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Naked singularities and the hoop conjecture: An analytic exploration

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The formation of a naked singularity during the collapse of a finite object would pose a serious difficulty for the theory of general relativity. The hoop conjecture suggests that this possibility will never happen provided the object is sufficiently compact ($\lesssim M$) in all of its spatial dimensions. But what about the collapse of a long, nonrotating, prolate object to a thin spindle? Such collapse leads to a strong singularity in Newtonian gravitation. Here we construct an analytic sequence of momentarily static, prolate, and oblate dust spheroids in full general relativity. In the limit of large eccentricity the solutions all become singular. However, when the spheroids are sufficiently large there are no apparent horizons, lending support to the hoop conjecture. These solutions thus suggest that naked singularities with matter could possibly form in asymptotically flat spacetimes.

I. INTRODUCTION

It is well known that general relativity admits solutions with singularities, and that such solutions can be produced by the gravitational collapse of nonsingular, asymptotically flat initial data. The cosmic censorship hypothesis[1] states that such singularities will always be clothed by event horizons and hence can never be visible from the outside (no naked singularities). If cosmic censorship holds, then there is no problem with predicting the future evolution outside the event horizon. Alternatively, if it does not hold, then the formation of a naked singularity during such collapse would be a disaster for general-relativity theory. In this situation, one cannot say anything precise about the future evolution of any region of space containing the singularity since new information could emerge from it in a completely arbitrary way.

But what guarantees are there that an event horizon will always hide a naked singularity? There are no completely definitive theorems as yet. In the case of spherical collapse, the solution outside the collapsing matter is the Schwarzschild metric. In all numerical and analytic studies, the singularity occurs inside[2] the event horizon at $r_S = 2M$. Results for nonspherical collapse are less complete. For this situation, Thorne[3] has proposed the *hoop conjecture:* Black holes with horizons form when and only when a mass M gets compacted into a region whose circumference in *every* direction is $\mathcal{C} \lesssim 4\pi M$.

If the hoop conjecture is indeed correct, aspherical collapse with one or two dimensions appreciably larger than the others might then lead to naked singularities. For example, consider the Lin-Mestel-Shu instability[4] for the collapse of a nonrotating, homogeneous spheroid of dust in Newtonian gravity. If the spheroid is slightly oblate, the configuration collapses to a pancake, while if the spheroid is slightly prolate, it collapses to a spindle. While in both cases the density becomes infinite, the formation of a spindle during prolate collapse is particularly worrisome. The gravitational potential, gravitational force, tidal force, potential, and kinetic energies all blow up. This behavior is far more serious than mere shell crossing, where the density alone becomes momentarily infinite. In the case of collisionless matter, prolate evolution is forced to terminate at the singular spindle state, while for oblate evolution the matter simply passes through the pancake state. In fact, having passed through the pancake, the oblate configuration then evolves to a spindle singularity.

Does this example have any relevance to general relativity? We already know that *infinite* cylinders do collapse to singularities in general relativity, and, in accord with the hoop conjecture, are not hidden by event horizons.[3,5] While it has been argued[3,5,6] that the ultimate singular state will be avoided by the presence of pressure (as long as the adiabatic index $\Gamma > 1$), we feel that this does not address the fundamental problem. Naked singularities could then still form in the case of *perfectly* collisionless matter. Does the possibility of forming naked singularities then depend on the details of the interactions affecting matter at high densities? Does the degree to which collapse becomes singular depend on the complete zoo of coupling constants, gauge fields, supersymmetric partners, and so on? We would be far more comfortable knowing that Einstein's equations automatically prevent naked singularities with only very weak conditions imposed on the matter stress-energy tensor. The key question thus is not about pressure but is whether singularities form during the prolate collapse of a finite object in asymptotically flat spacetime.

