Spherically symmetric solutions of general second-order gravity

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The general second-order gravity theory, whose Lagrangian includes higher powers of the curvature, is considered in arbitrary dimensions. It is shown that spherically symmetric solutions are static, except in certain, special, unphysical cases. Spherically symmetric solutions are found and classified. Each theory's solutions fall into a number of distinct branches, which may represent finite space with two singular boundaries, or an asymptotically either flat or (anti-)de Sitter space with one singular boundary. A theory may contain at most one branch of solutions in which all singularities are hidden by event horizons. Such horizons generally emit Hawking radiation, though in certain cases the horizon may have zero temperature. Black holes do not necessarily radiate away all their mass: they may terminate in a zero-temperature black hole, a naked singularity, or a hot black hole in equilibrium with a "cosmological" event horizon. The thermodynamics of black-hole solutions is discussed; entropy is found to be an increasing function of horizon area, and the first law is shown to hold.

I. INTRODUCTION

In Einstein's original paper on general relativity[1] he was able to deduce the simple Ricci scalar Lagrangian only by making certain simplifying assumptions. The gravity Lagrangian could, in fact, contain an arbitrary number of terms, consisting of the invariants which can be constructed from powers of the Riemann curvature tensor. And, because the curvature in all normal physical situations is so small, it is hard to argue on experimental grounds that such additional terms should not be present. However, terms of order n in the curvature lead, in general, to $2n$th-order field equations, which are difficult to analyze classically, and in most cases appear to lead to ghost problems in quantizing the theory. A fair amount of work has nevertheless been done on the $R+R^2$ theories, which contain the Einstein term together with curvature-squared terms.

When gravity is considered in a higher-dimensional space, a new situation arises. There are then certain combinations of the Riemann invariants which yield second-order field equations. These are the dimensionally continued Euler densities.[2,3] In lower dimensions such combinations correspond to topological invariants—and so do not contribute to field equations there.

Interest in theories of gravity whose Lagrangian contains such higher powers of the curvature, but which are nevertheless second-order theories, has increased in recent years, largely due to the interest in string theories, whose low-energy limit seems to contain such terms.[4,5] Such theories are also of particular interest because they admit the possibility of spontaneous compactfication.[6,7]

A detailed analysis of spherically symmetric solutions has been carried out for gravity theories which include terms no more than quadratic in the curvature (together with various matter fields).[8–13] Black holes for the theories up to quartic in the curvature (the maximum relevant to ten dimensions) have been considered,[14] though the results published are inaccurate in several respects.

Wheeler[15] has considered, in arbitrary dimensions, the general theory whose Lagrangian is composed of dimensionally continued Euler densities—that is, the most general second-order gravity theory. He analyzed their asymptotically flat spherically symmetric static solutions, and cosmological solutions.

In this paper we shall consider the same general second-order theory, and in Secs. II and III we shall follow procedures similar to those of Wheeler. In Sec. II we shall present the theory, and then calculate and solve its field equations for a general spherically symmetric space. We extend Wheeler's work to nonstatic spacetimes, and thus prove Birkhoff's theorem for the general second-order gravity theory. In Sec. III we shall characterize these solutions, analyzing their behavior near spatial infinity, their singularities, and horizons. This analysis represents an extension of Wheeler's work to include asymptotically nonflat solutions, and presents a correct analysis of the horizon structure of the solutions. (Wheeler's claim that asymptotically flat spherically symmetric static solutions have at most one horizon is shown to be false.) Finally, in Sec. IV we shall consider Hawking radiation and the thermodynamics of the black-hole solutions. We shall derive expressions for the temperatures and entropy, and a generalized first law.

II. THE GENERAL SECOND-ORDER THEORY

We shall consider in this paper the general second-order gravity theory in d dimensions, represented by the Lagrangian

$$L=\sum_{n=0}^{N}\frac{L_n}{(d-2n)(d-2)!}\times\epsilon^{i_1\cdots i_d}R_{i_1i_2}\wedge R_{i_3i_4}\wedge\cdots e_{i_{(2n+2)}}\wedge\cdots e_{i_d}\,, \tag{2.1}$$

where the R_{AB} are the curvature two-forms, corresponding to the vielbein one-forms, e_A, with the indices living in a d-dimensional Minkowski space with signature $(-+\cdots+)$. The divisor of the series coefficients, L_n, has been introduced for simplicity of the field equations. The maximum value of n in the sum is given by

$$N=\begin{cases}\frac{1}{2}d-1, & d \text{ even} ,\\ \frac{1}{2}d-\frac{1}{2}, & d \text{ odd} .\end{cases} \tag{2.2}$$

The field equations are obtained from the variation of (2.1) with respect to the vielbein,

$$\delta L=\sum_{n=0}^{N}\frac{L_n}{(d-2)!}\epsilon^{i_1\cdots i_d}R_{i_1 i_2}\cdots e_{i_{d-1}}\delta e_{i_d} , \tag{2.3}$$

where we have suppressed the wedge product symbols. (For simplicity, we shall suppress these, as well as the summation limits, in all future equations.) The variation assumes this simple form because the δR_{AB} terms are found to be total derivatives, as a consequence of the Bianchi identities.[15]

