Quantum kinematics of spacetime. II. A model quantum cosmology with real clocks

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Nonrelativistic model quantum cosmologies are studied in which the basic time variable is the position of a clock indicator and the time parameter of the Schrödinger equation is an unobservable label. Familiar Schrödinger-Heisenberg quantum mechanics emerges if the clock is ideal—arbitrarily accurate for arbitrarily long times. More realistically, however, the usual formulation emerges only as an approximation appropriate to states of this model universe in which part of the system functions approximately as an ideal clock. It is suggested that the quantum kinematics of spacetime theories such as general relativity may be analogous to those of this model. In particular it is suggested that our familiar notion of time in quantum mechanics is not an inevitable property of a general quantum framework but an approximate feature of specific initial conditions.

I. INTRODUCTION

The time parameter of the Schrödinger equation is one of the basic observables of familiar quantum theory. It is not an observable, of course, in the sense of being represented as an operator in Hilbert space for it is distinguished in the theoretical framework from other observables by not being so represented. Rather it enters the theory as a parameter describing the evolution of the state vector in the Schrödinger picture or the ordering of operators in the Heisenberg picture. But the time parameter *is* assumed to be an observable in the sense that differences in its value are determinable to arbitrary precision by suitable measurements. Indeed, so central is this assumption that *all* probabilities predicted directly by familiar, Schrödinger-Heisenberg quantum mechanics are for observations at a single instant of the time parameter. We may, through poor apparatus or neglect of data, be ignorant of the precise time difference between any two observations, but we assume that a precise difference *could* have been determined. Observations which do otherwise are incomplete.

What are the grounds for so strong an assumption? Empirically they arise from the fact that, as observed on all accessible scales, over the whole of the accessible universe, spacetime has a classical geometry. Classical distances between spacetime points are fixed by the metric. Within any quantum theory of gravity, however, this can only be an approximate fact which is a consequence of the particular quantum state of the universe. States generally will not predict observations correlated as in classical spacetime geometries because the metric is a quantum variable and a state generally will exhibit dispersion in it. Particular states *can* exhibit approximately the correlations of classical spacetimes. If this is the case, the special role played by time in familiar quantum mechanics would seem most naturally to have its origin, not in a preferred status in the formalism, but rather in particular properties of the quantum initial conditions of the universe. This series of papers is concerned with this point of view.

The discussion of the problem of time has a long history in connection with the quantization of spacetime.[1] The traditional view has been that a preferred time parameter is an essential part of any predictive quantum-mechanical framework.[2] However, the idea that one could move away from quantum mechanics with a preferred time by "including clocks in the system" or otherwise adjoining a dynamical time to the theory also has a long history (for a sampling of views see Refs. 3–8) and has recently been extensively discussed by Page and Wooters.[9] Limitations on such clocks and consequently on the precision with which time intervals could be defined were suggested by Salecker and Wigner.[10] The idea that our notion of time might be appropriate only to the late universe has been discussed by many but emerges especially clearly from the work of Lapchinsky and Rubakov,[11] Banks,[12] and Halliwell and Hawking.[13] In various ways these authors showed how the familiar quantum laws of evolution for states of matter fields can emerge from enforcing the constraints of quantum gravity in the limit in which spacetime behaves classically.[14]

To explore with precision the idea that the standard, Schrödinger-Heisenberg formulation of quantum mechanics is an approximation appropriate to those parts of the universe where spacetime behaves classically, a quantum framework more general than the standard one is needed. Schrödinger-Heisenberg quantum mechanics is a definite prescription for calculating probabilities for prediction. It is completely summarized by the fundamental formula[15] for the joint probability for "yes" answers to a series of "yes–no" questions $\alpha_1 \cdots \alpha_n$ at times $\tau_1 < \tau_2 < \cdots < \tau_n$,

$$p(\tau_n \alpha_n, \ldots, \tau_1 \alpha_1) = \mathrm{Tr}[P_{\alpha_n}(\tau_n) \cdots P_{\alpha_1}(\tau_1) \rho P_{\alpha_1}(\tau_1) \cdots P_{\alpha_n}(\tau_n)] \; . \tag{1.1}$$

Here, ρ is the density matrix of the universe in the Heisenberg picture and the $P_\alpha(\tau)$ are the projection operators in Hilbert space corresponding to the questions. The projections appear *time ordered* in (1.1). This ordering is the expression of causality in quantum theory. If a preferred ordering time is abandoned as a fundamental notion and with it any associated notion of causality, what replaces the formula (1.1) for the precise prediction of probabilities? This paper proposes an answer to this question.

The idea is sometimes advanced (see, e.g., Refs. 9 and 12) that it is sufficient for prediction to calculate probabilities for observations from a single wave function "on a spacelike surface" and that any notion of time is to be recovered from a study of the probabilities for correlations between indicators of clocks and other variables. While it is no doubt true that many interesting probabilities, especially in cosmology,[16] are for observations which are more or less on one spacelike surface, they do not exhaust those predicted by familiar quantum mechanics [as (1.1) shows], nor those which are important. For example, a physical system will behave as a good clock when the probability is high that the position of its indicator is strongly correlated with the location in spacetime of *successive* spacelike surfaces. Another example is the construction of history.[17] Most honestly the predictive consequences of history are correlations between present records. However, to give a probability to a history requires the calculation of probabilities of correlations between *present* information and events in the *past*. For both of these examples some generalization of (1.1) is needed.

It might be thought that a replacement for (1.1) is needed only when spacetime behaves classically and a notion of time can be established. All conceivable cosmological observations, for example, take place when the geometry of the universe is classical. But what is meant by classical behavior is that the probability is high that certain observations at different times are correlated according to classical laws. Just to define precisely what is meant by "spacetime behaving classically" a generalization of the formula (1.1) for multitime predictions is needed.

The sum-over-histories framework, properly stated, is a natural candidate for providing the generalization of the Schrödinger-Heisenberg formulation that we seek. With it, amplitudes for observation can be computed directly without the intervention of those parts of the standard machinery which are associated with a preferred time parameter, such as a Hilbert space of states on a spacelike surface. Seeing the sum-over-histories formulation as an alternative starting point for quantum mechanics (rather than as merely a computational tool) is a point of view which goes back to Feynman.[18] It has been advocated in quantum gravity by Teitelboim[19] who has proposed a formalism for constructing the necessary amplitudes. In this series of papers we develop the idea (sketched in Ref. 16) that the sum-over-histories formulation of quantum mechanics supplies an alternative to the fundamental formula of quantum mechanics (1.1) which coincides with it when the theory has a preferred time but which generalizes it when there is none.

Within the sum-over-histories framework the existence of a Schrödinger-Heisenberg formulation is an issue for investigation. An example of such an investigation for nonrelativistic quantum mechanics was given in paper I of this series.[20] There we concluded that a Schrödinger-Heisenberg formulation could be recovered on those spacetime hypersurfaces which the histories crossed once and only once and only on such surfaces. These surfaces define the preferred nonrelativistic time parameter. For observations restricted to definite values of the time parameter the probabilities predicted by the sum-over-histories formulation coincide exactly[18] with those of standard quantum mechanics as expressed in (1.1).

In this paper we shall apply the sum-over-histories formulation to two model nonrelativistic quantum cosmologies for which there are no surfaces in observable variables which the histories cross once and only once and for which, therefore, it is unlikely that there is a preferred time parameter. The universe of these models consists of two nonrelativistic particles. The only observables are their positions, and histories are curves in the configuration space of these positions. Both positions have the same status in the formalism which therefore does not distinguish a preferred time parameter. Given a theory of initial conditions, the sum-over-histories prescription predicts probabilities which are the generalizations of (1.1). From these probabilities we show how a notion of time and a Schrödinger-Heisenberg formulation of quantum mechanics is recovered in two cases: First, they are recovered exactly when the initial conditions are such that one of the particles functions as an ideal clock. This, however, requires unrealizable dynamics inconsistent with positive energy. Second, a Schrödinger-Heisenberg formulation is recovered *approximately* for more realistic dynamics when the initial conditions are such that one of the particles functions approximately as an ideal clock for an interesting length of time.

The examples discussed in this paper are not intended to be proposals for the modification of nonrelativistic quantum mechanics. As argued above, the existence of a time parameter in that theory reflects a true physical fact. It is a relic of an underlying dynamical spacetime treated in the limit in which it behaves classically. Rather these examples are intended as models of quantum mechanics for systems where there is no preferred time. They will thus serve as models for the quantum kinematics of general relativity which we shall consider next.

II. IDEAL CLOCKS AND REAL CLOCKS

Quantum mechanics can be used to describe the operation of mechanical systems which track the Schrödinger-Heisenberg time parameter. In this section we illustrate the standard quantum-mechanical description of such clocks using two simple models. In both cases we consider a system of interest together with a clock with which the dynamics of that system is studied. We denote the configuration variables of the system of interest by X, the clock indicator variable by T, and the Schrödinger-Heisenberg time parameter by τ. For simplicity we shall assume that the clock configuration space is one dimensional[21] and that the clock and system do not interact. Such a system is described by the wave function

$$\Psi = \Psi(X, T, \tau) . \tag{2.1}$$

Its evolution in τ is determined by the total Hamiltonian

$$h = h_C + h_X \ , \tag{2.2}$$

where h_X acts only on the configuration space of the system and h_C only on the configuration space of the clock.

A. Ideal clocks

An ideal clock is one for which, if the clock variable T is precisely correlated with τ at one time, it remains precisely correlated for all times. That is, one possible solution of the Schrödinger equation for Ψ is

$$\Psi(X,T,\tau) = \delta(T - T(\tau))\psi(X,\tau) \ , \tag{2.3}$$

where $T(\tau)$ is a one-to-one calibrating relation between T and τ. For such a state, two Schrödinger times τ and τ' are distinguishable by the corresponding values of T because

$$\int_R dT\, \delta(T - T(\tau))\delta(T - T(\tau')) = \delta(\tau - \tau')/\dot{T}(\tau) \ , \tag{2.4}$$

where an overdot denotes a τ derivative. The existence of such solutions for a complete set of ψ and the Hermiticity of h_C are enough to show that, up to an additive constant normalization,

$$h_C = -i\dot{T}(\tau(T))\frac{\partial}{\partial T} = \dot{T}(\tau(T))p_T \ . \tag{2.5}$$

(Throughout we use units where $\hbar = 1$.) The Hamiltonian of an ideal clock is thus linear in the momentum conjugate to the indicator variable.

