma)) = \int dp_\phi^+ \, \psi(p_\phi^+,\phi^+,\phi_i(\sigma)) e^{ip_\phi^+\phi^-}$$

and

$$\psi(p_\phi^+,\phi^+,\phi_i(\sigma)) = \int dp^i (2p_0 V_\phi)^{-1/2} e^{i(p_\phi^-\phi^+ + p_\phi^i \phi^i)} \delta\left[\sum_{i=1}^{D-2}\sum_{l=1}^{\infty} n_i(l) - \tilde{n}_l(l)\right]$$
$$\times \prod_{i=1}^{D-2}\prod_{l=1}^{\infty} H_{n_i(l)}(\phi_l^i) \prod_{i=1}^{D-2}\prod_{l=1}^{\infty} H_{\tilde{n}_i(l)}(\tilde{\phi}_l^i) e^{-[\phi_i(l)\phi_i(l)]+[\tilde{\phi}_i(l)\tilde{\phi}_i(l)]/2} \tag{C3}$$

(where $p_0 = p_\phi^+ + p_\phi^-$, and the mass-shell condition is given by

$$p_\phi^- = \frac{1}{2p_\phi^+}[p_\phi^i p_\phi^i + M^2(n,\tilde{n})]$$

and

$$M^2(n,\tilde{n}) = \sum_{l=1}^{\infty}\sum_{i=1}^{D-2}[l n_i(l) + l\tilde{n}_i(l)]$$

the analogue of our $\Lambda(n_1,n_2)$) a two-dimensional observer might wonder if there is another structure built out of the zero modes of the two-dimensional fields. The momenta conjugate to these zero modes would be seen to play a crucial role in the dynamics of this higher theory. Indeed in string theory they will be the ten-dimensional energy and momentum. We are in a similar situation in studying the second quantization of the WDW equation of the hypothetical two-dimensional observer. In four dimensions it is difficult to find the dynamical quantities which govern the behavior of the second-quantization WDW field whose arguments are the four-dimensional fields $\psi(\phi_I^0) I = 1,2,\ldots,4$ with the timelike zero mode ϕ_4^0 set to zero.

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