We are interested in spherically symmetric solutions of the theory (2.1), so we shall choose the basis

$$e^0=e^{\phi}dt,\quad \phi=\phi(t,r) , \tag{2.4a}$$

$$e^1=e^{\lambda}dr,\quad \lambda=\lambda(t,r) , \tag{2.4b}$$

$$e^i=r\left[\prod_{k=2}^{i-1}\sin\theta^k\right]d\theta^i,\quad i=2,\ldots,d , \tag{2.4c}$$

and so we may calculate the curvature two-forms by the Cartan procedure, to get

$$R_{01}=\left[e^{-\lambda-\phi}\frac{\partial}{\partial t}e^{\lambda-\phi}\frac{\partial}{\partial t}\lambda -e^{-\lambda-\phi}\frac{\partial}{\partial r}e^{-\lambda+\phi}\frac{\partial}{\partial r}\phi\right]e_0\wedge e_1 , \tag{2.5a}$$

$$R_{0i}=\left[-\frac{e^{-2\lambda}}{r}\frac{\partial}{\partial r}\phi\right]e_0\wedge e_i+\left[\frac{e^{-\phi-\lambda}}{r}\frac{\partial}{\partial t}\lambda\right]e_1\wedge e_i , \tag{2.5b}$$

$$R_{1i}=\left[\frac{e^{-2\lambda}}{r}\frac{\partial}{\partial r}\lambda\right]e_1\wedge e_i-\left[\frac{e^{-\phi-\lambda}}{r}\frac{\partial}{\partial t}\lambda\right]e_0\wedge e_i , \tag{2.5c}$$

$$R_{ij}=[\psi]e_i\wedge e_j , \tag{2.5d}$$

where

$$\psi=r^{-2}(1-e^{-2\lambda}) . \tag{2.6}$$

To write the field equations in explicit form, we need to insert these expressions for the vielbein and curvature forms into (2.3), and set the result equal to zero for arbitrary perturbations of the vielbein. We need consider, in fact, only perturbations of the form

$$\delta e_0=[\delta e_0^0]e_0+[\delta e_0^1]e_1 , \tag{2.7a}$$

$$\delta e_1=[\delta e_1^0]e_0+[\delta e_1^1]e_1 , \tag{2.7b}$$

$$\delta e_i=0 , \tag{2.7c}$$

to obtain the three independent field equations since the fourth equation, obtained by considering perturbations of δe_i, is related to these three by the Bianchi identities. We find

$$\begin{aligned}\delta L=\sum L_n\Bigg\{&2n\psi^{n-1}\left[\left[\frac{e^{-2\lambda}}{r}\frac{\partial}{\partial r}\lambda\right]\delta e_0^0-\left[\frac{e^{-\phi-\lambda}}{r}\frac{\partial}{\partial t}\lambda\right]\delta e_0^1\right]+(d-2n-1)\psi^n\delta e_0^0\\ &+2n\psi^{n-1}\left[\left[-\frac{e^{-2\lambda}}{r}\frac{\partial}{\partial r}\phi\right]\delta e_1^1+\left[\frac{e^{-\phi-\lambda}}{r}\frac{\partial}{\partial t}\lambda\right]\delta e_1^0\right]+(d-2n-1)\psi^n\delta e_1^1\Bigg\}\sqrt{-g}\,d^dx ,\end{aligned} \tag{2.8}$$

and hence the field equations may be written

$$\left[\sum nL_n\psi^{n-1}\right]\left[\frac{e^{-\phi-\lambda}}{r}\frac{\partial}{\partial t}\lambda\right]=0 , \tag{2.9a}$$

$$\left[\sum nL_n\psi^{n-1}\right]\left[\frac{e^{-2\lambda}}{r}\frac{\partial}{\partial r}(\lambda+\phi)\right]=0 , \tag{2.9b}$$

$$\left[\sum 2nL_n\psi^{n-1}\right]\left[\frac{e^{-2\lambda}}{r}\frac{\partial}{\partial r}\lambda\right]+\sum(d-2n-1)L_n\psi^n=0 , \tag{2.9c}$$

where the three equations are obtained, respectively, from the δe_0^1 variation, the $\delta e_0^0-\delta e_1^1$ variation, and the δe_0^0 variation. [It should be noted that these equations supersede those published by Wurmser[14] for the $d=10$ case, in which some of the permutation coefficients appear to be incorrect. His analysis of the character of the type-(i) solutions below is also flawed.]

We find two classes of solutions to these field equations.

(i) Suppose that

$$\sum nL_n\psi^{n-1}\equiv 0 ; \tag{2.10}$$

then both (2.9a) and (2.9b) are identically satisfied, while (2.9c) yields

$$\sum L_n\psi^n\equiv 0 . \tag{2.11}$$

To solve both (2.10) and (2.11) simultaneously we need $\psi=\psi_0$ where ψ_0 is a repeated root of $\sum L_n\psi_0^n\equiv 0$. Therefore, such solutions will exist only in the theories with special sets of values of the Lagrangian coefficients L_n. Moreover, these solutions are pathological since the field equations do not constrain the g_{00} component of the metric. The metric

$$ds^2=-e^{2\phi}dt^2+(1-\psi_0 r^2)^{-1}dr^2+r^2d\Omega^2 \qquad (2.12)$$

will be a spherically symmetric solution for any function ϕ. (These solutions are considered in some detail by Wheeler.[15])

(ii) Suppose that $\sum nL_n\psi^{n-1}\not\equiv 0$; then the field equation (2.9a) and (2.9b) yield, just as in the usual Einstein case,

$$\frac{\partial}{\partial t}\lambda=0 , \qquad (2.13a)$$

$$\frac{\partial}{\partial r}(\lambda+\phi)=0 , \qquad (2.13b)$$

which together imply

$$\lambda=\lambda(r) , \qquad (2.14a)$$

$$\phi=-\lambda(r)+g(t) , \qquad (2.14b)$$

and, since the arbitrary function $g(t)$ can be absorbed into a redefinition of t, we have thus extended Birkhoff's theorem to apply to the general second-order gravity theory: all spherically symmetric solutions [of theories which do not admit type-(i) solutions] are static.

