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Exponentiation of soft photons in Monte Carlo event generators: The case of the Bonneau-Martin cross section

Stanislaw Jadach
SLAC, P.O. Box 4349, Stanford, California 94309
*and Institute of Physics, Jagellonian University, Cracow, ul. Reymonta 4, Poland**

B. F. L. Ward
*Department of Physics and Astronomy, The University of Tennessee, Knoxville, Tennessee 37996-1200**
and SLAC, P.O. Box 4349, Stanford, California 94309
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It is shown, explicitly, how to proceed in the Monte Carlo program in order to include multiple-soft-photon emission. The method is based on the rigorous method of summing infrared contributions to the respective cross section by Yennie, Frautschi, and Suura. Procedures are illustrated on the example of the initial-state bremsstrahlung. One photon is allowed to be hard and an arbitrary number of real soft additional photons are confined to the neighborhood of the infrared point.

I. INTRODUCTION

The currently used Monte Carlo (MC) programs for the calculation of QED bremsstrahlung effects in high-energy lepton-lepton and lepton-hadron processes (see, for example, Ref. 1) are based on the single-bremsstrahlung calculations. They include, typically, an emission of a real single hard photon while the infrared point (photon energy equal zero) is excluded from the phase space by means of a traditional cutoff on the photon energy in the center-of-mass system. Events without a photon are also generated and they populate phase space precisely at the infrared point; i.e., they are distributed within a reduced phase space with one particle (three dimensions) less. Their cross section includes contributions from virtual- and real-photon emission, the result being infrared finite. On the other hand, there was in the past a variety of the calculations based on the summation of the contributions from the infinite number of the soft photons, i.e., on the so-called exponentiation procedure. The most extensive and complete discussion of exponentiation was exposed in the paper of Yennie, Frautschi, and Suura[2] (YFS). It provides a rigorous framework for the calculation in which one may improve the precision of the calculation step by step as in the traditional perturbative expansion. In most of the practical applications the common procedure was not to apply the YFS scheme precisely but rather to make an educated guess related to the YFS scheme. Typically that was done by an *ad hoc* modification of an analytical formula for the partly integrated cross section resulting from the single-bremsstrahlung (one-loop) calculation. An example of such a procedure may be found for instance in the paper of Jackson and Scharre,[3] where the calculation of Bonneau and Martin[4] is "exponentiated." This sort of procedure is regarded as a relatively easy method of introducing higher-order effects in the QED calculation. In fact, when the double-bremsstrahlung (two-loop) result is compared[5] with that of the "exponentiated" single bremsstrahlung (one loop), one finds that they are rather close.

The question which we address in this paper is the following: is it possible to find a corresponding procedure of introducing multiple soft photons in the Monte Carlo event generators? Our ambition is also not to rely on *ad hoc* procedures but rather to refer to the original YFS scheme. The answer is generally positive and the first complete recipe of how one answers our question (proposed examples of MC algorithms) was given in Ref. 6. Here we shall work out an example of adding in the Monte Carlo generator multiple soft photons in addition to the one hard photon. All photons are emitted from the initial-state beams in the e^+e^- annihilation. This will be roughly analogous to the "exponentiation" made on the integrated cross section in Ref. 3. It should be stressed however that the procedure used in our MC calculation is based on the rigorous prescriptions of Ref. 2 whereas Ref. 3, and numerous other works related to it, involve various departures from rigor.[5,7] There will be no major obstacle in improving our calculations by inclusion of a second hard photon in the future. In some preliminary form it was done even in this work. It is needless to mention that in addition to the necessity of calculating and/or correcting cross sections due to QED effects there is another reason for including multiple-soft- and -hard-photon emission in the MC generators. They may be seen in the detector and it is essential to include them in the MC sample for apparatus acceptance studies.

The plan of the paper is the following. In the next section we consider the Bonneau-Martin cross section and its naive exponentiation in the spirit of the original work of Jackson and Scharre and of the recent improvements[5,8,9] of Jackson and Scharre's idea. (Accordingly, we feel that Ref. 3 and its improvements in Refs. 5, 8, and 9 are a complete representation of the naive "exponentiation" procedure; they will not misrepresent the pedagogical relationship between the naive procedure and our rigorous methods.) In Sec. III we review the relevant aspects of the YFS program from the standpoint of our

Monte Carlo methods. In Sec. IV we describe the essential ingredients in our Monte Carlo realization of the YFS program for the Bonneau-Martin case. Section V contains the numerical results which we use to illustrate the effects of including the multiple photons in the respective final state. It also contains our concluding remarks.