Interestingly, there exists a class of static axisymmetric solutions of the vacuum Einstein equations that correspond to the exterior fields of prolate and oblate spheroidal configurations.[7] These configurations are finite in extent and the spacetimes are asymptotically flat. Yet, they have the *same* singularity structure that characterizes the corresponding Newtonian spheroids. Because these solutions do not contain matter, however, their relevance to the formation of naked singularities during gravitational collapse is not clear.

More troublesome are the simulations of axisymmetric fluid collapse by Nakamura and co-workers.[8] They simu-

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lated the general-relativistic collapse of deformed stars with internal pressure. The found that if the initial internal energy was appreciable, then apparent horizons always formed, no matter how large the initial deformation was. However, if the initial internal energy was small and the initial deformation large, the results were different. Specifically, they describe an example of prolate collapse that appears to be evolving to a singular state, despite the presence of pressure obeying a very stiff equation of state (asymptotically $\Gamma=2$). By the time the simulation was terminated, no apparent horizon had appeared. Of course it is always possible that an event horizon had already formed, but this was not studied because it is tedious in two dimensions.

Given all this, it is clearly desirable to perform detailed numerical simulations of the prolate collapse of realistic initial configurations with matter and to probe the resulting spacetimes for the growth of singularities and the development of event horizons. As a first step, one must construct a suitable family of initial configurations for such an evolution. Accordingly, in this paper we solve the initial-value problem in general relativity for a class of axisymmetric prolate and oblate spheroids containing matter. The configurations are finite-size, inhomogeneous dust spheroids. The matter is instantaneously at rest at $t=0$ (moment of time symmetry) and the dynamical components of the gravitational field are set equal to zero. For the cases we consider, the Hamiltonian constraint equation reduces to Poisson's equation. For the adopted density profile, the solutions are analytic and are determined from the solutions for homogeneous Newtonian spheroids.

We analyze these solutions for apparent horizons to assess the validity of the hoop conjecture. We are particularly interested in extreme configurations with eccentricities approaching unity as candidates for singularities. Should these configurations show singularities that are not hidden by horizons, this would suggest that naked singularities could actually arise in dynamical collapse. We explore this possibility by considering a sequence of momentarily static configurations of fixed rest mass but increasing eccentricity. We find that highly eccentric prolate and oblate spheroids are indeed singular, and in agreement with the hoop conjecture, extended configurations have no apparent horizons. Hence, the validity of the unqualified cosmic censorship hypothesis is somewhat suspect. Further dynamical calculations are urgently needed.

II. HOMOGENEOUS NEWTONIAN SPHEROIDS

In this section we summarize the key equations governing the gravitational field of homogeneous spheroids in Newtonian gravity. They will be used below in constructing fully relativistic spheroids. We set $G=c=1$.

The fundamental equation is Poisson's equation for the potential Φ_N,

$$\nabla^2\Phi_N=4\pi\rho_N \ , \tag{2.1}$$

where the rest-mass density is

$$\rho_N=\begin{cases}\dfrac{M_N}{4\pi a^2c/3}, & \dfrac{R^2}{a^2}+\dfrac{z^2}{c^2}\le 1 \ ,\\ 0 \ \text{elsewhere} \ . \end{cases} \tag{2.2}$$

Here M_N is the total Newtonian rest mass, a is the equatorial radius, c is the polar radius, and R and z are the usual cylindrical coordinates.

A. Oblate spheroids $c\le a$

For oblate spheroids, we define the eccentricity e by

$$e=(1-c^2/a^2)^{1/2} \ . \tag{2.3}$$

The potential can be written[9] in terms of the radial and vertical components of the force

$$K_R=-\frac{3M_N}{2(ae)^3}R(\beta-\sin\beta\cos\beta) \ , \tag{2.4}$$

$$K_z=-\frac{3M_N}{(ae)^3}z(\tan\beta-\beta) \ , \tag{2.5}$$

as

$$\Phi_N=-\frac{3M_N}{2ae}\beta-\tfrac{1}{2}(RK_R+zK_z) \ . \tag{2.6}$$

Here β is given by

$$\sin\beta=e \tag{2.7a}$$

for a point inside the spheroid, while

$$R^2\sin^2\beta+z^2\tan^2\beta=a^2e^2 \tag{2.7b}$$

for a point outside the spheroid. Note that

$$\Phi_N(0)\to-\frac{3\pi M_N}{4a} \quad (\text{finite}), \quad e\to 1 \ . \tag{2.8}$$

