Exact ground state, mass gap, and string tension in lattice gauge theory

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We make use of the arbitrariness in defining lattice gauge theory to propose a new form of lattice Hamiltonian with exact ground state. Four such Hamiltonians are obtained. Since the ground state is exactly known, a variational method is applied to obtain rigorous upper bounds of the mass gap in $(2+1)$-dimensional U(1), SU(2), and SU(3) lattice gauge theories. Trial wave functions for excited states contain loop variables up to 30×30 Wilson loops. Nice scaling behaviors are obtained. The scaling behaviors of the mass gap and string tension in non-Abelian theory are in agreement with that predicted by weak-coupling perturbation theory and the Monte Carlo method, but is extended to a much weaker coupling region $1/g^2\sim 7$. For non-Abelian theory, universality is confirmed.

I. INTRODUCTION

One of the most interesting problems in particle physics is to clarify the low-energy behavior of gauge theory. There are many approaches to investigate it. The Monte Carlo method is one approach. It has been successful in giving us numerical values of various physical qualities. However, up to now, it cannot give us the wave functions of the vacuum and excited states which are important in the study of the breakdown of chiral symmetry, confinement, and so on. Therefore, the need for analytic approximations in the investigation of lattice gauge theories (LGT's) is generally accepted. The variational method presents itself as one of the promising nonperturbative approximation schemes. In particular, in the case in which the ground state is exactly known, we can obtain rigorous upper bounds of elementary excitation energies by this method.

In this paper, we adopt the Hamiltonian formalism of LGT and make use of canonical transformations and the arbitrariness in defining lattice gauge theories to obtain four Hamiltonians for which the ground states are exactly known. Since the ground states are exactly known, we can obtain unambiguously the upper bound of elementary excitation energies by the variational method. The mass gaps of U(1), SU(2), and SU(3) lattice gauge theories in $2+1$ dimensions are studied in this paper. Trial wave functions for excited states contain loop variables up to 30×30 Wilson loops, whose coefficients we take as variational parameters. It is found that the coefficients of large Wilson loops increase with $1/g^2$. We also calculate the string tensions using the exact vacuum wave functions. All results mentioned above show good scaling behavior extending to a very deep weak-coupling region $(1/g^2\sim 7)$. This region is far beyond those investigated in present Monte Carlo and analytic calculations. The universality in LGT is confirmed in this paper.

The main motivation for studying the $(2+1)$-dimensional pure Yang-Mills theory is its relevance for QCD at finite temperature (hot QCD) as the high-temperature limit of it. Therefore, there are many papers[1-18] extensively discussing $(2+1)$-dimensional LGT. It was shown that the theory was confined and asymptotically free. It should be mentioned, however, that Yang-Mills theory in $2+1$ dimensions has no true instantons, but possesses Wu-Yang-type monopoles. Furthermore, in $2+1$ dimensions, the square of the bare coupling constant has dimension of mass, so that the theory is super-renormalizable.

In $2+1$ Hamiltonian LGT, the expectation value of any operator in the vacuum can be calculated exactly. This exact integrability is the reason why we choose the Hamiltonian formalism in $2+1$ dimensions.

The theoretical predictions[3] of asymptotic behavior for the mass gap m and string tension σ in $(2+1)$-dimensional QED are

$$m^2a^2=8\pi\beta\exp[-2\pi^2 v(0)\beta] , \tag{1.1a}$$

$$\sigma a^2=(4\sqrt{2}g/\pi)\exp[-\pi^2 v(0)\beta] , \tag{1.1b}$$

where a is the lattice spacing, the bare coupling g is related to charge e by $g^2=e^2a$, $v(0)=0.2527$, and $\beta=1/g^2$.

The scaling behavior for mass gap m and string tension σ of $(2+1)$-dimensional SU(2) LGT obtained by weak-coupling perturbation[12] is

$$ma\propto g^2 , \tag{1.2a}$$

$$\sigma a^2\propto g^4 . \tag{1.2b}$$

Monte Carlo simulations confirm the above behavior in the intermediate-coupling region.

The plan of this paper is as follows. In Sec. II we proposed a new form of Hamiltonian with exact ground state, and obtain four such Hamiltonians. The general procedure to obtain the mass gap of $(d+1)$-dimensional SU(N) LGT ($d=2,3$) by the variational method and to obtain the string tension is investigated in Sec. III. The mass gap and string tension of $(2+1)$-dimensional U(1), SU(2), and SU(3) is discussed in Secs. IV–VI, respectively. Section VII is devoted to a summary and further discussions.