II. BONNEAU-MARTIN CROSS SECTION AND ITS EXPONENTIATION

The effect of the initial-state bremsstrahlung in the e^+e^- annihilation on the total cross section can be summarized in a simple formula usually referred to as a Bonneau-Martin formula. It includes an integral over the photon-energy spectrum convoluted with the lowest-order cross section at the reduced c.m.-system (c.m.s.) energy (here, the process under study is $e^+e^-\to f\bar{f}+n\gamma$, $f=\mu, \tau, u, d, s, c, b$ or t, $n=0,1$):

$$\sigma_{\rm BM}(s)=\sigma^B(s)[1+\delta_{SX}(s/m_e^2,x_0)] + \frac{2\alpha}{\pi}[\ln(s/m_e^2)-1]\int_{x_0}^{1}dx\frac{1+(1-x)^2}{2x} \times\sigma^B((1-x)s)\ , \tag{1}$$

where s is c.m.s. energy squared, x is photon energy in units of $E_{\rm beam}=\sqrt{s}/2$, and

$$\delta_{SX}(s/m_e^2,x_0)=2\alpha\tilde{B}(s/m_e^2,x_0)+2\,{\rm Re}F_1(s/m_e^2) \tag{2}$$

consists of the virtual-photon (vertex) correction

$$2\,{\rm Re}F_1(s/m_e^2)=\frac{\alpha}{\pi}\left[[\ln(s/m_e^2)-1]\ln(m_\gamma^2/m_e^2) -\tfrac{1}{2}\ln^2(s/m_e^2)+\tfrac{3}{2}\ln(s/m_e^2) -2+\frac{2\pi^2}{3}\right] \tag{3}$$

and of the real-soft-photon contribution [$p_{1\,(2)}$ is the four-momentum of e ($\bar{e}$) and $P\equiv p_1+p_2$]

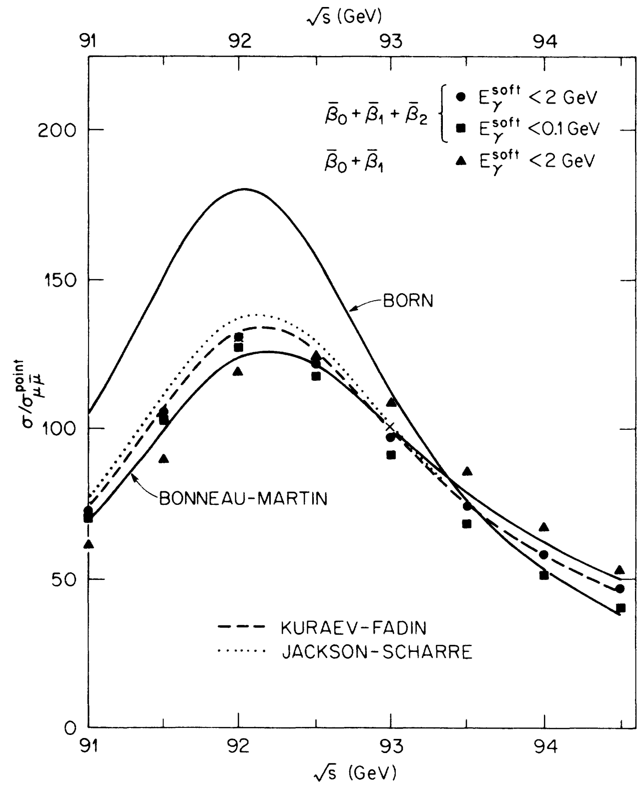

FIG. 1. Two solid curves represent the Born and Bonneau-Martin cross sections. The dotted curve is according to Jackson-Scharre and dashed curve is from the Kuraev-Fadin result. The Kuraev-Fadin result is defined as follows:

$$\sigma_{\rm KF}=\int_0^1 dx\,\sigma^B[s(1-x)][\alpha A x^{\alpha A-1}(1+\delta_R)+\alpha A(-1+x/2)],$$

$$\delta_R=\frac{3}{2}\frac{\alpha}{\pi}[\ln(s/m_e^2)-1]+\frac{\alpha}{\pi}\left[\frac{\pi^2}{3}-2\right],$$