B. Real clocks

Ideal clocks do not exist. The spectrum of the Hamiltonian (2.5) is unbounded below, and this is not observed in nature. Typically, the realistic Hamiltonians of nonrelativistic quantum mechanics are quadratic in momenta and bounded below. A real clock is a system governed by such a realistic Hamiltonian in a particular *state* whose features *approximate* those of an ideal clock. Specifically, such states are solutions of the Schrödinger equation of the form

$$\Psi(X,T,\tau) = \phi(T,\tau)\psi(X,\tau) \ , \tag{2.6}$$

in which $\phi(T,\tau)$ is sharply peaked about a calibrating relationship $T = T(\tau)$ for an interesting length of time $\mathcal{T}$. That is, in this period

$$\phi(T,\tau) \approx \chi_\epsilon(T - T(\tau)) \ , \tag{2.7}$$

where $\chi_\epsilon(x)$ is sharply peaked about $x = 0$ with characteristic width ϵ. It then follows that in this period

$$h_C\phi \approx \dot{T}(\tau(T))p_T\phi \tag{2.8}$$

and

$$\int_{-\infty}^{+\infty} dT\, \phi(T,\tau)\phi(T,\tau') \approx 0 \ , \tag{2.9}$$

whenever τ and τ' differ by much more than ϵ. Equation (2.8) and (2.9) are approximate versions of (2.5) and (2.4). Thus, to the extent Eq. (2.7) is satisfied over an interval $\mathcal{T}$, the system in such a state will represent a good clock which keeps track of the Schrödinger time with an accuracy ϵ. We call such a state a "good clock state."

It is not difficult to realize the condition (2.7). Consider as the simplest example a particle of mass M moving in one dimension T. The Hamiltonian is

$$h_C = \frac{P_T^2}{2M} + MV_C(T) \ . \tag{2.10}$$

(Here, we have written the clock's potential as MV_C for later convenience.) Classically, if the particle is moving on a trajectory $T(\tau)$, its position T determines τ over any interval, this relation is single valued. Quantum mechanically, if the particle is in a wave-packet state whose center tracks this classical motion, the same relation can be used to connect position and time to an approximation set by the width of the packet. In quantum mechanics, however, there is a limit to the amount of time a particle can serve as a good clock.[10] If the classical motion is unbounded, the wave packet will inevitably spread. If the system is closed, the position will assign a unique time only over the period to cross the potential.

Such limitations may be simply illustrated for the case of a free particle where V_C is constant. Classically, if moving with speed v, the particle's position measures Newtonian time according to the calibrating relationship

$$\tau = T/v \ . \tag{2.11}$$

The corresponding good clock state is one in which the particle is initially ($\tau = 0$) in a wave packet centered about $T = 0$ and $p_T = Mv$ with initial widths $(\Delta T)_0$ and $(\Delta p_T)_0$ consistent with the uncertainty principle. In the subsequent evolution, the peak of the packet follows the classical law

$$\tau = \langle T \rangle_\tau / v \ . \tag{2.12}$$

Thus the scaled position (2.11) measures time with an error given by the current width of the packet

$$\text{(clock error in tracking Schrödinger time)} = (\Delta T)_\tau / v \ . \tag{2.13}$$

Initially the clock can be made as accurate as desired by choosing $(\Delta T)_0$ small. Inevitably, however, the wave packet will spread as a consequence of the quadratic dependence on momentum of the Hamiltonian (2.10). At time τ,

$$(\Delta T)_\tau^2 = (\Delta T)_0^2 + \tau\langle \{p_T, \Delta T\} \rangle_0 / M + [(\Delta p_T)_0/M]^2\tau^2 \ . \tag{2.14}$$

If a maximum tolerable error ϵ is fixed, Eqs. (2.13) and (2.14) determine the length of time $\mathcal{T}$ which the clock can run keeping within this error.

The general wave-packet solution of the free particle Schrödinger equation may be written

$$\phi(T,\tau) = \int_{-\infty}^{+\infty} dV\, A(V) \exp\left[iMV(T - v\tau) - \frac{i}{2}MV^2\tau \right] \ . \tag{2.15}$$

A wave packet with the properties we have been describing corresponds to an $A(V)$ which is localized about $V=0$ with width $(\Delta p_T)_0/M$ and whose Fourier transform is localized about the origin with width $M(\Delta T)_0$. For times in the interval $\mathcal{T}$ determined by ϵ and (2.14), the second term in the exponent is approximately negligible and one will have

$$\phi(T,\tau)\approx\phi(T-v\tau)\ , \tag{2.16}$$

where $\phi(x)$ is sharply peaked about $x=0$ with width $\epsilon=(\Delta T)_0$. Such wave-packet states will therefore satisfy Eqs. (2.7)–(2.9) over the period $\mathcal{T}$. In this way the free particle can make a good clock.

A free particle clock can be made arbitrarily accurate over an arbitrarily long period of time by making its energy large. Put differently, for a fixed ϵ and $\mathcal{T}$ there is a lower bound on the kinetic energy, $E=\frac{1}{2}Mv^2$, of a free particle clock. This bound can be found by finding the $(\Delta p_T)_0$ and $(\Delta T)_0$ consistent with the uncertainty principle which minimize the accumulated error from (2.13) and (2.14) over the range $\mathcal{T}$ and then demanding that this be less than ϵ. The result is

$$E\gtrsim(1/\epsilon)(\mathcal{T}/\epsilon)\ . \tag{2.17}$$

This is but a special case of the general bounds on the mass of a one-dimensional clock considered by Salecker and Wigner.[10] These bounds are not trivial. For example, for a free particle clock to match the performance of the best atomic clocks ($\mathcal{T}\sim10^3$ sec, $\epsilon\sim10^{-15}$ sec) it would need a kinetic energy in excess of 10^6 ergs or, at the speed of light, a mass in excess of 10^9 proton masses.

C. Quantum dynamics with real clocks

The probabilities predicted directly from the state vector in standard quantum mechanics are for observations at a known moment of the time parameter τ. That is, they are conditional probabilities in which one of the conditions is the value of τ. In the simple models we have been discussing, suppose the state of the total system arising from initial conditions $\mathcal{C}$ is $\Psi_{\mathcal{C}}(X,T,\tau)$. Then we write

$$p(X,T\mid\tau,\mathcal{C})dX\,dT=|\Psi_{\mathcal{C}}(X,T,\tau)|^2dX\,dT \tag{2.18}$$

as the conditional probability for the system to be observed at X and T in small intervals dX and dT *given* the initial conditions $\mathcal{C}$ *and* the value of the time parameter τ. The important point to note is that it is probabilities conditioned on τ that are predicted, not joint probabilities for X, T, *and* τ in intervals dX, dT, and $d\tau$. Indeed, from the point of view of probability theory, it is just in this sense that time enters quantum mechanics as a parameter.

We have access to the value of τ only through measurements of the positions of clock indicators. It must, therefore, be sufficient for prediction in physics to include the clock in the system discussed and to calculate probabilities for correlations between clock indicators and variables of the system of interest *given* that the clock is in a good clock state, and *given also* that the measurement interaction is such that the observations of these variables occur at some one value of the time parameter τ. An example of such a directly accessible probability in the cases we have been considering is the conditional probability $p(X\mid T,\mathcal{C})dX$ for observing X at a given value of T at an unknown moment τ.

Probability densities such as $p(X\mid T,\mathcal{C})$ are calculated from the basic prediction (2.18) by the standard rules of classical probability theory.[22] Specifically, assume clock and system are independent and represented by a wave function of the form (2.6). From an analysis of the clock state (like that in Sec. II B) one can deduce the probability $p(\tau\mid T,\mathcal{C})d\tau$ that the value of the time parameter is τ, given that the clock reads T. The probability for X and τ given T can then be inferred from (2.18) and (2.6) as

$$p(X,\tau\mid T,\mathcal{C})dX\,d\tau=[p(X\mid\tau,\mathcal{C})dX](p(\tau\mid T,\mathcal{C})d\tau]\ . \tag{2.19}$$

The probability density $p(X\mid T,\mathcal{C})$ then follows as

$$p(X\mid T,\mathcal{C})=\int_{-\infty}^{+\infty}d\tau\,p(X,\tau\mid T,\mathcal{C})\ . \tag{2.20}$$

If the clock is such that a measurement of T yields only the information that τ is in some interval $\Delta_\tau(T)$ then, in particular

$$p(\tau\mid T,\mathcal{C})=1/\Delta_\tau(T) \tag{2.21}$$

and

$$p(X\mid T,\mathcal{C})=\frac{1}{\Delta_\tau(T)}\int_{\Delta_\tau(T)}d\tau\,|\psi(X,\tau)|^2\ . \tag{2.22}$$

When the clock state is good (2.22) becomes approximately

$$p(X\mid T,\mathcal{C})\approx|\psi(X,\tau(T))|^2\ , \tag{2.23}$$

where $\tau(T)$ is the calibrating relation between T and τ. Equations (2.20) and its special cases (2.22) and (2.23) display how quantum mechanics makes predictions when there are imperfect measurements of the parameter τ.

III. OBSERVABLES AND LABELS

The universe of the model quantum cosmologies we shall consider consists of two nonrelativistic particles. For simplicity we shall take them to move in one dimension although generalization to more is straightforward. The observable position X and T define the configuration space of these models. The physical histories are all possible curves in this space.

There are four basic elements needed to construct the sum-over-histories quantum mechanics of any system. They were reviewed in paper I. They are the histories, the action, the measure, and the basic observables. With them one can sum $\exp[i(\text{action})]$ to form joint probability amplitudes for observations.

The dynamics of many physical theories of interest are most conveniently described, not directly in terms of physical configuration space, but rather in terms of physical configuration space augmented by redundant variables.[23] Examples are the use of gauge potentials to describe the electromagnetic field and the use of metric functions in particular coordinates to describe spacetime geometry. The use of such redundant variables is often the only transparent way of expressing the invariances of a theory and its locality.

To allow for analogies with such theories, in particular general relativity, we shall consider models which possess an extended configuration space of variables (X,T,N) where N is a redundant variable. Histories are curves $(X(\tau),T(\tau),N(\tau))$ in this extended configuration space where τ is a parameter running along the curve. The action is a functional of these curves which we take to be of the familiar local form

$$S[X(\tau),T(\tau),N(\tau),\tau'',\tau'] = \int_{\tau'}^{\tau''} d\tau\, L(\dot{X}(\tau),\dot{T}(\tau),X(\tau),T(\tau),N(\tau)) , \quad (3.1)$$

where an overdot denotes a τ derivative. The two models we shall consider will be specified by particular forms for L.