The Newtonian gravitational binding energy is given by

$$W_N=-\tfrac{1}{2}\int\rho_N\Phi_N d^3x \tag{2.9a}$$

$$=\frac{3}{5}\frac{M_N^2}{a}\frac{\arcsin(e)}{e} \ , \tag{2.9b}$$

which is also finite for all eccentricities.

B. Prolate spheroids $a\le c$

For prolate spheroids, we define the eccentricity e by

$$e=(1-a^2/c^2)^{1/2} \ . \tag{2.10}$$

The potential can be obtained from the oblate case by the transformation

$$e\to i\frac{c}{a}e, \quad \beta\to i\beta \ . \tag{2.11}$$

The potential can again be written in terms of the radial and vertical components of the force

$$K_R=\frac{3M_N}{2(ce)^3}R(\beta-\sinh\beta\cosh\beta) \ , \tag{2.12}$$

$$K_z=\frac{3M_N}{(ce)^3}z(\tanh\beta-\beta) \ , \tag{2.13}$$

as

$$\Phi_N = -\frac{3M_N}{2ce}\beta - \tfrac{1}{2}(RK_R + zK_z) \; . \tag{2.14}$$

Here β is given by

$$\sinh\beta = \frac{c}{a}e \tag{2.15a}$$

for a point inside the spheroid, while

$$R^2 \sinh^2\beta + z^2 \tanh^2\beta = c^2 e^2 \tag{2.15b}$$

for a point outside the spheroid. Note that

$$\Phi_N(0) \to \frac{3M_N \ln(1-e)}{4c} \to -\infty, \quad e \to 1 \; . \tag{2.16}$$

The Newtonian gravitational binding energy is given by

$$W_N = \frac{3}{10}\frac{M_N^2}{ce}\ln\frac{1+e}{1-e} \; , \tag{2.17}$$

which also diverges as $e \to 1$.

III. MOMENTARILY STATIC SPHEROIDS IN GENERAL RELATIVITY

The most general nonrotating axisymmetric three-metric can be written in the form[10]

$$^{(3)}ds^2 = \psi^4[e^q(dR^2 + dz^2) + R^2 d\phi^2] \; . \tag{3.1}$$

For a moment of time symmetry, the extrinsic curvature satisfies $K_{ij}=0$. The only remaining dynamical component of the field is the metric function q. We will look for initial data that satisfy $q=0$. The line element then becomes conformally flat:

$$^{(3)}ds^2 = \psi^4 \delta_{ij} dx^i dx^j \; . \tag{3.2}$$

When convenient, we will utilize Cartesian coordinates (x,y,z), cylindrical coordinates (R,z,ϕ), or spherical polar coordinates (r,θ,ϕ) for the Euclidean metric δ_{ij}.

The stress-energy tensor for dust is

$$T_{\mu\nu} = \rho u_\mu u_\nu \; , \tag{3.3}$$

where ρ is the rest-mass energy density and u_μ is the four-velocity.

The only Einstein equation that the initial data must satisfy nontrivially is the Hamiltonian constraint

$$^{(3)}R = 16\pi\rho^* \; , \tag{3.4}$$

where $\rho^* = T_{\mu\nu}n^\mu n^\nu$ and n^μ is the normal vector to the initial hypersurface $t=0$. For a configuration momentarily at rest, $u^\mu = n^\mu$, and so $\rho^* = \rho$. Thus the Hamiltonian constraint becomes

$$\nabla^2\psi = -2\pi\psi^5\rho \tag{3.5}$$

with the boundary conditions

$$\nabla\psi = 0, \quad r=0, \quad \psi \to 1 + \frac{M}{2r}, \quad r \to \infty \; , \tag{3.6}$$

where M is the total mass energy of the configuration.