II. THE HAMILTONIAN WITH EXACT GROUND STATE

The Hamiltonian of LGT is not uniquely determined. The lattice Hamiltonian should satisfy gauge invariance and possesses a correct continuum limit, otherwise it is arbitrary. By using this arbitrariness, we can obtain four suitably modified Hamiltonians of LGT which have exact ground states and also have the same continuum limit as the Wilson form of lattice Hamiltonians.

Let the Hamiltonian of $(d+1)$-dimensional SU(N) LGT be ($d=2,3$)

$$H=\frac{g^2}{2a}\sum_l E_l^2-\frac{1}{ag^2}\sum_p \mathrm{tr}(U_p+U_p^\dagger)+\Delta H \ , \qquad (2.1)$$

where ΔH is some function of plaquette variable U_p. If ΔH is Hermitian and vanishes in the continuum limit $a\to 0$, this H has the same continuum limit as the standard Wilson Hamiltonian and can be taken as a new Hamiltonian of LGT.

For the convenience of finding the exact ground state of H, we rewrite H in the new form

$$\begin{aligned}H&=\frac{g^2}{2a}\exp(-R)E_l^a\exp(2R)E_l^a\exp(-R)\\&=\frac{g^2}{2a}[\exp(R)E_l^a\exp(-R)]^\dagger[\exp(R)E_l^a\exp(-R)] \ ,\end{aligned} \qquad (2.2)$$

where R is some function of plaquette variables and satisfies the following conditions: (1) $[R,[R,E_l]]=0$; (2) $R=R^\dagger$; (3) H possesses correct classical limit.

Obviously, H in the form (2.2) is Hermitian, positive, and has an exact ground state $|\Psi_0\rangle$:

$$|\Psi_0\rangle=\exp(R)|0\rangle \ , \qquad (2.3)$$

where $|0\rangle$ is the state defined by $E_l^a|0\rangle=0$. The energy of the ground state is zero, that is, $H|\Psi_0\rangle=0$.

It should be mentioned that in the lattice Hamiltonian corresponding to the continuum theory, the magnetic term should be

$$\frac{1}{ag^2}\sum_p \mathrm{tr}(2-U_p-U_p^\dagger) \ .$$

We have omitted the constant term $(1/ag^2)\sum_p \mathrm{tr}2$ in (2.1) to yield the ground-state energy exactly zero. Had we included this term in the Hamiltonian, the ground-state energy would be ultraviolet divergent in the continuum limit, in agreement with the continuum theory.

Using the relation

$$\exp(\hat{L})\hat{A}\exp(-\hat{L})=\hat{A}+[\hat{L},\hat{A}]+\frac{1}{2!}[\hat{L},[\hat{L},\hat{A}]]+\cdots \ ,$$

we obtain

$$H=\frac{g^2}{2a}\{E_l^2-[E_l^a,[E_l^a,R]]-[E_l^a,R][E_l^a,R]\} \ . \qquad (2.4)$$

In this form of H, its continuum limit is easy to be decided. Comparing Eqs. (2.1) and (2.4), the sufficient conditions for H possessing the correct classical continuum limit are

$$\frac{g^2}{2a}[E_l^a,[E_l^a,R]]-\frac{1}{ag^2}\sum_p \mathrm{tr}(U_p+U_p^\dagger)\to 0 \quad \text{as } a\to 0 \qquad (2.5)$$

and

$$\Delta H=-\frac{g^2}{2a}[E_l^a,R][E_l^a,R]\to 0 \quad \text{as } a\to 0 \ . \qquad (2.6)$$

Four kinds of functions R are discussed in the following. First, a simpler form of R can be taken as

$$R_1=\sum_p \alpha_1\mathrm{tr}(U_p+U_p^\dagger) \ . \qquad (2.7)$$

Making use of the commutation relations

$$[E_l^a,U_l]=\Lambda^a U_l, \quad [E_l^a,U_l^\dagger]=-U_l^\dagger\Lambda^a \ , \qquad (2.8)$$

where Λ^a is a representation matrix of the generator T^a of the gauge group, we obtain

$$-\frac{g^2}{2a}[E_l^a,[E_l^a,R_1]]=-\frac{2g^2}{a}C_N\alpha_1\sum_p \mathrm{tr}(U_p+U_p^\dagger) \ ,$$

where C_N is the Casimir invariant of the gauge group in the fundamental representation.