$$\alpha A=\frac{2\alpha}{\pi}[\ln(s/m_e^2)-1]=t\ .$$

Three types of points come from our Monte Carlo data, 10^4 events, statistical error below the size of the dots. Round and square dots represent the Monte Carlo result for $\bar{\beta}_0+\bar{\beta}_1+\bar{\beta}_2$ and triangular points represent the $\bar{\beta}_0+\bar{\beta}_1$ result. The most energetic photon is allowed everywhere in the phase space and the other photons are confined within a sphere $E_\gamma\le\bar{E}_\gamma^{\rm soft}$. Two values for the $\bar{E}_\gamma^{\rm soft}$ cutoff are used: 2 and 0.1 GeV. The crosses show the effect of renormalization-group improvement on the round dots.

$$2\alpha\tilde{B}(s/m_e^2,x_0)=-(\alpha/4\pi^2)\int_{|\mathbf{k}|<x_0\sqrt{s}/2}\frac{d^3k}{(\mathbf{k}^2+m_\gamma^2)^{1/2}}\left[\frac{p_1}{kp_1}-\frac{p_2}{kp_2}\right]^2$$

$$\equiv\int_{|\mathbf{k}|<x_0\sqrt{s}/2}\frac{d^3k\,\tilde{S}(k)}{(\mathbf{k}^2+m_\gamma^2)^{1/2}}=\frac{\alpha}{\pi}\left[[\ln(s/m_e^2)-1]\ln\left[\frac{m_\gamma^2x_0^2}{m_e^2}\right]+\tfrac{1}{2}\ln^2(s/m_e^2)-\frac{\pi^2}{3}\right]\ . \tag{4}$$

Here m_γ is a photon mass introduced temporarily in order to regulate the infrared singularity. It drops out in the sum as is seen from the explicit expression

$$\delta_{SX}(s/m_e^2,x_0)=\frac{\alpha}{\pi}[\tfrac{3}{2}\ln(s/m_e^2)-2+\pi^2/3] + \frac{2\alpha}{\pi}[\ln(s/m_e^2)-1]\ln x_0 . \quad (5)$$

(x_0 then gives the usual separation of hard and soft photons.) The exponentiated formula of Jackson and Scharre (neglecting the contribution from the vacuum polarization) reads

$$\sigma_{\rm JS}(s)=\delta_{SX}(x_0=1)\sigma^B(s) + t\int_0^1 dx[x^{t-1}-(1-x/2)]\sigma^B((1-x)s) , \quad (6)$$

where

$$t=\frac{2\alpha}{\pi}[\ln(s/m_e^2)-1] \quad (7)$$

and it is obtained by means of the replacement

$$1+t\left[\frac{1}{x}\right]_+ \rightarrow tx^{t-1} . \quad (8)$$

Note that both distributions when integrated in the range from 0 to 1 give precisely one.

As an introductory numerical exercise we plot in Fig. 1 the result from the Bonneau-Martin formula and from the Jackson-Scharre formula for the Z^0 resonance near the top of the cross section (for $\tau\bar{\tau}$ production). It is worthwhile mentioning that the result is not very sensitive to the way the exponentiation is done. For example, one gets practically the same curve from another exponentiation ansatz:

$$\sigma'_{\rm JS}(s)=\delta_{SX}(x_0=1)\sigma^B(s) + t\int_0^1 dx\, x^{t-1}[1-x(1-x/2)]\sigma^B((1-x)s) . \quad (9)$$

As was mentioned this result is not far from the result of the exact second-order calculation.

Recently, an improved version of the Jackson-Scharre idea (8) for summing the effects of soft quanta has been obtained by several authors. A detailed comparison of these results is given by Alexander *et al.*,[8] where it is shown that they are all reasonably consistent with one another at the $\lesssim 1\%$ level. Here, we present in Fig. 1 the Kuraev-Fadin version of this improved implementation of (8) in order to provide a complete view of the naive exponentiation of the Bonneau-Martin cross section. Upon comparing the Kuraev-Fadin formula in Fig. 1 with (6), we see that the Jackson-Scharre soft-photon kernel tx^{t-1} is also renormalized by higher-order contributions in the δ_R in Fig. 1 whereas in (6) the approximation is made that such higher-order corrections only affect the contribution to the right-hand side (RHS) of (6) at $x=0$. (See Ref. 9 for a more detailed discussion of this point.) Since the Jackson-Scharre kernel is peaked at $x\rightarrow 0$, we expect that (6) and the Kuraev-Fadin formula will not diverge too much. And, indeed, this is borne out in Fig. 1; however, at the level of accuracy required for the long-term physics objectives at the SLAC Linear Collider and CERN LEP (see Ref. 7 for detailed references to the respective literature), the difference between (6) and the Kuraev-Fadin-type formula is unacceptable. See also Cahn and Greco in Ref. 8.