To obtain an analytic solution, set the density profile to be

$$2\pi\psi^5\rho \equiv 4\pi\rho_N \; , \tag{3.7}$$

where ρ_N is defined in Eq. (2.2). Comparing Eq. (2.1) with (3.5), we immediately conclude that

$$\psi = 1 - \Phi_N \; . \tag{3.8}$$

Since at large radii $\Phi_N \to -M_N/r$, we have, from Eqs. (3.6) and (3.8),

$$M = 2M_N \; . \tag{3.9}$$

The total rest-mass energy is given by

$$M_0 = \int \rho\psi^6 d^3x = \int 2\rho_N(1-\phi_N)d^3x \; . \tag{3.10}$$

Hence, using Eq. (2.9a), we find that

$$M_0 = 2M_N + 4W_N \; , \tag{3.11}$$

and so

$$M = \frac{2M_0}{1+(1+\alpha M_0)^{1/2}} \; , \tag{3.12}$$

where

$$\alpha = \begin{cases} \dfrac{12}{5ae}\arcsin(e) & \text{oblate} \; , \\[2ex] \dfrac{6}{5ce}\ln\dfrac{1+e}{1-e} & \text{prolate} \; . \end{cases} \tag{3.13}$$

The density profile ρ resulting from Eq. (3.7) is inhomogeneous. It is constant in these coordinates on self-similar spheroids and increases outwards from the center.

Geometric probes

If an apparent horizon exists in the three-metric (3.2), then by axisymmetry it is a surface of revolution $r=r(\theta)$ satisfying[11]

$$r_{,\theta\theta} + \left[\frac{r_{,\theta}^3}{r^2} + r_{,\theta}\right]\left[\frac{4\psi_{,\theta}}{\psi} + \cot\theta\right] - r_{,\theta}^2\left[\frac{3}{r} + \frac{4\psi_{,r}}{\psi}\right] - 2r - r^2\frac{4\psi_{,\theta}}{\psi} = 0 \; , \tag{3.14}$$

where

$$r_{,\theta} = 0 \quad \text{at } \theta = 0, \pi/2 \; . \tag{3.15}$$

While the coefficients in Eq. (3.14) are analytic, we have to integrate the equation numerically. However, since it is an ordinary differential equation (ODE), it can be solved to essentially arbitrary accuracy. A simple method is to start at the pole with a trial value $r=r_0$ and integrate to the equator. Vary r_0, starting at large $r_0 \gg M$, and search for a sign change of $r_{,\theta}$ at the equator. If there is no sign change for any r_0, there is no apparent horizon. If there is a sign change, one can iterate until $r_{,\theta}=0$ to locate r_0 precisely. In practice, because of the cotangent singularity in Eq. (3.14), one actually begins the integration at finite θ using

$$r = r_0 + r_2\theta^2, \quad \theta \ll 1 \; , \tag{3.16}$$

where r_2 is obtained by series expansion of the coefficients in the differential equation.

To aid in evaluating the hoop conjecture, we will want to compute certain invariant measures of the apparent horizon (AH). Its equatorial and polar circumferences are given by

$$\frac{\mathcal{C}_{\rm eq}^{\rm AH}}{2\pi(2M)}=\frac{r(\theta)\psi^2[r(\theta),\theta]}{2M}\bigg|_{\theta=\pi/2}, \tag{3.17}$$

$$\frac{\mathcal{C}_{\rm pole}^{\rm AH}}{2\pi(2M)}=\frac{1}{\pi M}\int_0^{\pi/2}\psi^2(r,\theta)(r^2+r_{,\theta}^2)^{1/2}d\theta , \tag{3.18}$$

while its surface area is

$$\frac{\mathcal{A}}{4\pi(2M)^2}=\frac{1}{4M^2}\int_0^{\pi/2}\psi^4(r,\theta)r^2\sin\theta\, d\theta . \tag{3.19}$$