The action (3.1) is to be a functional of curves in the extended configuration space. It does not depend, therefore, on the particular way these curves are parametrized. Put differently, the action must be invariant under reparametrizations $\tau \to f(\tau)$. Under these

$$X(\tau)\to\overline{X}(\tau)=X(f(\tau)) , \quad (3.2a)$$

$$T(\tau)\to\overline{T}(\tau)=T(f(\tau)) , \quad (3.2b)$$

and N may transform in some more complicated way. The redundancy of N must also correspond to some symmetry of the action. For our examples we will take this always to be reparametrization symmetry. As a consequence of reparametrization invariance the models will exhibit constraints.

For the measure we take the standard nonrelativistic time-slicing measure (see, e.g., Ref. 24).

The remaining part of the sum-over-histories framework is the specification of the basic observables. In particular, the variables of a history must be divided into observables and labels, and then restrictions on the observables which isolate a history must be imposed. The basic observables we take to correspond to a determination of whether or not the system's history crosses a given region of physical (X,T) configuration space. Thus, X and T are the observable parts of the history and the rest $N(\tau),\tau'',\tau'$ are unobservable labels. As restrictions on observations and conditions which isolate a history we shall take the following: At one end the histories shall satisfy conditions which we shall call the initial conditions of the model quantum cosmology. We leave these unspecified. At the other end we require an observation which locates the end of a history. It is only for histories so restricted that joint probabilities are predicted directly by sum-over-histories quantum mechanics. If, for example, an experiment does not determine the end of a history precisely, as most will not, the joint probabilities must be summed (incoherently) over the unresolved range.

Unobservable labels such as occur in these models are familiar elsewhere in quantum mechanics. Examples besides the redundant dynamical variables already noted are the use of labels for identical particles and the use of an unobservable proper time to formulate the theories of a relativistic particle. The distinguishing characteristic of unobservable labels is that amplitudes are always summed over them before being squared to yield joint probabilities for prediction. For example, amplitudes are symmetrized or antisymmetrized over the labels of identical particles before being squared to yield joint probabilities. This sum is automatic in the sum-over-histories formulation if we only compute amplitudes for observables.

The central feature of the model quantum cosmologies under discussion here is that the variable τ, the natural candidate for the time parameter of Schrödinger-Heisenberg quantum mechanics, is treated as an unobservable label. The only access to time is through the observable positions X and T. Having thus abandoned a notion of preferred ordering time and any associated notion of causality we shall now see for what initial conditions these familiar features of our world may be recovered approximately.

IV. QUANTUM MECHANICS WITHOUT TIME—IDEAL CLOCKS

Consider a nonrelativistic system whose histories $(X(\tau),T(\tau))$ are restricted to paths in which X moves forward in T in the sense that each T determines a unique X and whose dynamics is summarized by the action

$$S[X(\tau),T(\tau),\tau'',\tau'] = \int_{\tau'}^{\tau''} d\tau\, \dot{T} l\left[\frac{\dot{X}}{\dot{T}},X\right] , \quad (4.1)$$

where an overdot denotes a derivative with respect to τ. This is a simple but instructive example of the general class of models discussed in the preceding section. In particular, the action (4.1) is invariant under reparametrizations of τ. Its simplicity arises from the restriction to forward-moving paths. For then, the parameter τ can be chosen to coincide with T. With this parametrization the action is

$$S[X(T),T'',T'] = \int_{T'}^{T''} dT\, l\left[\frac{dX}{dT},X\right] \quad (4.2)$$

which is the action for a nonrelativistic system in which T is the Newtonian time. The process of passing from (4.2) to (4.1) is called "parametrizing the time" by Kuchař[25] who has studied the resulting model in depth both classically and quantum mechanically. The action (4.1) describes an ideal clock because, although T has the same formal status as X as a dynamical variable, it is clearly fully equivalent to the Newtonian time through (4.2) and the restriction to forward-moving paths. In this section we shall consider the sum-over-histories quantum mechanics of this ideal clock.

To illustrate the quantum mechanics of this ideal clock we calculate, according to the path integral prescription, the joint probability amplitude $\Phi(X,T,\mathcal{C})$ for an observation of X and T given that the system was prepared by some conditions $\mathcal{C}$. The histories are the paths

$(X(\tau),T(\tau))$ in which X moves forward in T. The action is (4.1). We choose the standard path measure which correctly reproduces the quantum mechanics of the system described by (4.2). (The details of this, readily available,[24] will not be important for us). The observables are X and T. The parameter τ is an unobservable label. The amplitude $\Phi(X,T,\mathcal{C})$ is then the sum of $\exp(iS)$ over all physically distinct paths which meet the conditions $\mathcal{C}$ and end at (X,T).

Some care is indeed in carrying out the sum which defines $\Phi(X,T,\mathcal{C})$. The classical action (4.1) is invariant under changes in the parametrization of the paths. That is, the action is unchanged by the substitutions (3.2) for arbitrary increasing $f(\tau)$ as long as the end points are also changed to $\bar{\tau}''$ and $\bar{\tau}'$ with $f(\tau'')=\bar{\tau}'', f(\tau')=\bar{\tau}'$. Classically, this invariance under reparametrization of the time gives rise to the constraint that the total Hamiltonian vanish:

$$H=p_T+h_X(p_X,X)=0 \ . \tag{4.3}$$

Here, p_T and p_X are the momenta congugate to T and X implied by (4.1) while h_X is the Hamiltonian constructed from the Lagrangian l in the canonical way. The constraint (4.3) is easily verified by constructing the Hamiltonian implied by the action (4.1).

Quantum mechanically, the invariance under reparametrizations means that paths which differ by reparametrization are not physically distinct and should be counted only once in the sum over histories. This can be done by summing over all paths with appropriate "gauge-fixing" conditions. Thus, following the sum-over-histories prescription, we write

$$\Phi(X,T,\mathcal{C})=\int_{\mathcal{C}}\delta X\,\delta T\left|\det\left[\frac{dF}{dT}\right]\right|\delta[F(T)-\tau] \times\exp\{iS[X(\tau),T(\tau)]\} \ . \tag{4.4}$$

The sum is over all forward-moving paths which satisfy the conditions $\mathcal{C}$ and end at X and T. $F(T)$ is an arbitrary function such that $F(T)=\tau$ assigns a unique T to each τ. This restriction an F ensures the paths are forward moving. The inverse of F determines the range of parameter integration in terms of the range of T. The functional δ function may be thought of as a "gauge-fixing δ function" enforcing the gauge condition $F(T)=\tau$. The factor $\det(dF/dT)$ is the associated "Faddeev-Popov" determinant.

The path integral (4.4) is most easily carried out in the case $F=T$. Then the δ function can be used to carry out the T integration easily. Since the parameter τ is the Newtonian time T in this gauge we may write

$$\Phi(X,T,\mathcal{C})=\int_{\mathcal{C}}\delta X\exp\{iS[X(T)]\} \ . \tag{4.5}$$

The sum in (4.5) is over forward-moving paths $X(T)$ which satisfy the conditions $\mathcal{C}$ and end at X at time T. This is the standard sum-over-histories expression for the joint amplitude to observe X at T given that the system was prepared with conditions $\mathcal{C}$. That is, Φ is the Schrödinger wave function. In this way we recover familiar quantum mechanics from the theory of ideal clocks. The Schrödinger equation

$$H\Phi=\left[-i\frac{\partial}{\partial T}+h_X\left[-i\frac{\partial}{\partial X},X\right]\right]\Phi(X,T,\mathcal{C})=0 \ , \tag{4.6}$$

is a consequence[24] of the functional integral representation (4.5). In the present context, the Schrödinger equation is the operator form of the classical constraint (4.3) that the total energy vanish

$$H\Phi=0 \ . \tag{4.7}$$

Because the paths in (4.5) move forward in time T they intersect a surface of constant T at one and only one position. The values X of this intersection are thus a set of exhaustive and exclusive possibilities given T and the conditions $\mathcal{C}$. The probability of any one value is then given, according to the rule of sum-over-history quantum mechanics, by

$$p(X\mid T,\mathcal{C})=|\Phi(X,T,\mathcal{C})|^2\Big/\int dX\,|\Phi(X,T,\mathcal{C})|^2 \ . \tag{4.8}$$

As (4.8) shows, the conditions $\mathcal{C}$ must be such that the integral of $|\Phi(X,T,\mathcal{C})|^2$ over all X is finite. There is no requirement that $|\Phi(X,T,\mathcal{C})|^2$ be normalizible over the whole configuration space. The values of T at a given value of X, for example, are not an exhaustive and exclusive set of possibilities. The path may cross a given value of X at a great many different T. A further consequence of forward-moving paths is that, as described in paper I, sums over histories can be factored about a surface of constant T into a sum before that surface, a sum after that surface, and a sum over the point of intersection with that surface. This permits the construction of a Hilbert space of states on a surface of constant T with $\Phi(X,T,\mathcal{C})$ representing the state vector and an inner product

$$(\Phi,\Psi)=\int dX\,\Phi^*(X,T)\Psi(X,T) \tag{4.9}$$

induced by the measure in the sum over histories.

The discussion of this section shows that familiar quantum mechanics may be viewed as a theory in which time enters, not a Schrödinger parameter, but as an indicator variable on a certain type of ideal clock—that with the coupling of (4.1). The variable T is singled out by the requirement, implicit in the restriction to forward-moving paths, that it serve as a good parameter along them. This special role is reflected in the form of constraint where p_T is the only momentum which enters linearly. However, as discussed in Sec. II such ideal clocks do not exist in nature. We, therefore, turn to a model with more realistic clocks in the next section.

V. QUANTUM MECHANICS WITHOUT A PREFERRED TIME—REAL CLOCKS

A model quantum cosmology in which the indicators of clocks have dynamically the same status as any other positions in the theory may be constructed in the framework of Sec. III by choosing histories and action which give no preferred role to either T or X. The model discussed by Banks[12] provides a particularly instructive example.