What we consider in the following is a rigorous approach to the exponentiation of the Bonneau-Martin cross section at the level of a Monte Carlo event generator. This approach is based on the YFS formalism. We therefore review this formalism in the next section.

III. YENNIE-FRAUTSCHI-SUURA EXPANSION

In the following we review briefly the essential ingredients of the YFS formalism which are necessary for further discussion of our Monte Carlo calculation. Let us start with the YFS expansion truncated on the first two $\bar{\beta}$ terms:

$$\sigma_{\rm YFS}(s)=\exp(2\,{\rm Re}\alpha B)\left[\sum_{n=0}^{\infty}\frac{1}{n!}\int d\tau_{n+2}(P;q_1,q_2,k_1,\ldots,k_n)\prod_{l=1}^{n}\tilde{S}(k_l)\bar{\beta}_0(q_1,q_2) + \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{j=1}^{n}\int d\tau_{n+2}(P;q_1,q_2,k_1,\ldots,k_n)\prod_{\substack{l=1\\ l\neq j}}^{n}\tilde{S}(k_l)\bar{\beta}_1(q_1,q_2,k_j)\right] , \quad (10)$$

where $q_{1\,(2)}$ is the four-momentum of $f\,(\bar{f})$ and where

$$d\tau_n(P;\bar{p}_1,\ldots,\bar{p}_n)=\prod_{l=1}^{n}\frac{d^3\bar{p}_l}{\bar{p}_l^{\,0}}\delta^4\left[P-\sum_{i=1}^{n}\bar{p}_i\right] . \quad (11)$$

In the above formula soft-virtual-photon contributions are sitting in the $\exp(2\,{\rm Re}\alpha B)$ factor where

$$2\,{\rm Re}\alpha B={\rm Re}\left[\frac{i\alpha}{4\pi^2}\int\frac{d^4k}{k^2-m_\gamma^2+i\epsilon}\left[\frac{2p_1+k}{k^2+2kp_1}-\frac{2p_2+k}{k^2+2kp_2}\right]^2\right] = \frac{\alpha}{\pi}\{[\ln(s/m_e^2)-1]\ln(m_\gamma^2/m_e^2)-\tfrac{1}{2}\ln^2(s/m_e^2)+\tfrac{1}{2}\ln(s/m_e^2)-1+\pi^2/6\} \quad (12)$$

and the real-photon emission cross section is rearranged in such a way that the formula as a whole is infrared finite. (The distributions $\bar{\beta}_i$ are finite and will be discussed below.) In order to see this more clearly let us introduce temporarily a photon mass regulator and take advantage of the explicit factorizability of the infrared part of the formula (10):

$$\sigma_{\rm YFS}(s)=\exp(2\,{\rm Re}\alpha B+2\alpha\tilde{B})\times\left[\int d\tau_2'(P;q_1,q_2)\bar{\beta}_0(q_1,q_2)+\int d\tau_3'(P;q_1,q_2,k)\bar{\beta}_1(q_1,q_2,k)\right], \tag{13}$$

where we define

$$d\tau_n'(P;\bar{p}_1,\ldots,\bar{p}_n)=\prod_{i=1}^{n}\frac{d^3\bar{p}_i}{\bar{p}_i^0}\int\frac{d^4y}{(2\pi)^4}\exp\left[iy\left[P-\sum_{i=1}^{n}\bar{p}_i\right]+D\right] \tag{14}$$

for

$$D\equiv\int\frac{d^3k}{k}[e^{-iyk}-\theta(K_{\rm max}-k)]\tilde{S}(k) \tag{15}$$

so that $\tilde{B}$ depends on $K_{\rm max}$. The sum in the exponent is finite (m_γ cancels out) and, assuming $K_{\rm max}\simeq\sqrt{s}/2$, i.e., $x_0=1$, we get

$$2\alpha(\tilde{B}+{\rm Re}B)=\frac{\alpha}{\pi}[\tfrac{1}{2}\ln(s/m_e^2)-1+\pi^2/3]\ . \tag{16}$$