The third field equation (2.9c) may be written in the form

$$r^{-d+2}\sum L_n\frac{\partial}{\partial r}(r^{d-1}\psi^n)=0 . \qquad (2.15)$$

Thus, by integration, the general spherically symmetric solution of the theory (2.1) may be written as

$$ds^2=-V\,dt^2+V^{-1}dr^2+r^2d\Omega^2 , \qquad (2.16a)$$

where

$$V=1-\psi r^2 \qquad (2.16b)$$

and

$$\sum L_n r^{d-1}\psi^n=\mu , \qquad (2.16c)$$

and the constant of integration μ is the mass parameter of our solutions.

III. CHARACTERIZATION OF SPHERICALLY SYMMETRIC SOLUTIONS

Following Wheeler,[15] let us consider the exterior solutions (2.16) by rewriting (2.16c) in the following form:

$$f(\psi)=\frac{\mu}{r^{d-1}} , \qquad (3.1a)$$

where

$$f(\psi)\equiv\sum L_n\psi^n . \qquad (3.1b)$$

Since each theory is completely determined by its Lagrangian coefficients L_n there is a one-to-one correspondence between the set of second-order gravity theories and the set of polynomial functions (3.1b).

When $\mu=0$, Eq. (3.1a) gives the vacua of a theory f, as $\psi=\psi_0$, where ψ_0 is a zero of f. (If f has no zeros, the theory will have no vacuum solutions.) In general, therefore, there will be as many distinct vacuum solutions as there are zeros of f. And these vacuum solutions will be anti–de Sitter space, flat space, or de Sitter space, according to whether the corresponding zero of f, ψ_0, is negative, zero, or positive. However, if any of the zeros of f is not a simple zero, then the corresponding vacuum solution is instead given by (2.12). Such vacua are pathological, and may be ruled out from further consideration.

When $\mu\neq 0$, Eq. (3.1a) represents, for any given theory f, a many-to-one map from ψ to r. This will map the ψ region where $\psi\rightarrow\psi_0$ from below, and the ψ region where $\psi\rightarrow\psi_0$ from above, to two distinct asymptotic spatial infinities of the solutions of the theory. (If f has no zeros, then any solutions of the theory contain no spatial infinity.) The asymptotic behavior of the solutions may be determined, in general, by examining the behavior of f close to ψ_0. This depends, by Taylor's expansion, only on the value of the derivative f with respect to ψ at ψ_0, $f'(\psi_0)$, which is nonzero in the cases of interest. We have, near spatial infinity,

$$V(r)\approx 1-\psi_0 r^2-\frac{\mu}{f'(\psi_0)r^{d-3}} . \qquad (3.2)$$

The form of (3.2) shows that the solutions are asymptotic to the vacuum which corresponds to $\psi=\psi_0$. And from (3.1a) we see that μ must have the same sign as f. So this represents a positive gravitational mass solution when $\psi\rightarrow\psi_0$ from above [f has the same sign as f', and so the last term of (3.2) is positive], and negative gravitational mass, otherwise. We may define an effective gravitational constant $[1/f'(\psi_0)]$ which, when negative, indicates an effective antigravity force—positive mass will repel and negative mass attract.

Turning now to the question of singularities, we see from the scalar curvature calculated from the curvature two-forms (2.5) that wherever $r^{-1}d^2V/dr^2$, $r^{-1}dV/dr$, or ψ are singular there must be a curvature singularity. The field equation (2.9c), moreover, implies that $r^{-1}dV/dr$ and $r^{-1}d^2V/dr^2$ are singular when and only when both $f'=0$ and $f\neq 0$. So every solution contains a curvature singularity: for, as ψ moves away from a zero of f (corresponding to moving radially inwards from spatial infinity), eventually either a stationary point of f is reached or else $|\psi|$ must reach infinity (and so we have hit a curvature singularity).

To represent a particular branch of solutions, we select a particular zero ψ_0, and then choose a point ψ_1 lying on one or the other side of ψ_0, which is the nearest such curvature singularity on that side. (Strictly speaking, ψ_1 is taken as the nearest stationary part of f on a particular side of ψ_0, and if none exists, then ψ_1 is taken as infinity.) The interval $[\psi_0,\psi_1]$ then represents the outer part of one of the spherically symmetric solutions and since, from the definition of f,

$$r=\left[\frac{\mu}{f(\psi)}\right]^{1/(d-1)} , \qquad (3.3)$$

so r will decrease monotonically from infinity at ψ_0 to its lowest value at the curvature singularity ψ_1.