Let $l_X[dX/dt,X]$ be the Lagrangian for the system and for the clock take

$$l_C\left[\frac{dT}{dt},T\right]=\frac{1}{2}M\left[\frac{dT}{dt}\right]^2-MV_C(T)\ , \tag{5.1}$$

where M is the clock's mass, and we have written the potential as MV_C for later convenience. For the total action of clock plus system write

$$S[X(\tau),T(\tau),N(\tau),\tau'',\tau']$$
$$=\int_{\tau'}^{\tau''}d\tau N\left[l_X\left[\frac{\dot{X}}{N},X\right]+l_C\left[\frac{\dot{T}}{N},T\right]-E\right]\ , \tag{5.2}$$

where an overdot denotes a derivative with respect to τ and E is an arbitrary constant. This action is invariant under reparametrizations of τ of the form (3.2) provided N, τ', and τ'' also transform as

$$N(\tau)\to\bar{N}(\tau)=N(f(\tau))/\dot{f}(\tau)\ , \tag{5.3a}$$

$$\tau'\to\bar{\tau}'=f(\tau'),\quad \tau''\to\bar{\tau}''=f(\tau'')\ . \tag{5.3b}$$

The possible histories are all curves in the (X,T) configuration space with X moving both forwards and backwards in T. Quantum mechanically, the central assumption of the model is that the basic observables of the theory are the variables X describing the system and the position T of the clock indicator. Histories may be parametrized by a parameter τ in which, by construction, the paths $(X(\tau),T(\tau))$ move forward. However, neither τ nor its total duration along the path is an observable. They are unobservable labels. The sole information about time comes from correlations between X and T.

This model can be said to represent a theory of real clocks because, as shown in Sec. II, a realistic model of a clock can be constructed from a particle whose dynamics are summarized by (5.1). However, we cannot expect to extract a notion of preferred time in the theory alone for neither the dynamics of the model nor the histories prefer a special role for T. Nor should they, because as argued in Sec. II a particle can keep accurate track of time only when it is in a wave packet with approximately defined position and a momentum. A preferred time will, therefore, emerge in this model, not generally, but only as an approximation appropriate to initial conditions which define a universe which contains good clocks. It is in this sense that we are dealing with a model quantum cosmology.

In paper III we shall see that the quantum mechanics of the model we are considering has certain important features analogous to the quantum mechanics of a closed general-relativistic cosmology. The variable T here is analogous to the three-geometry of a spacelike surface, X to matter fields on that surface, and the histories $(X(\tau),T(\tau))$ to four-geometries with matter fields upon them. The invariance under reparametrization of τ is analogous to the invariance under diffeomorphisms involving relabeling spacelike surfaces. The energy E which parametrizes these models is like the cosmological constant. Those readers for whom these brief remarks are suggestive may wish to keep these analogies in mind.

Before exhibiting the sum-over-histories quantum mechanics of the model we have defined, it is useful to consider it classically. As a consequence of the reparametrization invariance (5.3) there is a constraint. This is easily found as the classical equation of motion arising from varying N. A direct calculation shows that the constraint is

$$H=h_X+h_C-E=0\ , \tag{5.4}$$

that is, that the total energy of the system equals E. The remaining equations of motion follow from varying (5.2) with respect to X and T.

The invariance under reparametrization means that the form of N is arbitrary. In the gauge in which $N=1$ and $\tau'=0$, the action becomes

$$S[X(\tau),T(\tau),\tau]=s_C[T(\tau),\tau]+s_X[X(\tau),\tau]-E\tau\ , \tag{5.5}$$

where τ (formerly τ'') is the total parameter time duration, s_X is the action for the system

$$s_X[X(\tau),\tau]=\int_0^{\tau}d\tau' l_X[\dot{X},X]\ , \tag{5.6}$$

and s_C the action for the clock

$$s_C[T(\tau),\tau]=\int_0^{\tau}d\tau'[\tfrac{1}{2}M\dot{T}^2-MV_C(T)]\ . \tag{5.7}$$

In this gauge, the equations of motion are the familiar equations of motion of nonrelativistic physics in the Newtonian time. The constraint (5.4) emerges as the additional condition

$$\partial S/\partial\tau=0\ . \tag{5.8}$$

Thus, classically the theory specified by (5.2) is equivalent to familiar nonrelativistic mechanics with the additional constraint that the total energy be fixed at the value E.

In the special case where the Lagrangian l has the form

$$l_X\left[\frac{dX}{dt},X\right]=\frac{1}{2}g_{ij}(X)\frac{dX^i}{dt}\frac{dX^j}{dt}-V(X)\ , \tag{5.9}$$

so that the momentum from (5.2) is

$$p_i=N^{-1}g_{ij}(dX^j/d\tau) \tag{5.10}$$

the constraint (5.4) may be solved for N yielding

$$N=\mathcal{T}^{1/2}(E-\mathcal{V})^{-1/2}\ . \tag{5.11}$$

Here, $\mathcal{T}$ is the total kinetic energy

$$\mathcal{T}=\tfrac{1}{2}g_{ij}\dot{X}^i\dot{X}^j+\tfrac{1}{2}M\dot{T}^2\ , \tag{5.12}$$

and $\mathcal{V}$ the total potential energy

$$\mathcal{V}(X,T)=V(X)+MV_C(T)\ . \tag{5.13}$$

If (5.11) is used to eliminate N, the action takes the form

$$S_J[X(\tau),T(\tau),\tau]$$
$$=2\int_0^{\tau}d\tau'\{\mathcal{T}(\dot{X},\dot{T},X)[E-\mathcal{V}(X,T)]\}^{1/2}\ . \tag{5.14}$$

This is the Jacobi form of the action for classical mechanics. It also, because of the quadratic form of (5.12), is invariant under reparametrizations of the parameter τ. Variation with respect to X and T yield equations of motion which, if the parameter τ is chosen so that

$$H=\mathcal{T}+\mathcal{V}-E=0 \tag{5.15}$$

[i.e., $N=1$ through (5.11)] are the same as those of familiar Newtonian mechanics. The theory summarized by (5.14) is thus again Newtonian mechanics with the constraint that the total energy be fixed at E.

The Jacobi action (5.14) is the form arrived at by Barbour and Bertotti[26] in their search for a classical mechanics which would be "fully relational (and hence Machian)" in the sense of considering only relative distances as real and using a "relational" concept of time. This, they say, represents Leibniz's concept of time in which "instants are defined by the successive relative configurations of the universe" that is, in present terms, by the correlations between X and T. The theory of Barbour and Bertotti is the same as that of (5.2) in a special gauge. It is thus fully equivalent to Bank's model at the classical level. The quantum theory of real clocks that we will now present may therefore be regarded as a quantum cosmology implementing the theory of Barbour and Bertotti at least in its treatment of time.

We have presented three actions (5.2), (5.5), and (5.14) which yield the same classical theory. Which should be taken as the starting point for a sum-over-histories quantum theory? Only the actions (5.2) and (5.5) which are quadratic in the velocities are suitable starting points. The situation is similar to that for the relativistic particle where the actions[27]

$$S[X^\alpha(\tau),N(\tau),\tau]=\frac{m}{2}\int_0^\tau d\tau'\left[\frac{(\dot X^\alpha)^2}{N}-N\right], \tag{5.16a}$$

$$S[X^\alpha(\tau),\tau]=-m^2\tau+\tfrac{1}{4}\int_0^\tau d\tau'(\dot X^\alpha)^2\,, \tag{5.16b}$$

$$\begin{aligned}S[X^\alpha(\tau),\tau]&=-m\int_0^\tau d\tau\\&=-m\int_0^\tau d\tau'[-(\dot X^\alpha)^2]^{1/2}\end{aligned} \tag{5.16c}$$

are equivalent classically but the correct relativistic quantum mechanics is only found by starting from (5.16a) or (5.16b). To proceed most directly to answers we shall use (5.5). In this form the action is the same as familiar nonrelativistic quantum mechanics except that the duration τ is an unobservable label which must be summed over in constructing physical amplitudes. Thus in the following we are always working explicitly in the gauge $N=1$, $\tau'=0$.

Consider by way of an important example the joint amplitude $\Phi(X,T,\mathcal{C})$ that the combined system is found at (X,T) given that it was prepared with conditions $\mathcal{C}$. In the sum-over-histories framework this is the sum of $\exp(iS)$ over all paths which end at X,T and which satisfy the conditions $\mathcal{C}$. For each path the end point $\tau=0$ labels one of the conditions $\mathcal{C}$ and τ the end of the path at (X,T). It does not matter in what order the labeling of the conditions is taken for they must be independent of the unobservable label τ. This sum over paths may be carried out in two steps. First construct the sum over paths with fixed τ:

$$\Psi_{\mathcal{C}}(X,T,\tau)=\int_{\mathcal{C},\tau}\delta X(\tau)\delta T(\tau)\exp\{iS[X(\tau),T(\tau),\tau]\}\,. \tag{5.17}$$

Second, complete the sum over paths by integrating over all values of the labels τ:

$$\begin{aligned}\Phi(X,T,\mathcal{C})&=\int_{-\infty}^{+\infty}d\tau\,\Psi_{\mathcal{C}}(X,T,\tau)\\&=\int_{-\infty}^{+\infty}d\tau\int_{\mathcal{C},\tau}\delta X(\tau)\delta T(\tau)\\&\qquad\times\exp\{iS[X(\tau),T(\tau),\tau]\}\,.\end{aligned} \tag{5.18}$$

The sum in (5.18) includes both positive τ and negative ones. We are thus summing over both the amplitude for the process and the amplitude for the τ-reversed or path-reversed process. This is appropriate because they are not distinguishable. It implies an explicit abandonment of any notion of causality in the quantum theory. In making the sum a definite choice of phase has been made between these alternatives which is prescribed by the defining sum over histories (5.17). This assignment, unimportant and arbitrary in ordinary quantum mechanics because the alternatives are noninterfering, enters here in a fundamental way.

The amplitude $\Phi(X,T,\mathcal{C})$ satisfies a constraint. One sees this immediately because $\Psi_{\mathcal{C}}(X,T,\tau)$ defined by (5.17) is the familiar Schrödinger wave function for the system plus clock prepared in a state characterized by the conditions $\mathcal{C}$. The amplitude $\Phi(X,T,\mathcal{C})$ is, by (5.5), the projection of $\Psi_{\mathcal{C}}(X,T,\tau)$ onto an eigenstate of the total Hamiltonian with energy E. Thus,

$$H\Phi=(h_C+h_X-E)\Phi=0\,, \tag{5.19}$$

where h_C and h_X are the Hamiltonians constructed from the actions s_C and s_X, respectively, in the canonical fashion.