For the purpose of the Monte Carlo calculation we repeat this exercise once more but this time we split the integral over $x=2|\mathbf{k}|/\sqrt{s}$ from 0 to 1 into two parts: the first from 0 to δ and the second from δ to 1. The first contribution we include in the exponent

$$\exp\{2\alpha[{\rm Re}B+\tilde{B}(\delta)]\}=\exp\left[\frac{\alpha}{\pi}\{[\ln(s/m_e^2)-1]\ln\delta+\tfrac{1}{2}\ln(s/m_e^2)-1+\pi^2/3\}\right] \tag{17}$$

and the second we leave where it was, i.e., in the phase-space integral. In this way we split the phase space into two regions: below $x=\delta$ where virtual and real soft photons are combined analytically to yield an infrared-finite result and above where we have only real photons which will be generated in the Monte Carlo simulation. The energy limit $k_\delta=\delta\sqrt{s}/2$ which separates virtual and soft photons in the phase space may be set arbitrarily low. The resulting differential cross section reads [here, D in (14) is now $\int^{k\le k_\delta}(d^3k/k)\tilde{S}(k)(e^{-iyk}-1)\to 0$ for $k_\delta\to 0$]

$$\sigma_{\rm YFS}(s)=e^{2\alpha[{\rm Re}B+\tilde{B}(\delta)]}\left[\sum_{n=0}^{\infty}\frac{1}{n!}\int_{k_l^0>k_\delta}d\tau_{n+2}'(P;q_1,q_2,k_1,\ldots,k_n)\left[\prod_{l=1}^{n}\tilde{S}(k_l)\right]\bar{\beta}_0(q_1,q_2)+\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{j=1}^{n}\int_{k_l^0>k_\delta}d\tau_{n+2}'(P;q_1,q_2,k_1,\ldots,k_n)\left[\prod_{\substack{l=1\\l\neq j}}^{n}\tilde{S}(k_l)\right]\bar{\beta}_1(q_1,q_2,k_j)\right]. \tag{18}$$

The reader may worry that the above expression looks as if the four-momentum was not conserved. For example, the phase-space element $d\tau_3$ includes $\delta^4(P-q_1-q_2-k_j)$ and it seems that only one photon was included in the four-momentum conservation which would determine $\bar{\beta}_1$. Let us now clarify this point. In fact, for the sake of simplicity, some simplification in the notation was tacitly done. Our master formula should be better written as

$$\sigma_{\rm YFS}(s)=e^{2\alpha[{\rm Re}B+\tilde{B}(\delta)]}\left[\sum_{n=0}^{\infty}\frac{1}{n!}\int_{k_l^0>k_\delta}d\tau_{n+2}'(P;q_1,q_2,k_1,\ldots,k_n)\left[\prod_{l=1}^{n}\tilde{S}(k_l)\right]\bar{\beta}_0(\mathcal{R}q_1,\mathcal{R}q_2)+\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{j=1}^{n}\int_{k_l^0>k_\delta}d\tau_{n+2}'(P;q_1,q_2,k_1,\ldots,k_n)\left[\prod_{\substack{l=1\\l\neq j}}^{n}\tilde{S}(k_l)\right]\bar{\beta}_1(\mathcal{R}q_1,\mathcal{R}q_2,k_j)\right] \tag{19}$$