Note that the dimensionless area defined in Eq. (3.19) is always less than or equal to one. This is because

$$\frac{\mathcal{A}}{4\pi(2M)^2}\le\frac{\mathcal{A}^{\rm EH}}{4\pi(2M)^2}\le\frac{\mathcal{A}^{\rm EH}}{4\pi(2M_{\rm final})^2}\le\frac{\mathcal{A}^{\rm EH}_{\rm final}}{4\pi(2M_{\rm final})^2}=1 . \tag{3.20}$$

The first inequality follows because the apparent horizon always lies inside the event horizon (EH); the second because the mass $M_{\rm final}$ of the Schwarzschild black hole inevitably left at the end of the dynamical evolution must not exceed the initial mass; the third because the area of the event horizon can never decrease.

A similar chain of reasoning provides an upper limit to the amount of gravitational radiation that can be emitted during the dynamical evolution of the spheroid:

$$\frac{\Delta E_{\rm rad}}{M}=1-\frac{M_{\rm final}}{M}\le 1-\left[\frac{\mathcal{A}}{4\pi(2M)^2}\right]^{1/2} . \tag{3.21}$$

It is illuminating to compute the equatorial and polar circumferences of the matter surface. The equatorial circumference $\mathcal{C}_{\rm eq}^{\rm surf}$ is given by Eq. (3.17) with $r(\theta)$ replaced by a. The polar circumference $\mathcal{C}_{\rm pole}^{\rm surf}$ is given by Eq. (3.18) with

$$r=(R^2+z^2)^{1/2},\quad \theta=\arctan(R/z) . \tag{3.22}$$

To do the integral, it is convenient to parametrize the surface by η where

$$R=a\cos\eta,\quad z=c\sin\eta . \tag{3.23}$$

The matter circumferences can be arbitrarily large, even for compact configurations well inside their apparent horizons. To assess the hoop conjecture, we must calculate the *minimum* circumferences outside the matter. These minimum circumferences must be geodesics of the two-metric. The minimum equatorial circumference $\mathcal{C}_{\rm eq}^{\rm min}$ must be a circle, so we evaluate Eq. (3.17) at various radii outside the matter and search for the minimum. The geodesic equation for the minimum polar circumference $\mathcal{C}_{\rm pole}^{\rm min}$ is

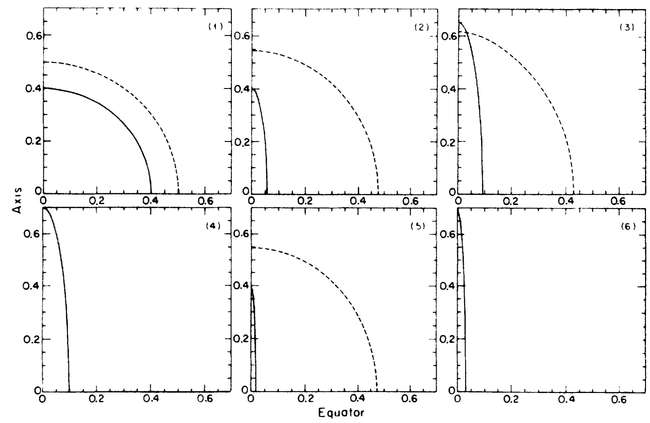

FIG. 1. Representative cases of fully relativistic, momentarily static prolate spheroids. The solid line shows the matter surface. The dashed line shows the apparent horizon if it is present. The coordinates are in units of M. Parameters for the specific cases are given in Table I. Note that whenever any dimension exceeds $\approx 0.5M$, no apparent horizon forms (hoop conjecture).

$$r_{,\theta\theta}-2\frac{r_{,\theta}^2}{r}-r+2\left[1+\frac{r_{,\theta}^2}{r^2}\right]\left[\frac{\psi_{,\theta}}{\psi}r_{,\theta}-\frac{\psi_{,r}}{\psi}r^2\right]=0 . \tag{3.24}$$

We vary the initial value of r and $r_{,\theta}$ on the semimajor axis and compute the circumference using the solution of Eq. (3.24) in Eq. (3.18). In this way we can locate the minimum circumference that is everywhere outside the matter.