The constraint $H\Phi=0$ expresses the unobservability of the duration τ. Only stationary states single out no value of τ over any other. The constraint (5.19) is a consequence of (5.18) only where the range of τ integration is from $-\infty$ to $+\infty$. Invariance is obtained, therefore, only at the sacrifice of causality.[28]

Despite the absence of any dependence on τ, the constraint $\Phi=0$ does not imply the absence of dynamics. Dynamics arise from the correlations implied by Φ between X and T. Indeed, as (4.6) shows, the constraint $H\Phi=0$ gives rise to the familiar dynamics of the Schrödinger equation in the case of ideal clocks. The constraint (5.19) has previously been considered as the starting point for a "quantum theory without time" although typically in very different frameworks from the one advocated here (see, e.g., Refs. 3, 6, 9, and 11–14).

The calculation of $\Phi(X,T,\mathcal{C})$ described above illustrates how to calculate amplitudes in this model quantum cosmology with real clocks. To connect amplitudes to probabilities a set of complete and exclusive possibilities given the condition $\mathcal{C}$ is needed. We can construct examples with detectors which locate the system in a given volume Δ of configuration space. We could imagine a row of such detectors labeled by i arrayed over all X at a given value of $\bar{T}$ (Fig. 1). We could ask the following: "Given the condition $\mathcal{C}$ and that only one of the detectors registers, what is the probability that it is the detector at $\bar{X}_i$?" Since the histories move forward and backward in T more than one detector may register in a general experiment, and we shall discuss the computation of probabilities for such outcomes in the next section. If, however, attention is restricted to a situation in which only a single detector is known to register, then the various values of its location are an exhaustive and exclusive set of possibilities. Whichever detector registers, it locates the end of the history. The joint probability that the history ends in the detector occupying volume Δ_i is therefore proportional to the square of the relevant amplitude, in this case $\Phi(X,T,\mathcal{C})$, integrated over Δ_i. The required conditional probability is then

$$P(\Delta_i \mid \mathcal{C})=\frac{\int_{\Delta_i} dX\,dT\,|\Phi(X,T,\mathcal{C})|^2}{\sum_i \int_{\Delta_i} dX\,dT\,|\Phi(X,T,\mathcal{C})|^2} . \tag{5.20}$$

If the conditions $\mathcal{C}$ are such that the T component of

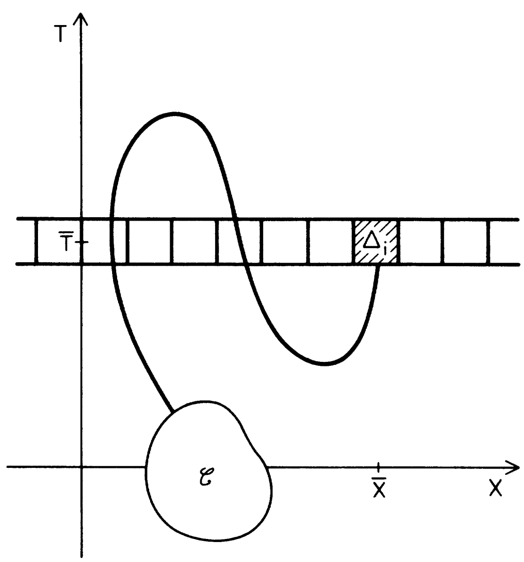

FIG. 1. A possible experiment whose outcomes are assigned probabilities in the model quantum cosmology discussed. The configuration space is the space of two positions X and T. Amplitudes are sums over paths which meet a set of cosmological initial conditions $\mathcal{C}$ on one end and are localized at a definite point on the other. The paths may move both forward and backward in T. The figure shows an array of detectors which spans all X at one T, each detector occupying a configuration-space volume Δ_i. If it is known that only one of these registers, then a complete and exclusive set of outcomes for the experiment are the locations $\bar{X}_i$ of the detector which does register. When the initial conditions $\mathcal{C}$ are such that T makes a good clock indicator, the probability that a detector centered at X registers is, to a good approximation, the same as the probability of Schrödinger-Heisenberg quantum mechanics that a measurement of X at time $\bar{T}$ yields the value $\bar{X}$.

the system is "prepared in a good clock state," then the probabilities (5.20) approximate those obtained from familiar Schrödinger quantum mechanics. We can illustrate this with the case of the free particle clock. There, the general solution to the constraint (5.19) may by separation of variables be written

$$\Phi(X,T,\mathcal{C})=\int_{-\infty}^{+\infty} de\,\phi_{E-e}(T)\psi_e(X) , \tag{5.21}$$

where ϕ_e and ψ_e satisfy

$$h_X\psi_e(X)=e\psi_e(X) , \tag{5.22a}$$

$$h_C\phi_e(T)=e\phi_e(T) , \tag{5.22b}$$

but are otherwise arbitrary. Introducing the Fourier transforms

$$\phi(T,\tau)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} de\, e^{-ie\tau}\phi_e(T) , \tag{5.23a}$$

$$\psi(X,\tau)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} de\, e^{-ie\tau}\psi_e(X) , \tag{5.23b}$$

Eq. (5.21) can be written

$$\Phi(X,T,\mathcal{C})=\int_{-\infty}^{+\infty} d\tau\, e^{iE\tau}\phi(T,\tau)\psi(X,\tau) . \tag{5.24}$$

The function $\phi(T,\tau)$ satisfies the Schrödinger equation with τ as the time parameter. From the analysis of Sec. II we know that there are solutions which represent "good clock states" in the sense that $\phi(T,\tau)$ remains peaked in T about a calibrating relation $T(\tau)$ for an interesting length of time. For such states one will have approximately

$$|\Phi(X,T,\mathcal{C})|^2 \approx \text{const}\times |\psi(X,\tau(T))|^2 . \tag{5.25}$$

The function $\psi(X,\tau)$ also satisfies the Schrödinger equation. If the T width of the regions Δ_i is chosen small compared to the scale over which ψ varies significantly, (5.25) inserted in (5.20) will reproduce the probabilities for this experiment predicted by standard quantum mechanics. One recovers these probabilities, not generally, but only for those model quantum cosmologies where the initial conditions $\mathcal{C}$ mandate that ϕ corresponds to a good clock state.

In this model of quantum cosmology it is not possible for initial conditions to mandate a feature of its Universe which tracks the Schrödinger time parameter arbitrarily accurately for an arbitrarily long interval. The limitation comes from the constraint (5.19) which limits the total energy to E. For example, a free particle clock is constrained by (2.17). In the model E is a parameter of the action. We are not free to vary it. Thus, when the initial conditions are appropriate one recovers the probabilities of familiar quantum mechanics through (5.25) but always approximately.

The agreement expressed by (5.25) does not exhaust the predictions of standard quantum mechanics that one needs to recover to claim that theory is an approximation to the quantum-mechanical framework presented here. There are most generally the multitime predictions of (1.1). It is to these that we now turn.

VI. APPROXIMATE RECOVERY OF SCHRÖDINGER-HEISENBERG QUANTUM MECHANICS

One possible outcome of the experiment illustrated in Fig. 1 is that only a single detector in the array at $\bar{T}$ registers. We calculated the relative probability for the possible locations of that detector in Sec. V. However, because the histories may move backward in T, there are many other possible outcomes of this experiment. Two or three or indeed any number of detectors may register. More generally, we can consider detectors arranged at several different values of T; an interesting case is illustrated in Fig. 2. Indeed, there is no need to consider only arrays of detectors which exhaust all values of X at a given T or detectors having common size volumes such as are used in these examples. Sum-over-histories quantum mechanics assigns joint probabilities to all possible outcomes of properly conditioned experiments. In this section we shall illustrate their calculation for some examples.

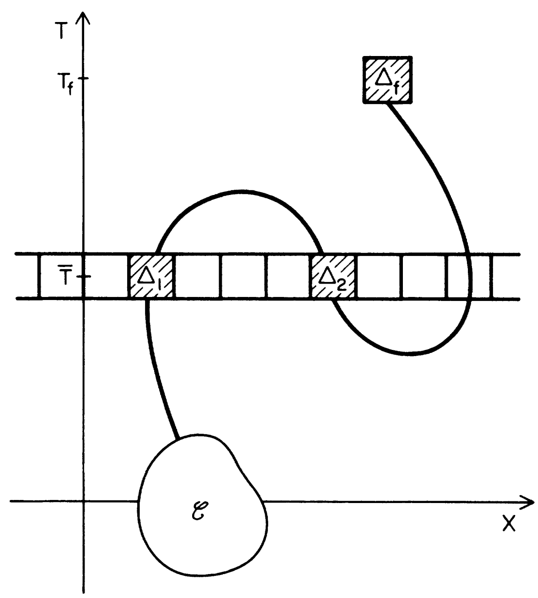

FIG. 2. A possible experimental arrangement corresponding to measurements at successive times in familiar quantum mechanics. There are initial conditions $\mathscr{C}$ defining one end of the histories and a condition that the detector at $(\bar{X}_f, \bar{T}_f)$ registers which defines the other end. As a third condition assume that at least one detector in the array at $\bar{T}$ registers (a measurement of X at $\bar{T}$). Probabilities for the possible outcomes, the number and locations of the detectors which do register, can be computed by sum-over-histories quantum mechanics. If the initial conditions $\mathscr{C}$ mandate that the position T is a good clock indicator then, for suitably large detector volumes, there is almost no amplitude for a path to move backward in T (as in the illustrated case). There is therefore a significant probability only for a single detector at $\bar{T}$ to register and this agrees approximately with that predicted by familiar quantum mechanics. Thus causality and unitarity are approximately restored by the choice of initial conditions.

Probabilities for the outcomes of experiments involving detectors at several values of T are important, for among them are the probabilities which must agree in the appropriate limit with the multitime probabilities (1.1) of familiar quantum mechanics. In this section we shall argue that when the initial conditions of our model quantum cosmology with real clocks are such as to mandate a good clock state then the formula (1.1) will be recovered approximately along with such features as a Hilbert space of states on surfaces of constant clock time.

A. Sensible conditions

Because of the large number of possible outcomes of a given arrangement of detectors there is a wide variety of conditions which define a set of exhaustive and exclusive outcomes. These sets can be determined from the detector arrangement in an elementary, essentially geometrical way, by enumerating the possible outcomes the conditions allow. However, rather greater care is needed than in familiar quantum mechanics to specify sensible conditions that lead to interesting outcomes.