In the framework of the YFS scheme one performs certain manipulations on the differential cross sections in which infrared singularities for real soft photons are extracted in singular $\tilde{S}(k_l)$ factors and $\bar{\beta}_i$ functions are the finite residua in this procedure at the singular point, i.e., at the point reached by putting the momenta of some photons to zero. The related fact is that, strictly speaking, $\bar{\beta}_0$ is defined within two-body phase space and $\bar{\beta}_1$ is defined inside three-body phase space. The operation $\mathcal{R}$ is defined such that in $\bar{\beta}_0$ q_i obeying $q_1+q_2+\sum_i k_i=P$ are transformed into "reduced momenta" $\mathcal{R}q_i$ which obey $\mathcal{R}q_1+\mathcal{R}q_2=P$. Similarly in $\bar{\beta}_1$ reduced momenta obey $\mathcal{R}q_1+\mathcal{R}q_2+k_j=P$ instead of $q_1+q_2+\sum_i k_i=P$. This corresponds exactly to going to the residue position. It amounts in practice to some manipulations on momenta in which momenta of some photons are excluded from the four-momentum balance. There is a certain degree of freedom on how it is actually done but there are also some restrictions. The previous formula and the actual master formula are numerically equivalent in the sense that in the previous one the momenta q_i should be really treated as new integration variables $q_i'=\mathcal{R}q_i$ used instead of the original ones. In practice, one may take advantage of the Lorentz invariance of the phase-space element $d\tau_n$ under boosts and rotations and use these transformations as the building blocks in the reduction operation $\mathcal{R}$. In general, one has to do at least one rescaling of the momenta and it turns out, not surprisingly, that the best is to do that in the rest frame of q_1+q_2.

Finally let us write our master formula once again in a form which will be useful for further discussion of the Monte Carlo algorithm:

$$\sigma_{\rm YFS}(s)=\sum_{n=0}^{\infty}\frac{1}{n!}\int_{k_l^0>k_\delta} d\tau'_{n+2}(P;q_1,q_2,k_1,\ldots,k_n)\left[\prod_{l=1}^{n}\tilde{S}(k_l)\right]b_n(q_1,q_2,k_1,\ldots,k_n)\,, \tag{20}$$

where

$$b_n(q_1,q_2,k_1,\ldots,k_n)=e^{2\alpha[{\rm Re}B+\tilde{B}(\delta)]}\left[\bar{\beta}_0(\mathcal{R}q_1,\mathcal{R}q_2)+\sum_{j=1}^{n}\bar{\beta}_1(\mathcal{R}q_1,\mathcal{R}q_2,k_j)/\tilde{S}(k_j)\right]. \tag{21}$$

IV. THE MONTE CARLO CALCULATION

For the purpose of the further discussion we need not only the Bonneau-Martin distribution which is integrated over angles but also the differential cross section. It may also be cast[10] into a semifactorizable form which involves the differential lowest-order distributions $d\sigma^B/d\Omega(s',\cos\theta)$ at the reduced c.m. energy squared $s'=(1-x)s$ multiplied by certain functions dependent on the photon momentum:

$$\sigma_{\rm BM}(s)=\sigma^B(s)[1+\delta_{SX}(s/m_e^2,x_0)]$$
$$+\int_{x_0}^{1}\frac{dx}{x}\int d\Omega_\gamma\left[\int d\Omega_1 g_1(x,\cos\theta_\gamma)\frac{d\sigma^B}{d\Omega_1}(s',\cos\theta_1)+\int d\Omega_2 g_2(x,\cos\theta_\gamma)\frac{d\sigma^B}{d\Omega_2}(s',\cos\theta_2)\right], \tag{22}$$

where $i=1,2$ in the parametrization of the final-state fermion direction $d\Omega_i$ correspond to two well-defined choices of the z axis in the rest frame of the final-state fermions (r.c.m.s.). In the first case ($i=1$) it points in the direction of the first beam (in r.c.m.s.) and in the second case ($i=2$) it points opposite to the second beam (also in r.c.m.s). Coefficient functions are given by the expressions

$$g_i(x,\cos\theta_\gamma)=\frac{\alpha}{2\pi^2}(1-\tfrac{1}{2}x\delta_i)^2 \times\left[\frac{1}{\delta_1\delta_2}-\frac{2m_e^2}{s}\frac{1-x}{1+(1-x)^2}\left[\frac{1}{\delta_1^2}+\frac{1}{\delta_2^2}\right]\right],$$
$$\delta_1=1-\cos\theta_\gamma(1-4m_e^2/s)^{1/2}\,, \tag{23}$$
$$\delta_2=1+\cos\theta_\gamma(1-4m_e^2/s)^{1/2}\,.$$

The Bonneau-Martin formula is easily recovered using the identity

$$\int d\Omega_\gamma[g_1(x,\cos\theta_\gamma)+g_2(x,\cos\theta_\gamma)]/x \simeq\frac{1+(1-x)^2}{2x}\frac{2\alpha}{\pi}[\ln(s/m_e^2)-1]\,. \tag{24}$$