As a diagnostic of the presence of singularities, we evaluate the Riemann invariant outside the matter:

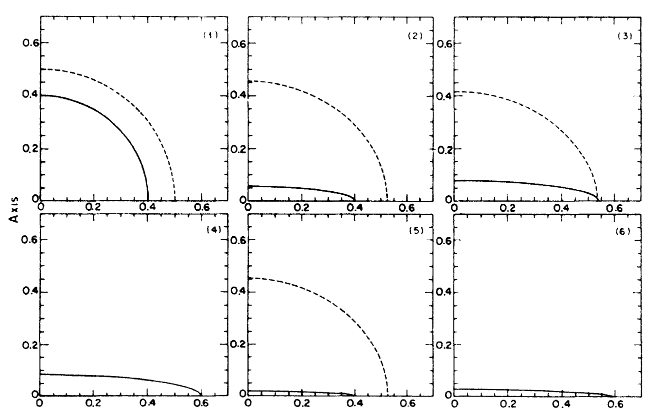

FIG. 2. Representative cases of fully relativistic, momentarily static oblate spheroids. The solid line shows the matter surface. The dashed line shows the apparent horizon if it is present. The coordinates are in units of M. Parameters for the specific cases are given in Table II.

TABLE I. Properties of prolate configurations in Fig. 1.

No.	e	c/M	M/M_0	Apparent horizon?	$\mathcal{C}_{\rm eq}^{\rm AH}/4\pi M$	$\mathcal{C}_{\rm pole}^{\rm Ah}/4\pi M$	$\mathcal{A}/16\pi M^2$	$\mathcal{C}_{\rm eq}^{\rm surf}/4\pi M$	$\mathcal{C}_{\rm pole}^{\rm surf}/4\pi M$	$\mathcal{C}_{\rm eq}^{\rm min}/4\pi M$	$\mathcal{C}_{\rm pole}^{\rm min}/4\pi M$
1	0.01	0.40	0.40	Yes	1.00	0.99	1.00	1.01	1.01	1.00	1.00
2	0.99	0.40	0.20	Yes	0.94	1.03	0.99	0.74	2.49	0.74	1.03
3	0.99	0.65	0.29	Yes	0.86	1.10	0.96	0.57	2.00	0.57	1.07
4	0.99	0.70	0.30	No				0.55	1.95	0.55	1.08
5	0.999	0.40	0.15	Yes	0.94	1.03	0.99	0.46	4.44	0.46	1.03
6	0.999	0.70	0.24	No				0.32	3.21	0.32	1.08

$$I \equiv R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = {}^{(3)}R^{ijkl\,(3)}R_{ijkl} \ . \tag{3.25}$$

This is most easily evaluated in Cartesian coordinates, where

$$I = 96\psi^{-12}(\psi_{,i}\psi^{,i})^2 - 96\psi^{-11}\psi_{,ij}\psi^{,i}\psi^{,j} + \psi^{-10}\psi_{,ij}\psi^{ij} \ . \tag{3.26}$$

We point out that trivial shell-crossing singularities give finite values of I in the matter exterior.

IV. TESTING THE HOOP CONJECTURE

We have done a large survey of the solutions described in the previous section for different values of e and a/M (oblate) or c/M (prolate). A representative sample of the results for the prolate case is tabulated in Table I and displayed in Fig. 1. The corresponding sample for the oblate case is in Table II and Fig. 2.