First of all, the conditions must allow for *some* outcomes. The question "Given initial conditions $\mathscr{C}$ for the model universe, what is the probability that there are no observations of it?" makes no sense in quantum cosmology. Probabilities are only relevant for an observer contained within the system who makes *some* observations.[17] This is reflected in any quantum-mechanical formalism. In Schrödinger-Heisenberg quantum mechanics there are no conditional probabilities which can be computed from (1.1) if it contains no P_α's. In the sum-over-histories framework all amplitudes are for conditions which define both ends of a history. The limitations of the quantum cosmologies under discussion do not permit much of a model of an observer to be constructed but the restriction that there must be some outcome is the restriction that at least one of the detectors in the arrangement must register.

In the absence of any built-in notion of preferred time and any associated notion of causality, the complete details of the experimental arrangement over the whole configuration space must be specified. In standard quantum mechanics the probabilities for outcomes up to time τ are independent of any experiments done at a later time provided their outcomes are unknown. The sum $p(\tau_n\alpha_n, \ldots, \tau_{k+1}\alpha_{k+1}, \tau_k\alpha_k, \ldots, \tau_1\alpha_1)$ over all $\alpha_{k+1}, \ldots, \alpha_n$ is the same as $p(\tau_k\alpha_k, \ldots, \tau_1\alpha_1)$. This will not be the case here, although it will be approximately the case when the conditions $\mathscr{C}$ specify a good clock state.

Some care must also be taken to ask questions which have sensible answers. For example, in the experiment considered in Fig. 2, were we to ask, say, for the amplitude for any finite number of detectors of *arbitrarily small* fiducial volume to register on the intermediate surface at $\bar{T}$, we would obtain a vanishing result. This is because the expected number of crossings of such a surface is infinite (cf. the discussion in paper I). The formalism gives correct answers to such questions but they are not very interesting.

Any quantum-mechanical framework specifies basic observables in terms of which it is assumed that all actual observations can be modeled. In Schrödinger-Heisenberg quantum mechanics the basic observables are assumed to be the Hermitian operators in Hilbert space. As discussed more fully in paper I, in the sum-over-histories formulation the basic observables are assumed to correspond to regions of configuration space. In either formulation, whether measurements carried out with actual apparatus can be so modeled and whether apparatus can be constructed which will register each basic observable are further issues of measurability. They are not treated here and certainly a universe with two nonrelativistic particles has far too few degrees of freedom for that purpose. The models are therefore most honestly viewed as models of a kinematics for quantum cosmology within which such questions could be attacked.

B. An example

As an example of experiment with outcomes more diverse than that shown in Fig. 1 consider the situation illustrated in Fig. 2. Detectors Δ_i of dimensions Δ_X and Δ_T are arrayed over all X at a time $\overline{T}$. There is a further detector Δ_f at (X_f,T_f). For conditions we take, in addition to initial conditions $\mathcal{C}$, that the detector at (X_f,T_f) registers and that two of the detectors in the array register. The interesting probability is then for a particular pair in the array, Δ_1 and Δ_2, to register given these conditions. This can be constructed as follows: First consider paths which start with $\mathcal{C}$ and end in Δ_f. For these take the following steps: (1) Sum $\exp(iS)$ over all paths which start with the conditions $\mathcal{C}$ at $\tau=0$, proceed first to a position (X_1,T_1) at parameter time $\tau_1>0$, then to (X_2,T_2) at $\tau_2>\tau_1$, and finally to (X_f,T_f) at $\tau_f>\tau_2$. (2) Integrate this amplitude over (X_1,T_1) in detector volume Δ_1 and over (X_2,T_2) in detector volume Δ_2. (3) Add to this the result with the volumes Δ_1 and Δ_2 interchanged because the system might have gone through Δ_2 first on its path before Δ_1. (4) Sum the result over the unobserved labels $\tau_f>\tau_2>\tau_1>0$. (5) Add to this the result with the path traversed in the opposite order because, while each path is ordered by the requirement that they move forward in τ, there is no observable distinction between $\mathcal{C}$ being first and (X_f,T_f) last or vice versa. (6) Square the resulting amplitude and integrate over the volume Δ_f. To obtain the probability for a particular pair of detectors to register this process must be repeated for paths which start at $\mathcal{C}$ but end on Δ_1, again for those which end on Δ_2, and the results summed.

The above construction may sound more familiar when expressed in the equivalent wave mechanical terms. The result of steps (1) and (2) above are mathematically equivalent to beginning with the wave function $\Psi_{\mathcal{C}}(X,T,\tau)$ of the state prepared with conditions $\mathcal{C}$ and carrying out incomplete measurements of (X,T) in the volumes Δ_1 at τ_1 and Δ_2 at τ_2, followed by a measurement yielding (X_f,T_f) at τ_f. One then sums this result over the unobserved labels τ_1, τ_2, and τ_f. The resulting joint amplitude is

$$\int_{-\infty}^{\infty} d\tau_f \int_{-\infty}^{+\infty} d\tau_2 \int_{-\infty}^{+\infty} d\tau_1 \int_{\Delta_2} dX_2 dT_2 \int_{\Delta_1} dX_1 dT_1 K_+(X_f,T_f,\tau_f;X_2,T_2,\tau_2) K_+(X_2,T_2,\tau_2;X_1,T_1,\tau_1) \Psi_{\mathcal{C}}(X_1,T_1,\tau_1) . \tag{6.1}$$

Here, $K_+(X'',T'',\tau'';X',T',\tau')$ is the forward Schrödinger propagator for the system, nonzero only for $\tau''>\tau'$. The amplitude with Δ_1 and Δ_2 reversed must be added to (6.1) and then the τ reversed of the resulting sum to get the complete joint probability amplitude for the paths which end at Δ_f.

Because of the τ-translation invariance the expression (6.1) is simpler if Fourier transforms are used. Define

$$K_+(X''T'';X',T') = \int_{-\infty}^{+\infty} d\tau'' K_+(X'',T'',\tau'';X',T',0) \tag{6.2}$$

(only the range $\tau''\geq 0$ contributes) and

$$\Phi(X,T,\mathcal{C}) = \int_{-\infty}^{+\infty} d\tau\, \Psi_{\mathcal{C}}(X,T,\tau) . \tag{6.3}$$

An expression equivalent to (6.1) is then

$$\int_{\Delta_2} dX_2 dT_2 \int_{\Delta_1} dX_1 dT_1 K_+(X_f T_f;X_2 T_2) K_+(X_2 T_2;X_1 T_1) \times \Phi(X_1,T_1,\mathcal{C}) , \tag{6.4}$$

where Φ was defined by (5.18). Such expressions can be developed into a "propagator theory" for computing amplitudes similar to that used in relativistic quantum mechanics and for much the same reasons.

C. Approximate recovery of a Schrödinger-Heisenberg formulation

The variables X and T have the same formal status in this model quantum cosmology. They are both positions. They enter symmetrically into the action (5.2), the constraint (5.19), the measure, and the histories. One cannot, therefore, expect to recover a notion of "state" and associated Hilbert space on a surface of constant T. The arguments of paper I show why the natural route to constructing such a Hilbert space does not work. Briefly, one can derive a Hilbert space on any surface for which there is a composition law expressing amplitudes from the past of the surface to its future as a composition of amplitudes from the past to the surface with amplitudes from the surface to the future. On an (X,T) lattice the amplitude in the above example to pass from the conditions $\mathcal{C}$ to (X_f,T_f) may be factored into sums of products of amplitudes to go from $\mathcal{C}$ to a constant T surface and then from the constant T surface on to (X_f,T_f). The individual amplitudes involve a different number of intersections with the surface. In the "continuum" limit, however, the expected number of crossings of a path is infinite, and the amplitudes for a finite number of crossings relatively vanish. There is thus no composition law and no Hilbert space which is naturally constructable on a constant T surface. As the above example shows, this does not alter our ability to predict the probabilities of the outcomes of experiments which include detectors arrayed along a constant T surface although rather greater care is needed to arrange experiments which correspond to questions with sensible answers.

Such is the general situation. However, when the initial conditions are such that T tracks the label τ to a good accuracy, then appropriately conditioned probabilities are approximated by those of familiar quantum mechanics with T playing the role of time. We saw a simple example in the preceding section. To see this for "multitime observations" consider again the example shown in Fig. 2. Assume for conditions the following: initial conditions $\mathcal{C}$, the detection in Δ_f and that at least one of the detectors in the intermediate array registers. We shall show first that, in such a situation, the amplitude that two or more suitable detectors register is very small and that the probability for a single detection in the detector at $(\overline{X},\overline{T})$ is approximately the probability of the familiar theory that a position measurement at $\overline{T}$ gives a result in the range Δ_X.

Given just conditions $\mathcal{C}$ and detection in Δ_f there is the possibility that no detectors in the array at $\overline{T}$ register. The amplitude for this need not be small compared to the other possiblities and will in general interfere with them. The array of detectors is thus not perfectly efficient. This universe does not have sufficient degrees of freedom to make a model of an efficient detector for which the probability of no registration is small. The requirement that a measurement was actually carried out at $\overline{T}$ is modeled here by the condition that at least one of the detectors registers.

If the initial conditions $\mathcal{C}$ mandate a good clock state, then the amplitude for the system to propagate between one detector and another on a given constant $\overline{T}$ surface is negligibly small provided the dimensions of the detector Δ_X and Δ_T are suitably large. To see this let us calculate the amplitude for the propagation between two detectors Δ_1 and a point (X_2,T_2) inside detector Δ_2. Δ_1 and Δ_2 are centered at the same value of $\overline{T}$ but at two different values of X. Let us call the central values $(\overline{X}_1,\overline{T})$ and $(\overline{X}_2,\overline{T})$. According to the discussion above this amplitude is

$$\int_{-\infty}^{+\infty} d\tau_2 d\tau_1 \int_{\Delta_1} dX_1 dT_1 K_+(X_2,T_2,\tau_2;X_1,T_1,\tau_1) \times \Psi_{\mathcal{C}}(X_1,T_1,\tau_1) \ . \tag{6.5}$$

Assume that the conditions $\mathcal{C}$ are such that $\Psi_{\mathcal{C}}(X,T,\tau)$ factors as

$$\Psi_{\mathcal{C}}(X,T,\tau)=\phi(T,\tau)\psi(X,\tau) \ , \tag{6.6}$$

where $\phi(T,\tau)$ is a good clock state in the sense of Sec. II. That is, it is sharply peaked about $T=T(\tau)$ and remains peaked for a significant period of time. For a noninteracting clock and system the propagator K also factors

$$K_+(X_2,T_2,\tau_2;X_1,T_1,\tau_1)=k_C(T_2,\tau_2;T_1,\tau_1) \times k_X(X_2,\tau_2;X_1,\tau_1) \tag{6.7}$$

so that the clock and system evolve independently. The evolution described by (6.5) is, in familiar terms, just that produced by an incomplete measurement of X and T at time τ_1 followed by an integration over the possible values of this time. Let us analyze what happens in this language.