So far we have considered only solutions containing a spatial infinity. An interval $[\psi_1,\psi_2]$, where ψ_1 and ψ_2 are neighboring curvature singularities which are not separated by a zero of f, also represents a solution branch, but one which is not asymptotic to a vacuum solution, and which terminates in a singularity for both large and small r. We shall return only briefly to such solutions later in this section. The total number of solution branches, with or without spatial infinity, will always be one more than the sum of the number of zeros and the number of stationary points of f, which is at most $2N$.

The existence of a singularity at some small value of r raises the question: When will this singularity be surrounded by an event horizon? An event horizon is characterized by the vanishing (and change of sign) of the g_{00} component of the metric (2.16a), and hence we have from (2.16b) $\psi_h=r_h^{-2}$, where the subscript h will denote values at the horizon. So at a horizon

$$f(\psi_h)=\mu\psi_h^{(d-1)/2},\quad \psi_h\geq 0 \tag{3.4}$$

or defining

$$\tilde{f}(\psi)\equiv\frac{f(\psi)}{\psi^{(d-1)/2}}\ , \tag{3.5}$$

we have at a horizon

$$\tilde{f}(\psi_h)=\mu,\quad \psi_h\in I\equiv[\psi_0,\psi_1]\cap[0,\infty]\ , \tag{3.6a}$$

and, hence,

$$r_h=[\tilde{f}^{\,-1}(\mu)]^{-1/2}\ . \tag{3.6b}$$

Clearly, there will be a horizon only for $\min_I(\tilde{f})\leq\mu\leq\max_I(\tilde{f})$, where the maximum and minimum functions are defined on the interval I. From the expansion of $\tilde{f}$,

$$\tilde{f}(\psi)=\begin{cases}L_{(d-1)/2}+L_{(d-3)/2}\psi^{-1}+\cdots+L_0\psi^{-(d-1)/2}, & d\text{ odd}\ ,\\ L_{(d-2)/2}\psi^{-1/2}+L_{(d-4)/2}\psi^{-3/2}+\cdots+L_0\psi^{-(d-1)/2}, & d\text{ even}\ ,\end{cases} \tag{3.7}$$

we can see that $|\tilde{f}(\psi)|$ will be bounded above on the interval I, unless the interval includes 0. Therefore, there will, in general, be an upper limit for $|\mu|$, above which solutions will have marked singularities. Similarly, there will be a lower bound on $|\mu|$, below which there may be naked singularities, if $\tilde{f}$ does not have a zero on the interval I. The definition of $\tilde{f}$, (3.5), indicates that $\tilde{f}(\psi_0)=0$, except that $\psi_0=0$, so if $\psi_0>0$ then $\tilde{f}$ will have a zero on I. The expansion (3.7) also indicates that $\tilde{f}$ will have a zero at infinity, provided that if d is odd then $L_{(d-1)/2}$ vanishes. So if $\psi_1=\infty$ (and this proviso is met) there will be a zero of $\tilde{f}$ on I, for any ψ_0.

Of course, there may be more than one event horizon in a solution, for $\tilde{f}^{\,-1}$ may be multivalued if $\tilde{f}$ has any maximum or minimum on the interior of I. It is interesting to note that Eq. (2.9c) may be written as

$$\left[\frac{1}{r}\frac{dV}{dr}\right]f'=\tfrac{1}{2}\tilde{f}\,'\psi^{(d+1)/2}\ , \tag{3.8}$$

and, hence, the stationary points of $\tilde{f}$ on (ψ_0,ψ_1) correspond to the stationary points of the g_{00} component of the metric V. Thus, solution branches with a monotonic $\tilde{f}$ will consist of solutions for which V is monotonic and which have at most one event horizon. This also implies that we may transform the metric (2.16a) to Kruskal-type coordinates near a horizon, except if $\tilde{f}\,'(\psi_h)=0$, which can be the case only at the limiting value of $|\mu|$ above or below which the horizon does not exist.

In the asymptotically de Sitter cases (when $\psi_0>0$), it is, in fact, essential that both the "cosmological" and the "black-hole" horizons exist, since in de Sitter space the metric at spatial infinity has the wrong signature and must be hidden from the physical part of the solution, just as the singularity must be. The precise number of horizons in a particular branch will depend on the details of the Lagrangian coefficients, but we can rule out the existence of more than one horizon in two significant cases. First, if $\psi_1<\psi_0$ then $|f|$ decreases monotonically from ψ_1 to ψ_0 and, from (3.5), then so must $|\tilde{f}|$, so there must be at most one horizon. Second, when $\tilde{f}$ has only single zero on I, there will be only a single solution of (3.6b), and hence only one horizon, for small enough $|\mu|$.

We may now write down a procedure for characterizing the solutions of any second-order gravity theory. First, write down the function f for the theory, and calculate its zeros and singular points as described above. Then, for each solution branch $[\psi_0,\psi_1]$ the following is true.

(1) If $f'(\psi_0)=0$, the branch is ruled out, as it has a pathological vacuum.

(2) It is asymptotic to anti–de Sitter, flat, or de Sitter space according to whether ψ_0 is negative, zero, or positive.

(3) It has positive or negative mass according to $f(\psi_1)$ being positive or negative.

(4) It has positive or negative gravitational mass according to whether $\psi_1-\psi_0$ is positive or negative.

(5) It has a positive or negative effective gravitational constant according to whether $f'(\psi_0)$ is positive or negative.