Consider the implications of the solutions for the hoop conjecture. As the figures show, all prolate spheroids with $c/M \lesssim 0.5$ have apparent horizons, regardless of the eccentricity. The same is true for all oblate spheroids with $a/M \lesssim 0.5$. Recall that the event and apparent horizons coincide for a static metric; for a Schwarzschild black hole they are situated at $r/M = 0.5$ in these coordinates. For these spheroids, when there is an apparent horizon, $r(\theta)/M \approx 0.5$, independent of θ. An invariant expression of this result is the statement that $\mathcal{C}_{\rm eq}^{\rm AH}/2\pi(2M) \approx 1$ and $\mathcal{C}_{\rm pole}^{\rm AH}/2\pi(2M) \approx 1$, independent of e. The precise values of these quantities for the illustrated cases are given in Tables I and II. Also tabulated are the areas, Eq. (3.19).

Table I shows that there exists a configuration (case 4) with a minimum circumference (in units of $4\pi M$) of 1.08 that has no apparent horizon, while Table II shows that there is another configuration (case 3) with a minimum circumference of 1.09 that has an apparent horizon. From this we conclude that it is impossible to prove the hoop conjecture with a universal numerical coefficient, at least for apparent horizons. Furthermore, Bonnor[12] has already shown that charged spheres can exist without collapsing to black holes even if their greatest circumference is somewhat less than 1.

Note that even when there is no apparent horizon, there *may* still be an event horizon. Only a fully dynamical evolution of the given initial data locate the presence or absence of an event horizon. Of course, the existence of an apparent horizon implies the existence of an event horizon.

As we see from the tables, even when there is an apparent horizon the circumference of the matter surface can be arbitrarily large. For example, a sphere with $a/M \to 0$ has $\mathcal{C}_{\rm eq}^{\rm surf} \to \infty$. Clearly, the hoop conjecture is concerned with the minimum circumference in every direction being $\lesssim 4\pi M$. All of the configurations we have constructed satisfy this limit to within 10%.

What is worrisome about this confirmation of the hoop conjecture are the extended limiting configurations with $e \to 1$ and no apparent horizons, such as cases 4 and 6 in the figures and tables. In fact, we have constructed similar cases with e far closer to unity than those shown in the figures. In this limit the spheroids become singular, as measured, for example, by the invariant I (Fig. 3). In the oblate case, the singularity is an equatorial ring just outside the edge of the matter. This singularity is rather innocuous: it is Newtonian in origin, and it is already clear from the Newtonian theory that there are no problems integrating geodesics through it. In the prolate case, by contrast, the singularity is a spindle located just outside the matter. It too is Newtonian in origin, but as

TABLE II. Properties of oblate configurations in Fig. 2.

No.	e	a/M	M/M_0	Apparent horizon?	$\mathcal{C}_{\rm eq}^{\rm AH}/4\pi M$	$\mathcal{C}_{\rm pole}^{\rm AH}/4\pi M$	$\mathcal{A}/16\pi M^2$	$\mathcal{C}_{\rm eq}^{\rm surf}/4\pi M$	$\mathcal{C}_{\rm pole}^{\rm surf}/4\pi M$	$\mathcal{C}_{\rm eq}^{\rm min}/4\pi M$	$\mathcal{C}_{\rm pole}^{\rm min}/4\pi M$
1	0.01	0.40	0.40	Yes	1.00	0.99	1.00	1.01	1.01	1.00	1.00
2	0.99	0.40	0.32	Yes	1.07	0.96	0.99	1.21	1.05	1.05	0.96
3	0.99	0.54	0.38	Yes	1.17	0.94	0.98	1.17	0.98	1.09	0.95
4	0.99	0.60	0.41	No				1.17	0.97	1.11	0.95
5	0.999	0.40	0.30	Yes	1.07	0.96	0.99	1.22	1.10	1.06	0.96
6	0.999	0.60	0.40	No				1.18	1.01	1.11	0.94