Having in hand the above distributions we may proceed to constructing the functions $\bar{\beta}_0$ and $\bar{\beta}_1$ which are necessary to complete our master formula for the Monte Carlo calculation. The relevant definitions may be found in Ref. 7:

$$\bar{\beta}_0(q_1,q_2)=\frac{d\sigma^B}{d\tau_2(P;q_1,q_2)}(1+2\,{\rm Re}F_1-2\,{\rm Re}\alpha B) =\frac{2}{\beta'}\frac{d\sigma^B}{d\Omega}(s,\cos\theta)\left[1+\frac{\alpha}{\pi}[\ln(s/m_e^2)-1]\right], \tag{25}$$

where $\beta'=(1-4m_f^2/s)^{1/2}$ and

$$\bar{\beta}_1(q_1,q_2,k)/\tilde{S}(k) =\frac{2}{\beta'}\left[\frac{g_1(x,\cos\theta_\gamma)}{g_0(x,\cos\theta_\gamma)}\frac{d\sigma^B}{d\Omega}(s',\cos\theta_1)+\frac{g_2(x,\cos\theta_\gamma)}{g_0(x,\cos\theta_\gamma)}\frac{d\sigma^B}{d\Omega}(s',\cos\theta_2)-\frac{1}{2}\left[\frac{d\sigma^B}{d\Omega}(s,\cos\theta_1)+\frac{d\sigma^B}{d\Omega}(s,\cos\theta_2)\right]\right], \tag{26}$$

where

$$g_0(x,\cos\theta_\gamma)=\frac{\alpha}{2\pi^2}\left[\frac{1}{\delta_1\delta_2}-\frac{m_e^2}{s}\left[\frac{1}{\delta_1^2}+\frac{1}{\delta_2^2}\right]\right] \tag{27}$$

is up to a normalization constant equal to $\tilde{S}(k)$. In $\bar{\beta}_1$ above, the last two terms represent $\bar{\beta}_0(\mathcal{R}q_1,\mathcal{R}q_2)$. The $\mathcal{R}$ procedure in that case amounts to taking $\cos\theta_i$ in the rest frame of q_1+q_2 system and the average over i is taken in order to have a symmetric solution. It should be stressed also that in these two terms s is taken in contrast with the first two where s' is used instead. In the presence of the additional photons one has to provide for the $\mathcal{R}$ procedure to produce $\mathcal{R}q_1$, $\mathcal{R}q_2$, and k-momenta to be plugged in as arguments in the above $\bar{\beta}_1$ expression. In the actual Monte Carlo simulation it is done in the following way. One transforms q_i to the $\mathbf{q}_1+\mathbf{q}_2=0$ frame, then $\mathbf{q}_1$ and $\mathbf{q}_2$ are rescaled by the factor which corresponds to exclusion of additional photons and the momenta are boosted back to the c.m.s. system, taking again a boost parameter which takes the exclusion of additional photons into account. The resulting momenta obey $\mathcal{R}q_1+\mathcal{R}q_2+k=P$ where k is the momentum of the only one photon which actually was not touched in the $\mathcal{R}$ reduction procedure. (Using entirely analogous procedures we have also constructed the leading-logarithmic approximation to $\bar{\beta}_2$.)

The Monte Carlo algorithm is organized in such a way that there are two distinct levels in it. There is a low-level MC generator which generates events according to our master formula with a simplified b_n function. It is simply

$$b_n'(q_1,q_2,k_1,\ldots,k_n)=\frac{1}{2\pi\beta'}\sigma^B((q_1+q_2)^2)\ . \tag{28}$$

The events are generated on the four-momentum level using this simplified differential cross section and next the real distribution is recovered with help of the rejection procedure with the rejection weight

$$w=\frac{b_n'(q_1,q_2,k_1,\ldots,k_n)}{b_n(q_1,q_2,k_1,\ldots,k_n)}\ . \tag{29}$$

The advantage of this arrangement is that the low-level MC part will remain the same even if more $\bar{\beta}$'s are included in the future in the YFS expansion in b_n. One will need to change only the model-dependent part of the program. In a sense the low-level MC part is a sort of universal phase-space MC program for QED initial-state bremsstrahlung.