(6) There is a curvature singularity at

$$r_{\text{sing}}=\left[\frac{\mu}{f(\psi_1)}\right]^{1/(d-1)}>0\ , \tag{3.9}$$

which is timelike or spacelike according to whether $f^2(\psi_1)$ is greater than or less than $\mu^2\psi_1^{d-1}$.

(7) If $\psi_0>0$ and $\psi_1\leq 0$, there is a cosmological horizon but no black-hole horizon.

(8) If $\psi_0 > 0$ and $\psi_1 > 0$, there is a cosmological horizon except for large enough $|\mu|$, and possibly also at least one more horizon, but not for small $|\mu|$ unless $\psi_1 = \infty$ (and if d is odd, then $L_{(d-1)/2} = 0$).

(9) If $\psi_0 \leq 0$ and $\psi_1 \leq 0$, there is no horizon.

(10) If $\psi_0 \leq 0$ and $\psi_1 > 0$, there is at least one horizon except for small enough $|\mu|$, unless $\psi_1 = \infty$ (and if d is odd, then $L_{(d-1)/2} = 0$), in which case there is at least one horizon for all $|\mu|$.

Similar considerations imply the following for each branch $[\psi_1, \psi_2]$.

(1) There are curvature singularities at both (3.9) and $r_{\text{sing}} = [\mu / f(\psi_2)]^{1/(d-1)}$.

(2) No horizons exist for large $|\mu|$, unless $0 \in [\psi_1, \psi_2]$.

(3) No horizons exist for small $|\mu|$, unless $\infty \in [\psi_1, \psi_2]$ (and if d is odd, then $L_{(d-1)/2} = 0$).

(4) There is at most one horizon if $0 \in [\psi_1, \psi_2]$, or in other cases where $|f|$ is decreasing with ψ on I.

In the Appendix, this procedure is demonstrated for the Einstein-Gauss-Bonnet Lagrangian previously studied by others.[8,9,11–13]

In general, there is only one case of physical interest—namely, having positive mass, positive gravitational mass, and no naked singularities. We cannot accept an upper limiting mass above which naked singularities occur, because classically we could always add mass to a black hole to turn it into a naked singularity; we cannot accept a lower limiting mass because we would expect Hawking radiation to cause higher-mass black holes to radiate away leaving a naked singularity (though we shall see in the next section that this is not *always* the case). So a physically acceptable solution branch would seem to require that $\psi_0 \leq 0$, $\psi_1 = \infty$ (and, if d is odd, $L_{(d-1)/2} = 0$). Then, the singularity is spacelike and at the origin, and the solutions are asymptotically Schwarzschild–anti–de Sitter (when the cosmological constant is negative) or asymptotically Schwarzschild (when the cosmological constant vanishes). In order for such a branch to exist, the theory must be such that f has at least one zero, $f > 0$ for $\psi > 0$, and f has no stationary points for ψ greater than its greatest zero. So L_0 must be non-negative (and hence the usual cosmological constant must be nonpositive), L_1 must be strictly positive (and hence the usual gravitational constant, the coefficient of the Einstein term), any negative coefficients L_n must be sufficiently small, and the terminating coefficient of the series must be positive.

IV. BLACK-HOLE THERMODYNAMICS

In this section, except where explicitly stated otherwise, we shall consider only the physically acceptable solutions found in the previous section (namely, the asymptotically Schwarzschild–anti–de Sitter ones with positive mass, positive gravitational mass and no naked singularities). We shall derive and discuss expressions for their mass, temperature, thermodynamic energy, and entropy.

The mass of an asymptotically flat black-hole solution may be determined from the asymptotic form of V in the metric (2.16a). From (3.2) we have, in the asymptotically flat case,

$$V \approx 1 - \frac{\mu}{L_1 r^{d-3}} , \tag{4.1}$$

and, hence, following Myers and Perry,[16] we have

$$m = A_{d-2}\mu , \tag{4.2}$$

where A_{d-2} is the area of a unit $(d-2)$-sphere. This mass is the generalization of the Arnowitt-Deser-Misner (ADM) mass to higher dimensions, and represents both the inertial and gravitational mass of the black hole as viewed from infinity.

Turning now to the Euclidean section of the metric, and adopting a method analogous to that originally followed for the Schwarzschild metric,[17,18] we have

$$ds^2 = V\,d\tau^2 + V^{-1}dr^2 + r^2 d\Omega^2 , \tag{4.3}$$

which may be written

$$ds^2 = \left[\frac{V_h' R}{1} \right]^2 d\tau^2 + \left[\frac{V_h'}{V'} \right]^2 dR^2 + r^2 d\Omega^2 , \tag{4.4}$$

where $R = 2V^{1/2}/V_h'$, and V_h' is the derivative of V with respect to r at the horizon (which is proportional to the surface gravity of the black hole). This metric will be regular at the horizon only if we treat τ as an angular coordinate, identifying it with period $\beta = 4\pi / V_h'$. In general, the reciprocal of the period of the imaginary-time coordinate τ can be identified with the temperature of a solution, so the temperature of our black-hole solutions is readily found to be

$$T = \frac{1}{4\pi r_h} \frac{\sum (d-2n-1) L_n r_h^{-2n}}{\sum n L_n r_h^{-2n}} , \tag{4.5a}$$