At τ_1 the wave function $\Psi_{\mathcal{C}}$ is projected onto the detector region Δ_1 of dimensions Δ_T and Δ_X. For subsequent τ, it evolves by the Schrödinger equation. If the peaking of the wave packet $\phi(T,\tau)$ is small compared to Δ_T, then the packet after projection will be essentially unchanged from what it was before. A good clock state will remain. In particular, the amplitude (6.5), evaluated at a value T_2 inside a second detector Δ_2, will be negligible at any parameter time τ_2 which is $\tau'(\overline{T})\Delta_T$ later than τ_1. The question of whether there is any amplitude at all therefore concerns the X evolution between $\tau_1=\tau(\overline{T})$ and $\tau_1=\tau(\overline{T}+\Delta_T)$.

The wave function describing the system after projection will be a localized state of width Δ_X centered about $\overline{X}_1$. If Δ_X is small compared to the characteristic wavelengths in $\psi(X,\tau_1)$, the associated spread in momentum will be of order $1/\Delta_X$. After projection the wave function will spread. If it is unable to spread by a distance Δ_X in the parameter time $\tau'(\overline{T})\Delta_T$ then there will be a negligible overlap *at any* τ with the volume of the second detector Δ_2. Since the dispersion in momentum is of order $1/\Delta_X$ this will be the case if Δ_X is sufficiently large. Thus, if Δ_T is larger than the accuracy of the clock so that the system always remains in a good clock state, and Δ_X is large enough so the wave packet does not spread from one detector to another in the time Δ_T, there will be a negligible amplitude for two detectors at the same $\overline{T}$ to register.

A more quantitative idea of the restrictions on Δ_X and Δ_T can be obtained if the system is a free particle of mass M moving in one dimension. There the width of the wave packet after projection is

$$(\Delta X)^2_\tau=\Delta_X^2+\Delta\tau\langle \{p,\Delta x\}\rangle_{\tau_1}/M+(\Delta p)^2_{\tau_1}(\Delta\tau)^2/M^2 \ , \tag{6.8}$$

where $\Delta\tau=\tau-\tau_1$. If $\Delta\tau\sim\Delta_T/v$ and $(\Delta p)_{\tau_1}\sim 1/\Delta_X$ we require

$$\Delta_X \gtrsim (\Delta_T/Mv)^{1/2} \ . \tag{6.9}$$

We have already seen in Sec. II that for fixed running time $\mathcal{T}$, Δ_T can be made small by making the free clock massive. Equation (6.9) shows that in the large-M regime, Δ_X can be taken small also.

The only possibility with a significant amplitude is that a single detector registers in the array at $\overline{T}$. The joint amplitude for $\mathcal{C}$, this single detection at Δ_1, and finally the detection at Δ_f is

$$\int_{-\infty}^{+\infty} d\tau_f d\tau_1 \int_{\Delta_1} dX_1 dT_1 K_+(X_f,T_f,\tau_f;X_1T_1\tau_1) \times \Psi_{\mathcal{C}}(X_1,T_1,\tau_1) \qquad (6.10)$$

plus a similar amplitude with the order of the τ's reversed. However, this latter contribution is negligible because the good clock state enforces a correlation between positive values of T and positive values of τ. Equation (6.10) is therefore the only significant amplitude and causality is restored. Taking account of (6.6) and (6.7) the joint probability for $\mathcal{C}$ and registration by the detectors at $(\bar{X},\bar{T})$ and (X_f,T_f) is

$$\int_{\Delta_f} dX_f \, | \, k_X(X_f,T_f;\bar{X},\bar{T})\psi(\bar{X},\bar{T}) \, |^2 \, . \qquad (6.11)$$

This is the familiar result of standard quantum mechanics.

The preceding discussion might be summarized by saying that in a good clock state the amplitude for the clock to run backward in its indicator variable is very small for appropriately crude measurements. Thus the only paths which contribute significantly to relevant joint amplitudes are those which move forward in T. Thus, for conditions $\mathcal{C}$ which produce sufficiently accurate clocks and for sufficiently large Δ_X and Δ_T there are approximate notions of unitarity and causality for the experiment described in Fig. 2. In this situation the amplitude to excite more than one intermediate detector is negligible. The joint amplitude for at least one detector in the array to register given the conditions $\mathcal{C}$ and registration at (X_f,T_f) is approximately the sum for any one of the detectors to register. Further this is approximately

$$\int d\bar{X} \, k_X(X_f,T_f;\bar{X},\bar{T})\psi(\bar{X},\bar{T}) \, . \qquad (6.12)$$

This is the composition law of the standard theory.

VII. THE BORN-OPPENHEIMER APPROXIMATION

The results of Sec. II show that when the mass of a clock is sufficiently large, it can be made to behave classically for an interesting length of time. The results of the preceding two sections show that if the initial conditions of our model quantum cosmologies contain such good clocks then the probabilities for appropriate experiments are approximated by those of Schrödinger-Heisenberg quantum mechanics. In a theory in which it is fundamentally absent, we recover a notion of Schrödinger-Heisenberg time τ from the classical trajectories $T(\tau)$ along which the centers of wave packets representing good clock states move. Where it is valid, one therefore expects to recover a notion of Schrödinger-Heisenberg time in a systematic approximation in which the clock indicator variables are treated classically while the system variables are treated quantum mechanically. That this is indeed the case was shown in Refs. 11–14. We shall very briefly review these arguments here.

The approximation in which part of a system is treated classically and another part quantum mechanically is the Born-Oppenheimer approximation of molecular physics. There, the nuclei are a "heavy" system whose motion is described classically. The electrons move quantum mechanically in the potential of the nuclei. In the present case the clock is the "heavy" system. The controlling parameter is the clock's mass M. If the clock's Lagrangian is written in the form (5.7), the limit $M\to\infty$ is the limit in which the clock action is much larger than $\hbar$. This is the classical limit for the clock. It is to describe this limit supply that we have written the clock's potential in the form MV_C in (5.1).

The result of this approximation for the amplitude $\Phi(X,T,\mathcal{C})$ is immediate from the defining sum over histories (5.18). In the large-M limit the stationary phase or semiclassical approximation may be used to evaluate the sum over the clock paths, $T(\tau)$. If the initial conditions mandate a good clock state only a *single* stationary (classical) path $T(\tau)$ will contribute to the sum. One then has

$$\Phi(X,T,\mathcal{C})\approx[ds_C(T)/dT]^{-1/2}\exp[is_C(T)] \times \int_{\mathcal{C},\tau} \delta X \exp\{is_X[X(\tau),\tau]\} \, . \qquad (7.1)$$

Here, $s_C(T)$ is the action implied for the classical path by the initial conditions as a function of the path's end points. The classical path runs between its end points in a definite parameter interval τ. This is the interval used to define the remaining sum over paths, $X(\tau)$. These paths move forward in T because T moves forward in τ for a classical path. The sum over paths $X(\tau)$ in (6.1) is then just that which defines the familiar Schrödinger wave function. We may therefore write

$$\Phi(X,T,\mathcal{C})\approx\left[\frac{ds_C}{dT}\right]^{-1/2}\exp[is_C(T)]\psi_{\mathcal{C}}(X,\tau) \, , \qquad (7.2)$$

where $\psi_{\mathcal{C}}$, defined by the sum in (7.1), satisfies

$$i\frac{\partial\psi_{\mathcal{C}}}{\partial\tau}=h_X\psi_{\mathcal{C}} \, . \qquad (7.3)$$

In this way we recover the results (5.25) of the theory of a good clock initial condition directly as the limit in which the clock is treated classically.

The results (7.2) and (7.3) could also be obtained from the constraint (5.19). One looks for solutions of the form

$$\Phi(X,T)=\exp[is_C(T)]\psi(X,T) \qquad (7.4)$$

in which the functions s_C and ψ have systematic expansions in powers of M

$$s_C(T)=Ms_0(T)+s_1(T)+O(M^{-1}) \, , \qquad (7.5a)$$

$$\psi(X,T)=\psi_0(X,T)+O(M^{-1}) \, . \qquad (7.5b)$$

The result of enforcing the constraint $H\Phi=0$ to the lowest two nontrivial orders in M^{-1} is that Ms_0 satisfies the classical Hamiltonian-Jacobi equation, that ψ_0 (with suitable choice of energy normalization) satisfies the Schrödinger equation, and that s_1 can be calculated from s_0. One thus recovers (7.2) and (7.3) by systematically solving the constraint. (See Refs. 12 and 16 for details.) This equivalence between a steepest-descent approximation to a sum over histories and a WKB analysis of the associated constraint is, of course, not unexpected.

In view of its directness, a possible point of view might be that one should predict all probabilities in quantum cosmology by solving the constraint in the semiclassical approximation, recovering thereby a Schrödinger-Heisenberg time and then invoking familiar quantum-mechanical rules. Such an approach leaves untouched, however, a number of issues which can only be resolved by having a more fundamental framework in which there is at least *in principle* an exact connection between formalism and observation. Examples are the status of the approximation itself (how semiclassical does Φ have to be?), and the existence of Hilbert space (is Φ normalizable?). It is to answer such questions that we have investigated the present framework.

VIII. AN OPERATOR FORMALISM

Standard quantum mechanics can be expressed either in the language of sums over histories or in the language of operators on Hilbert space. In this section we show that the model quantum cosmology with real clocks that we have been discussing in sum-over-histories terms can be given an equivalent operator formulation.