The question is now, however, how events are generated in the low-level MC according to our simplified distribution. The solution is quite similar to that proposed in Ref. 6. The integration may be written as

$$\sigma'_{\rm YFS}=\int_0^1 dv\,\sigma^B((1-v)s)\sum_{n=0}^{\infty}\frac{1}{n!}\int_{k_l^0>k_\delta}\left[\prod_{l=1}^{n}\frac{d^3k_l}{k_l^0}\tilde{S}(k_l)\right]\delta\left[v-\frac{2KP-K^2}{P^2}\right]\ , \tag{30}$$

where $K=\sum k_i$. Photon momenta are generated quite similarly as in the algorithm 2 in Ref. 6. (This algorithm is described in detail in Ref. 7 also.) In this algorithm the energy conservation is obtained by rescaling momenta of all photons by a certain factor. Here the method is the same but the condition to be satisfied, that inside the δ function in the above expression, is slightly more complicated. Because of that one picks up a Jacobian factor in the integrand which has to be removed again by the rejection method. The details on that will be given elsewhere.

V. NUMERICAL RESULTS AND CONCLUSIONS

In Fig. 1 we plot the total cross section in R units for τ pair production at the vicinity of the Z^0 resonance. Included are the Born cross section, the cross section from the Bonneau-Martin formula, and the exponentiated results obtained using the Jackson-Scharre formula (as we have noted) and the formula of Kuraev and Fadin.[9] The results from the Monte Carlo calculations of the type described in this paper are represented by dots. They are obtained from samples of 10^4 events. The statistical error is of the size of the dot. At each energy the two cross sections correspond to two possible upper limits on the energy of the soft photon $E_\gamma^{\rm soft}$. There is no limitation on the energy of the most energetic photon but all other ones must stay below $\bar{E}_\gamma^{\rm soft}$. One result (higher cross section) is obtained using $\bar{E}_\gamma^{\rm soft}=2$ GeV and the other one using $\bar{E}_\gamma^{\rm soft}=0.1$ GeV. Generally, the result of the MC comes close to the result of the naive exponentiation and it depends rather weakly on $\bar{E}_\gamma^{\rm soft}$. It is very essential, however, that this dependence is included in the calculations.

The $\bar{\beta}_0$ and $\bar{\beta}_1$ in (25) and (26) do not include the effects of renormalization-group improvement. In the case at hand such improvement may be effected as follows. In $\bar{\beta}_0$, the prescription in Ref. 7 requires, here, the substitution [$\alpha(2m_{\mu,\rm phys})$ is α at the scale $2m_{\mu,\rm phys}$]

$$\alpha\to\alpha(\lambda)=\alpha(2m_{\mu,\rm phys})/[1-8\pi\alpha(2m_{\mu,\rm phys})b_0\ln\lambda] \tag{31}$$

with the understanding that, in the Z^0 squared coupling G^2, we write

$$\frac{G^2}{4\pi}=\frac{g_W^2(M_{Z^0})}{4\pi}+(M_{Z^0}^2/M_W^2)\alpha(\lambda)\ , \tag{32}$$

where, here, $\lambda=M_{Z^0}/2m_{\mu,\rm phys}\simeq 4.353\,621\times10^2$ and g_W is the SU(2)$_L$ coupling evaluated at the scale M_{Z^0} so that, from Ref. 11, we may take it to be 0.656 26. Here, $M_W=80.8$ GeV and $M_{Z^0}\simeq 92$ GeV. Note also that $b_0=11/48\pi^2$ here.

Similarly, in $\bar{\beta}_1$, the prescription in Ref. 7 requires that we leave $\alpha\simeq1/137.036\,04$ in $g_{1,2}$ but that we make the substitutions in (31) and (32) in $d\sigma^B$. This then accounts for the renormalization-group improvement of the results in (22)–(27). The improvement of $\bar{\beta}_2$ is effected in an analogous manner.

The respective renormalization-group-improvement effects on the cross section represented by the round dots in Fig. 1 are shown by the crosses in that figure. We see that these effects are significant if one wants high-precision Monte Carlo simulations. Such high precision is relevant to the SLC-LEP Physics objectives.

Note added. It has recently been verified [Wim de Boer (private communication)] that the total integrated cross sections associated with the Monte Carlo procedure described in this paper are in fact consistent, to three or more significant figures, with the total integrated cross sections in the second-order results of Berends *et al.*[5] near $\sqrt{s}=M_{Z^0}$. This is an important check on the global aspects of our Monte Carlo methods. More checks of this type will be taken up elsewhere.

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*Permanent address.

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