where r_h, the radial coordinate at the horizon, is given implicitly by

$$\mu = \sum L_n r_h^{d-2n-1} . \tag{4.5b}$$

Or, in the notation of the previous section, we may write the temperature as

$$T = -\frac{\tilde{f}'(\psi_h)}{2\pi f'(\psi_h)} \psi_h^{(d-1)/2} , \tag{4.6a}$$

and we may note that

$$\frac{d\mu}{dr_h} = -2\psi_h^{3/2} \tilde{f}'(\psi_h) . \tag{4.6b}$$

Consider any positive-mass solution of a theory which has $\psi_0 \leq 0$, $\psi_1 > 0$. At $\psi = 0$, $\tilde{f}$ is infinite, and it declines (though not necessarily monotonically) as ψ increases. Since the outermost horizon will be at the lowest positive value of ψ for which $\tilde{f} = \mu$, it is clear that $\tilde{f}'(\psi_h)$ must be negative. The outermost black-hole horizon has, therefore, a positive temperature T and will emit Hawking radiation at that temperature. As it does so, its mass must decline. But from (4.6b) we see that as the mass declines, the horizon will shrink (that is, ψ_h will increase). Now, it is not necessarily the case that the temperature will increase monotonically as the mass decreases, as in the usual Einstein theory, but we can be sure that the temperature will be finite and positive until either $\tilde{f}'(\psi_h)$ or $f'(\psi_h)$ go to zero, or ψ goes to infinity.

If the solution branch allows only a single horizon, we have seen that $\tilde{f}'$ cannot go to zero, so the temperature does not go to zero, and as $\psi \rightarrow \psi_1$ the temperature will go to infinity—a black hole will, therefore, radiate away in a finite time, leaving a naked singularity if $\psi_1 \neq \infty$. So solution branches which contain naked singularities for small masses should in reality lead to such naked singularities, since any larger-mass black hole would radiate away enough mass to lose its horizon within a finite time. There is an exception to this, however. In odd dimensions, it is easy to see from (4.5a), provided $L_{(d-1)/2} \neq 0$, that $T \sim r_h$ as the horizon shrinks to zero. So if $\psi_1 = \infty$, d is odd, and $L_{(d-1)/2} \neq 0$, then a zero-temperature black hole separates the higher-mass black holes from the lower-mass naked singularities; a higher-mass black hole will radiate away getting cooler and cooler but never quite reaching the zero-temperature case, and never becoming a naked singularity. Such solution branches, therefore, might be physically acceptable despite containing naked-singularity solutions, since we would expect classically formed singularities to have a mass greater than the limiting mass, and so be surrounded by a black-hole horizon which could not radiate away to a naked singularity. (Apart from this feature, they are not qualitatively different from the other physically acceptable branches.)

If the solution branch allows more than one horizon, $\tilde{f}'$ must have a zero on the interior of the interval I, and another possibility arises. If ψ_h is initially smaller than some zero of $\tilde{f}'$, then as the horizon shrinks the temperature will eventually tend towards zero. The black-hole mass will tend towards the mass below which this particular horizon does not exist—the mass at which this horizon merges with the next horizon in. After an infinite time, we would be left with a finite-mass, zero-temperature black hole.

In the asymptotically de Sitter cases (where $\psi_0 > 0$) the outermost (cosmological) horizon has positive $\tilde{f}'$, and so negative temperature as seen from infinity. In other words, it radiates Hawking radiation *inwards,* towards the next horizon in. This next horizon, the outermost black-hole horizon, has positive $\tilde{f}'$ and hence positive temperature. When the black-hole horizon temperature is greater than the cosmological horizon temperature, the net effect of the radiation will be for the black hole to lose mass. And as it does so, (4.6b) tells us that the black-hole horizon will shrink and the cosmological horizon will grow. Provided the black-hole temperature does not fall to equal the cosmological horizon temperature, the black hole will radiate away until there is nothing, or a naked singularity. (The zero-temperature black hole is not a possible end point unless the cosmological horizon temperature simultaneously goes to zero, or else the black-hole temperature would first have to fall below the cosmological horizon temperature.) If the black-hole temperature does ever become equal to the cosmological horizon temperature, the system will be in equilibrium at that mass, and the black hole will no longer shrink. So, in the asymptotically de Sitter case, it is possible to end up with a finite-mass hot black hole. Moreover, if the black-hole temperature is lower than the cosmological temperature for some masses, then black holes of those masses will *increase* in mass because there will be net absorption of radiation. The black-hole horizon will grow, and the cosmological horizon shrink. Such black holes will continue to gain mass either until an equilibrium mass is reached, when the cosmological temperature equals the black-hole temperature, or until the black-hole horizon and cosmological horizon tend towards the same size, at which point the temperatures of both tend towards zero (since $\tilde{f}'=0$).