Using the condition in Eq. (2.5), we obtain

$$\alpha_1=1/(2g^4C_N) \ . \qquad (2.9)$$

We now show that, for $(3+1)$-dimensional LGT,

$$\Delta H_1=-\frac{g^2}{2a}[E_l^a,R_1][E_l^a,R_1]\to 0 \quad \text{as } a\to 0 \ .$$

For R_1 in the form (2.7), ΔH_1 contains a double sum over all plaquettes p and p' sharing the same link l:

$$\sum_{p,p'\supset l}[E_l^a,\mathrm{tr}(U_p+U_p^\dagger)][E_l^a,\mathrm{tr}(U_{p'}+U_{p'}^\dagger)]=\sum_{p,p'\supset l}\mathrm{tr}[\Lambda^a(U_p-U_p^\dagger)]\mathrm{tr}[\Lambda^a(U_{p'}-U_{p'}^\dagger)] \ .$$

In the above expression, the positive orientation of p is taken to be that induced by the positive direction of l. For a plaquette p on the lj plane, the continuum limit of U_p is

$$U_p=1+iga^2F_{lj}+O(a^4) \ . \qquad (2.10)$$

Let the two plaquettes on opposite sides of l be $p(n)$ and $p(n-e_j)$, respectively, then

$$\sum_{p\supset l} \mathrm{tr}[\Lambda^a(U_p-U_p^\dagger)]$$

$$=\sum_j iga^2[F_{lj}^a(n)-F_{lj}^a(n-e_j)]+O(a^4)$$

$$=\sum_j iga^3\partial_j F_{lj}+O(a^4)\ . \quad (2.11)$$

Substituting (2.7), (2.9), and (2.11) into ΔH_1, we obtain

$$\Delta H_1=[1/(8g^6C_N^2a)]\sum_n g^2a^6\partial_k F_{lk}\partial_j F_{lj}$$

$$\to[1/(8g^4C_N^2)]\int d^3x\ a^2\partial_j F_{lj}\partial_k F_{lk}=O(a^2)$$

and is irrelevant in the continuum limit. Actually, the form of ΔH_1 is similar to the regularization term in the method of regularization by higher-order derivatives. Therefore, we can certainly take H_1 as a substitute of the standard Hamiltonian in (3+1)-dimensional LGT.

Unfortunately, things are more complicated in (2+1)-dimensional LGT. In this case, $g^2=ae^2$, where e is the invariant gauge coupling.

The continuum limit of U_p is

$$U_p=1+iea^2F_{lj}+O(a^4) \quad (2.12)$$

and ΔH_1 becomes

$$\Delta H_1=[1/(8g^6C_N^2a)]\sum_n e^2a^6\partial_j F_{lj}\partial_k F_{lk}$$

$$\to[1/(8e^4C_N^2)]\int d^2x\ \partial_j F_{lj}\partial_k F_{lk}=O(a^0)\ .$$

Therefore, ΔH_1 is not negligible and H_1 does not possess the same continuum limit as the standard Wilson Hamiltonian in (2+1)-dimensional LGT.

In order to find some Hamiltonians with exact ground state in (2+1)-dimensional non-Abelian SU(N) LGT, some more complicated functions R must be chosen: three of them are

$$R_2=\sum_p \{\alpha_2\mathrm{tr}(U_p+U_p^\dagger)+\beta_2[\mathrm{tr}(U_p+U_p^\dagger)]^2\}\ , \quad (2.13)$$

$$R_3=\sum_p \{\alpha_3\mathrm{tr}(U_p+U_p^\dagger)+\beta_3[(\mathrm{tr}U_p)^2+(\mathrm{tr}U_p^\dagger)^2]\}\ , \quad (2.14)$$

$$R_4=\sum_p [\alpha_4\mathrm{tr}(U_p+U_p^\dagger)+\beta_4(\mathrm{tr}U_p)(\mathrm{tr}U_p^\dagger)]\ . \quad (2.15)$$