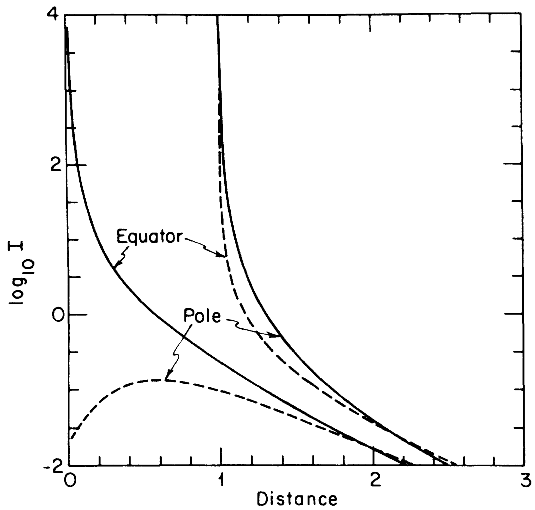

FIG. 3. The logarithm to the base 10 of the Riemann invariant I is plotted as a function of distance in units of M outside a relativistic spheroid of eccentricity $e = 0.9999$. The solid lines are for a prolate spheroid with $c/M = 1$, the dashed lines are for an oblate spheroid with $a/M = 1$. In the limit $e \rightarrow 1$, the oblate spheroid has a "mild" equatorial ring singularity, while the prolate spheroid has a "stronger" spindle singularity.

in the Newtonian case, it if far more serious than the oblate ring [cf. Eqs. (2.8) and (2.9) versus (2.16) and (2.17)]. World lines of particles approaching the spindle are likely to terminate, by analogy with the Newtonian result.

Are the extended configurations with $e \rightarrow 1$ naked singularities? We certainly do not expect horizons to form if the hoop conjecture is true: for the oblate configuration, $\mathcal{C}^{\min}_{\mathrm{eq}}/2\pi(2M) = 1.25$, while for the prolate configuration $\mathcal{C}^{\min}_{\mathrm{pole}}/2\pi(2M) = 1.15$. And what about configurations such as case 3? Here there is an apparent horizon, but it does not enclose all of the matter. Does the event horizon itself contain all the matter, or does some matter stick out of the horizon and become a "seminude" singularity in the limit $e \rightarrow 1$?

V. COLLAPSE SCENARIO

We can employ our momentarily static models to illustrate the plausible dynamical evolution of a collapsing prolate spheroid. Consider a sequence of momentarily static configurations with fixed rest mass M_0 and polar extent c/M_0. This sequence is parametrized by decreasing values of equatorial radius a and increasing eccentricity e. Such a sequence might crudely model the late stages of collapse of a prolate spheroid to a spindle singularity. In the Newtonian limit, the sequence is an accurate guide to the outcome of Lin-Mestel-Shu collapse, even though it does not contain kinetic energy. In this case, the time scale for collapse in the equatorial direction is much shorter than the time scale for c to change.

This sequence makes plausible that nonsingular, asymptotically flat, prolate initial data can evolve to singularities. Equation (3.12) shows that $M \rightarrow 0$ as $e \rightarrow 1$ for fixed M_0 and c/M_0. This has two major implications. First, it is likely that copious gravitational radiation will be emitted during prolate collapse—in principle, all the mass energy is being radiated away. Second, $c/M \rightarrow \infty$, so according to our earlier discussion on the hoop conjecture, no horizon will form and a naked singularity will appear.

Without a dynamical calculation, the above scenario is speculative. The sequence does not truly model dynamical collapse. In real collapse the density profile evolves, time symmetry does not hold after $t = 0$, gravitational waves are present, and there is appreciable kinetic energy. Moreover, it is not obvious that the quantity c/M_0 is the axial distance measure that remains constant at late times during relativistic collapse. Accordingly, the final state may be nonsingular. It is nevertheless interesting that the few numerical examples of prolate collapse,[8] which even included pressure, seem to suggest the formation of a naked singularity. The resolution of this crucial issue is a high priority for future numerical work.

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