In order to calculate thermodynamic quantities for the asymptotically flat metrics, we may use the partition function approach of Gibbons and Hawking.[18] We first find an expression for the Euclidean action, which has two parts: the volume integral of the Lagrangian over the Euclidean section of the metric (with τ identified with period β, and with $r_h \leq r \leq \infty$); and a surface integral over the boundary of the manifold. The curvature two-forms are given by

$$R_{01} = \left[\psi + 2r\psi' + \frac{r^2}{2}\psi''\right] e_0 \wedge e_1 \;, \tag{4.7a}$$

$$R_{0i} = \left[\psi + \frac{r}{2}\psi'\right] e_0 \wedge e_i \;, \tag{4.7b}$$

$$R_{1i} = \left[\psi + \frac{r}{2}\psi'\right] e_1 \wedge e_i \;, \tag{4.7c}$$

$$R_{ij} = (\psi) e_i \wedge e_j \;, \tag{4.7d}$$

and hence, from the Euclidean Lagrangian, the volume part of the action is

$$\begin{aligned} I_{\rm vol} = -\int \sum \frac{L_n}{d-2n} \Bigg[2n \left[\psi + 2r\psi' + \frac{r^2}{2}\psi''\right] \psi^{n-1} + 2n(2n-2)\left[\psi + \frac{r}{2}\psi'\right]^2 \psi^{n-2} \\ + 4n(d-2n)\left[\psi + \frac{r}{2}\psi'\right]\psi^{n-1} + (d-2n)(d-2n-1)\psi^n \Bigg] \sqrt{g}\, d^d x \\ = -\beta \int \left[\sum \frac{L_n}{d-2n} (r^d \psi^n)'' r^{2-d}\right] \sqrt{g}\, d^{d-1}x = -\beta A_{d-2} \left[\sum \frac{L_n}{d-2n}(r^d\psi^n)'\right]_{r=r_h}^{r=\infty} . \end{aligned} \tag{4.8}$$

To calculate the correct action for the path-integral approach we need to add surface terms to our original action (2.1), so that it becomes first order in the metric; these surface terms, by definition, exactly cancel the contribution to the above integral from the surface at infinity, so we are left with the contribution from the surface at the horizon:

$$I=\beta A_{d-2}\sum\frac{L_n}{d-2n}(r^d\psi^n)'\Big|r=r_h\ . \qquad (4.9)$$

This may be simplified, using (4.5), to

$$I=\beta m-4\pi A_{d-2}\sum\frac{nL_n}{d-2n}r_h^{d-2n}\ . \qquad (4.10)$$

The thermodynamic energy may now be calculated:

$$\langle E\rangle=\frac{dI}{d\beta}=m+\left[\beta\frac{dm}{dr_h}-4\pi A_{d-2}\sum nL_n r_h^{d-2n-1}\right]\frac{dr_h}{d\beta}$$
$$=m\ , \qquad (4.11)$$

and so the entropy

$$S=\beta\langle E\rangle-I=4\pi A_{d-2}\sum\frac{nL_n}{d-2n}r_h^{d-2n}\ . \qquad (4.12)$$

The thermodynamic energy, therefore, does equal the usual mass-energy (4.2), which is the conserved quantity associated with the timelike Killing vector, but the entropy is no longer simply related to the horizon area. Since the Lagrangian coefficients L_n need not all be positive, it is not obvious from (4.12) that the entropy need be positive. However, the differential of the entropy is given by

$$dS=4\pi A_{d-2}\left[\sum nL_n r_h^{d-2n-1}\right]dr_h\ , \qquad (4.13)$$

or, in our earlier notation,

$$dS=\frac{4\pi f'(\psi_h)}{d-2}dA\ . \qquad (4.14)$$

Since we have seen that $f'(\psi_h)$ is positive for any solution with both positive mass and positive gravitational mass, the entropy is an increasing function of the area (or radial coordinate) of the horizon. And we can see from (4.12) that the entropy vanishes when the horizon area vanishes, so for any solution branch whose singularity is at the origin the entropy must be positive for all black-hole solutions.

For comparison, we may follow Bardeen, Carter, and Hawking[19] and derive the mass formula for our black-hole solutions. We need to evaluate the integral

$$\sum_n\int\left[{}^{(n)}G_a^b-\frac{1}{d-2}{}^{(n)}G_e^e\delta_a^b\right]k^a d\Sigma_b\ , \qquad (4.15)$$

where ${}^{(n)}G_b^a$ represents the terms of the field equations arising from the nth-order terms of the Lagrangian, k^a is the timelike Killing vector, and the integral is evaluated over a spacelike hypersurface bounded by spatial infinity and the horizon. We shall not assume that the space outside the horizon is empty. This integral can be converted to two $(d-2)$-surface integrals, since ${}^{(n)}G_0^0$, and ${}^{(n)}G_1^1$ are pure divergences and ${}^{(n)}G_i^i$ (no summation implied) may be written as $L_n r^{-d+3}(r^{d-1}\psi^n)''$. Only the ${}^{(1)}G_a^b$ terms contribute to the integral at infinity, as the $n>1$ terms fall off too fast. At the horizon, however, the additional terms do make a simple contribution. The mass formula becomes

$$m=\frac{d-2}{d-3}\int\left[T_a^b-\frac{1}{d-2}T\delta_a^b\right]k^a d\Sigma_b$$
$$+A_{d-2}\sum L_n r_h^{d-2n-1}\ . \qquad (4.16)$$

The differential mass formula will be

$$dm=A_{d-2}\sum L_n(d-2n-1)r_h^{d-2n-2}dr_h$$
$$+\text{matter terms}\ . \qquad (4.17)$$

Thus the mass (4.2) is the real physical mass, in the sense that this corresponds to the total mass of all the matter which has collapsed in from the asymptotic region to form the black hole. The first law of black-hole mechanics is thus

$$\frac{dm}{T}=4\pi A_{d-2}\left[\sum nL_n r_h^{d-2n-1}\right]dr_h=dS\ , \qquad (4.18)$$

just as in Einstein gravity. The zeroth and second laws, however, cannot easily be proved, as the dominant energy condition is not satisfied by the Lagrangian (2.1). It is, therefore, not clear whether the entropy will always increase, and whether the thermodynamic quantities we have calculated have real physical significance.