Consider for definiteness the problem discussed in the preceding section and illustrated in Fig. 2. The amplitude to proceed from initial conditions forward through regions Δ_1, to (X_2,T_2) was expressed as a string of propagators in Eq. (6.1). The forward propagator K_+ was defined in terms of the familiar Schrödinger propagator by (6.2). This Schrödinger propagator is

$$K_+(X''T''\tau,X'T'0)=\langle X''T''\,|\,e^{-iH\tau}\,|\,X'T'\rangle \ . \quad (8.1)$$

Here, the Hilbert space of states $\mathcal{H}$ is the space of square-integrable functions on X and T while H is the operator $H=h_C+h_X-E$. Carrying out the integral over τ in (6.1), we have

$$K_+(X''T'',X'T')=\left\langle X''T''\left|\frac{-i}{H-i\varepsilon}\right|X'T'\right\rangle \ . \quad (8.2a)$$

In a similar way

$$K_-(X''T'',X',T)=\left\langle X''T''\left|\frac{i}{H+i\varepsilon}\right|X'T'\right\rangle \ . \quad (8.2b)$$

The first of these is the propagator which is the analog of that advocated by Teitelboim[28] for spacetime. Using these expressions, and the projections on a region Δ,

$$P_\Delta=\int_\Delta dX\,dT\,|\,XT\rangle\langle XT\,| \ , \quad (8.3)$$

amplitudes such as (6.4) can be reexpressed in operator form. For example, the amplitude to move both forward and backward from initial conditions $\mathcal{C}$ through Δ_1 to (X_2T_2) in Fig. 2 is

$$\left\langle X_2T_2\left|\frac{-i}{H-i\varepsilon}P_{\Delta_1}\frac{-i}{H-i\varepsilon}\right|\mathcal{C}\right\rangle + \left\langle X_2T_2\left|\frac{i}{H+i\varepsilon}P_{\Delta_1}\frac{i}{H+i\varepsilon}\right|\mathcal{C}\right\rangle \ . \quad (8.4)$$

The joint probability $p(\Delta_2,\Delta_1,\mathcal{C})$ is the square of this amplitude integrated over Δ_2 plus a similar expression with Δ_1 and Δ_2 interchanged. If $|\,\mathcal{C}\rangle$ is a good clock state, and Δ_2 and Δ_1 are disjoint regions centered about $T_2>T_1$, of not too narrow an extent in T, then of the four terms making up this probability only a single one contributes. This comes from the first term in (8.4), which, as we have argued in the preceding sections, yields the prediction of standard quantum mechanics. Characteristically, (8.4) involves a coherent superposition of an amplitude and its τ reversal. No quantum-mechanical arrow of time is therefore singled out. However for initial conditions $\mathcal{C}$ such that one element of this superposition is negligibly small we recover an approximate causality.

IX. CONCLUSION

We have presented a model quantum cosmology whose classical dynamics does not distinguish a preferred time from among its configuration-space variables. We have proposed how, using sum-over-histories quantum mechanics or an equivalent description on an extended Hilbert space, probabilities can be assigned to the outcomes of sensible experiments. These include experiments which involve detections on several different hypersurfaces in the classical configuration space. We have argued that there is no notion of state on a hypersurface, no notion of Hilbert space inner product on a hypersurface, and no notion of unitary evolution between hypersurfaces which will give these probabilities in general according to the familiar prescription of Schrödinger-Heisenberg quantum mechanics. However, we have shown that when the initial conditions of these model universes are such that part of the system behaves approximately as an ideal clock then there is an approximate notion of state, Hilbert-space inner product, unitary evolution, and a notion of causality using surfaces of constant clock time. The probabilities for suitably restricted experiments predicted by the sum-over-histories framework are then given approximately by the associated Schrödinger-Heisenberg quantum mechanics.

This model quantum cosmology is not proposed as a candidate for a more accurate quantum mechanics of isolated systems on familiar scales involving realistic clocks. If the clock is not very accurate the predictions of this model and the familiar quantum theory will disagree. Roughly, in familiar quantum-mechanics amplitudes for unresolved time differences are incoherently added (Sec. II C) while in the model they are coherently superposed. However, the models are incomplete. They do not contain variables to describe dynamical spacetime nor the universal gravitational coupling of all matter systems to it. As we shall argue in the next paper, the familiar formulation of quantum mechanics is the correct approximation for dealing with clocks, accurate or inaccurate, when these spacetime variables themselves are followed with precision much below the Planck scale in the late universe. What these models do illustrate in an elementary way is how the notion of time and causality can arise in a quantum theory which does not have them in general from the properties of specific initial conditions.

The interpretative framework of quantum mechanics loosely subsumed under the name "Copenhagen interpretation" contains two central assumptions which seem incompatible with a quantum cosmology built on covariant theories of spacetime. The first is a distinguished class of classical systems. The second is a distinguished time variable and its associated notion of causality. The first assumption seems incompatible with the application of quantum mechanics to the whole universe. The second seems incompatible with general covariance. The particular initial conditions of our universe could imply an approximate classical reality in the late universe.[29] The distinguished time and the associated causality of familiar scales could also arise, not as exact notions in the formalism of quantum theory, but as approximate notions in the late universe which are consequences of specific initial conditions.

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[1]For a review and discussion see J. A. Wheeler, in *Problems of Fundamental Physics,* proceedings of the International School of Physics "Enrico Fermi," Course LII, edited by G. Toraldo di Francia (North-Holland, Amsterdam, 1979); K. Kuchař, in *Quantum Gravity 2,* edited by C. J. Isham, R. Penrose, and D. W. Sciama (Clarendon, Oxford, 1981), p. 329ff.

[2]For some alternatives to the traditional canonical view discussed in Ref. 1 but still with a preferred time, see T. Padmanabhan, in *Gravitation, Gauge Theories and the Early Universe* (Reidel, Dordrecht, 1987); W. Unruh, in *Proceedings of the Fourth Moscow Quantum Gravity Seminar,* edited by M. A. Markov, V. P. Frolov, and V. A. Berezin (World Scientific, Singapore, 1988).

[3]P. A. M. Dirac, Proc. R. Soc. London **A111**, 405 (1926).

[4]R. Arnowitt, S. Deser, and C. W. Misner, Nuovo Cimento **19**, 668 (1961).

[5]H. Paul, Ann. Phys. (Leipzig) **9**, 252 (1962).

[6]B. DeWitt, Phys. Rev. **160**, 1113 (1967).

[7]G. R. Allcock, Ann. Phys. (N.Y.) **53**, 253 (1969); **53**, 286 (1969).

[8]A. Peres, Am. J. Phys. **48**, 552 (1980).

[9]D. Page and W. Wootters, Phys. Rev. D **27**, 2885 (1983); W. Wootters, Int. J. Theor. Phys. **23**, 701 (1984); in *Fundamental Questions in Quantum Mechanics,* edited by L. Roth and A. Inomata (Gordon and Breach, New York, 1986).

[10]E. Wigner, Rev. Mod. Phys. **29**, 255 (1957); H. Salecker and E. Wigner, Phys. Rev. **109**, 571 (1958).

[11]V. Lapchinsky and V. Rubakov, Acta Phys. Pol. **B10**, 1041 (1979).

[12]T. Banks, Nucl. Phys. **B249**, 332 (1985).

[13]J. Halliwell and S. W. Hawking, Phys. Rev. D **31**, 1777 (1985).

[14]For further developments, see S. Wada, Nucl. Phys. **B276**, 729 (1986); E. Joos, Phys. Lett. A **116**, 6 (1986); H. D. Zeh, *ibid.* **116**, 9 (1986); K. Kuchař and M. Ryan, in *Gravitational Collapse and Relativity,* edited by H. Sato and T. Nakamura (World Scientific, Singapore, 1986); R. Brout, G. Horwitz, and D. Weil, Phys. Lett. B **192**, 318 (1987); P. D. D'Eath and J. Halliwell, Phys. Rev. D **35**, 1110 (1987); J. Halliwell, *ibid.* **36**, 3626 (1987); R. LaFlamme, Phys. Lett. B **198**, 156 (1987); A. Vilenkin, Phys. Rev. D **37**, 888 (1988); T. Padmanabhan (unpublished); T. Padmanabhan and T. P. Singh (unpublished).

[15]See, e.g., E. P. Wigner, Am. J. Phys. **31**, 6 (1963); Y. Aharonov, P. G. Bergmann, and J. L. Lebowitz, Phys. Rev. **134**, B1410 (1964); M. Gell-Mann, Phys. Scr. **T15**, 202 (1987); W. Unruh, in *New Techniques and Ideas in Quantum Measurement Theory,* edited by D. M. Greenberger (New York Academy of Sciences, New York, 1986).

[16]See, e.g., J. B. Hartle, in *Gravitation in Astrophysics,* edited by B. Carter and J. B. Hartle (Plenum, New York, 1987).

[17]M. Gell-Mann, J. B. Hartle, and V. Telegdi (unpublished).

[18]R. P. Feynman, Rev. Mod. Phys. **20**, 367 (1948). See especially, C. Caves, Phys. Rev. D **33**, 1643 (1986); *ibid.* **35**, 1815 (1987); also J. Stachel, in *From Quarks to Quasars,* edited by R. G. Colodny (University of Pittsburgh Press, Pittsburgh, 1986), p. 331ff.

[19]C. Teitelboim, Phys. Rev. D **25**, 3159 (1983); **28**, 297 (1983); **28**, 310 (1983).

[20]J. B. Hartle, Phys. Rev. D **37**, 2818 (1988) (paper I).

[21]A complex system with a single collective coordinate serving as the clock indicator variable would have an essentially eqivalent description.

[22]W. Unruh and R. Wald have recently discussed the possibility of using these standard relations to recover a hidden ordering time in quantum cosmology. See W. Unruh, in *Proceedings of the Fourth Moscow Quantum Gravity Seminar* (Ref. 2).

[23]See J. B. Hartle and K. Kuchař, in *Quantum Theory of Gravity,* edited by S. Christensen (Hilger, Bristol, 1984), p. 315ff, for a discussion of this distinction in sum-over-histories quantum mechanics when spacetime is not a dynamical variable.

[24]See, e.g., R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[25]See, e.g., K. Kuchař, in *Relativity, Astrophysics, and Cosmology,* edited by W. Israel (Reidel, Dordrecht, 1973).

[26]J. Barbour and B. Bertotti, Proc. R. Soc. London **A382**, 295 (1982).

[27]See, e.g., J. B. Hartle and K. Kuchař, Phys. Rev. D **34**, 2323 (1986); J. B. Hartle and S. W. Hawking, *ibid.* **13**, 2188 (1976).

[28]The tension between these has been very clearly put by C. Teitelboim, Phys. Rev. Lett. **50**, 705 (1983). See also J. Halliwell, Phys. Rev. D **38**, 2468 (1988).

[29]See, for example, Ref. 17 and references therein.