V. CONCLUSION AND SUMMARY

In this paper we have seen that the most general second-order gravity theory in d dimensions can be solved in the spherically symmetric case, and that the solutions in general are static. (We are excepting those special theories which contain pathological solution branches—solutions which are not fully determined by the field equations, and, therefore, not necessarily static.) Each theory's solution set contains a finite number of solution branches, parametrized by their mass, which may represent either finite spaces with two singular boundaries, or else spaces asymptotic to anti–de Sitter, flat, or de Sitter space, with a singular boundary in their interior. The solution branches may represent positive or negative mass, and positive or negative gravitational mass.

A physically acceptable solution branch should have positive mass and positive gravitational mass, and in general contain no naked singularities. The exception to this last requirement is that in certain odd-dimensional theories naked singularities exist only with masses less than the Planck mass and higher-mass black holes are unable to radiate away to a naked singularity; in these solution branches an artificial limiting lower mass could consistently exclude naked singularities. In either case, only certain theories contain a physically acceptable solution branch, and each may contain at most one such branch. These branches represent solutions which behave very much like Schwarzschild–anti–de Sitter solutions, both at a classical and semiclassical level. The only qualitative physical differences we have found are the possibility of solutions with more than one horizon and the related possible existence of zero-temperature black holes, which would prevent some black holes from completely radiating away their mass. Even these differences can occur only if either the higher-degree terms (but not the highest) in the theory represents repulsive rather than attractive forces and are sufficiently strong, or else the number of dimensions is odd and the highest-degree term is nonzero.

For asymptotically flat black-hole solutions, the first law of black-hole mechanics still holds in the general second-order theory. For physically acceptable solution branches, the entropy is positive and is an increasing function of horizon area. However, the simple identification of entropy with horizon area is a feature of the Einstein theory alone, and we cannot show that the general second-order theory will not violate the zeroth or second laws.

Note added. After completing this paper, I have received a copy of a paper by R. C. Myers and J. Z. Simon [Phys. Rev. D **38**, 2434 (1988)], of the University of California, Santa Barbara, which presents an analysis of second-order gravity theories which is similar to the work described above.

APPENDIX

Here we shall consider the standard Einstein-Gauss-Bonnet Lagrangian

$$\frac{R}{16\pi G}+\alpha(R^{\mu\nu\sigma\tau}R_{\mu\nu\sigma\tau}-4R^{\mu\nu}R_{\mu\nu}+R^2)$$

and will characterize its solution according to the scheme outlined in Sec. III. This Lagrangian corresponds to the function

$$f(\psi)\equiv L_1\psi+L_2\psi^2 ,$$

where

$$L_1=\frac{d-2}{16\pi G},\quad L_2=(d-2)(d-3)(d-4)\alpha ,$$

where G is positive, and α is nonzero. The zeros of f are

$$\psi_0=0,\frac{-L_1}{L_2}$$

and the singular points are

$$\psi_1=-\infty,\frac{-L_1}{2L_2},\infty .$$

We may calculate, for further use,

$$r_{\text{sing}}=(-4\mu L_2/L_1^2)^{1/(d-1)},\quad f'(\psi_0)=L_1,-L_1$$

and

$$f(\psi_1)=\pm\infty,\frac{-L_1^2}{4L_2},\pm\infty ,$$

where the infinities take the sign of α.

We may immediately note that $f'(\psi_0)$ is nonzero, and hence there are no pathological solutions. Using the above calculations, it is easy to apply the other steps of the procedure to give the following characterizations.

If $\alpha<0$, then there are these four branches.

$[0,-\infty]$ asymptotically flat, negative mass, negative gravitational mass, positive effective gravitational constant, singular at the origin, no horizon.

$[0,-L_1/2L_2]$ asymptotically flat, positive mass, positive gravitational mass, positive effective gravitational constant, singular at r_{sing}, no horizon for low mass.

$[-L_1/L_2,-L_1/2L_2]$ asymptotically de Sitter, positive mass, positive gravitational mass, negative effective gravitational constant, singular at r_{sing}, no horizon for large mass.

$[-L_1/L_2,\infty]$ asymptotically de Sitter, negative mass, positive gravitational mass, negative effective gravitational constant, singular at origin, no horizon for large (negative) masses.

If $\alpha>0$, then there are these four branches.

$[-L_1/L_2,-\infty]$ asymptotically anti–de Sitter, positive mass, negative gravitational mass, negative effective gravitational constant, singular at origin, no horizon.

$[-L_1/L_2,-L_1/2L_1]$ asymptotically anti–de Sitter, negative mass, positive gravitational mass, negative effective gravitational constant, singular at r_{sing}, no horizon.

$[0,-L_1/2L_2]$ asymptotically flat, negative mass, negative gravitational mass, positive effective gravitational constant, singular at r_{sing}, no horizon.

$[0,\infty]$ asymptotically flat, positive mass, positive gravitational mass, positive effective gravitational constant, singular at origin, horizon *except* for low masses if $d=5$.

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