α_i,β_i ($i=2,3,4$) are determined by the correct classical continuum limit of H_i, that is the conditions in Eqs. (2.5) and (2.6). The condition in Eq. (2.5) gives the relation for α_i and β_i:

$$\alpha_i C_N+K_iN^2\beta_i=1/(2g^4)\ , \quad (2.16)$$

where $K_2=4$, $K_3=2$, and $K_4=1$. The condition in Eq. (2.6) gives another relation for α_i and β_i:

$$\alpha_i+K_iN\beta_i=0\ . \quad (2.17)$$

Therefore, the conditions that the (2+1)-dimensional non-Abelian SU(N) lattice Hamiltonian H_i has correct classical continuum limit are

$$\alpha_i=-N/[g^4(N^2+1)]\ , \quad (2.18)$$

$$\beta_i=1/[K_ig^4(N^2+1)]\ . \quad (2.19)$$

For Abelian U(1) group theory, we can also show that

$$H_2=\frac{g^2}{2a}\exp(-R_2)E_l^a\exp(2R_2)E_l^a\exp(-R_2)$$

possesses the correct continuum limit.

Up to now, four Hamiltonians have been found. In 3+1 dimensions, all four Hamiltonians possess the correct continuum limit. In 2+1 dimensions, H_2, H_3, and H_4 (which correspond to R_2, R_3, and R_4, respectively) possess the correct continuum limit, but H_1 does not. For the SU(2) theory, $\mathrm{tr}U_p=\mathrm{tr}U_p^\dagger$; therefore, $H_2=H_3=H_4$.

Since the ground state is exactly known, rigorous upper bounds of the mass gap can be obtained by the variational method, and the string tension can be obtained by calculating the vacuum expectation of Wilson loops, we will discuss them in the later sections.

Even in the case where H does not possess the correct continuum limit, e.g., H_1 in (2+1)-dimensional theory, we can still consider H as a model Hamiltonian with nontrivial self-interactions and study its spectrum. The theory may be of interest by itself or by comparing it with the standard theory. In Sec. V on SU(2) theory, we present calculations on both H_1 and H_2. The results give a detailed comparison of two theories with different continuum limits. In the U(1) and the SU(3) cases, we give results on H_1 only for simplicity. Calculations on H_2 in U(1) theory are in progress. Preliminary results are similar to those of H_1.

III. MASS GAP AND STRING TENSION

In the preceding section we have studied the exact ground state of the gauge field. Now, let us turn to the excited states. Especially we will investigate the lowest excited state with zero momentum, which gives us the so-called mass gap.

First, we describe the general procedure to obtain the lowest state of ($d+1$)-dimensional SU(N) LGT for given quantum numbers. Let $|\Psi_0\rangle$ denote the exact vacuum which is assumed to be normalized:

$$H|\Psi_0\rangle=E|\Psi_0\rangle=0\ .$$

Then we can write a trial wave function for the zero-momentum excited state in the following form:

$$|\Psi\rangle=(\Phi-\langle\Phi\rangle_0)|\Psi_0\rangle\ , \quad (3.1)$$

where $\langle\Phi\rangle_0$ means the expectation value in the vacuum state $|\Psi_0\rangle$ and Φ is a linear combination of gauge-invariant operators which are translationally invariant and which have the given quantum numbers. To make $|\Psi\rangle$ orthogonal to the ground state, the expectation value $\langle\Phi\rangle_0$ is subtracted from the operator Φ in Eq. (3.1),

$$\langle\Psi_0|\Psi\rangle=\langle(\Phi-\langle\Phi\rangle_0)\rangle_0=0\ .$$

The norm of the state $|\Psi\rangle$ can be written in terms of vacuum expectation values:

$$\langle\Psi|\Psi\rangle=\langle\Phi^\dagger\Phi\rangle_0-\langle\Phi^\dagger\rangle_0\langle\Phi\rangle_0\ , \tag{3.2}$$

where $\Phi^\dagger$ is the Hermitian conjugation of Φ.

The expectation value E of H in $|\Psi\rangle$ is

$$E=\langle\Psi|H|\Psi\rangle/\langle\Psi|\Psi\rangle\ , \tag{3.3}$$

where

$$\langle\Psi|H|\Psi\rangle=\frac{g^2}{2a}\langle 0|\exp(R)(\Phi^\dagger-\langle\Phi^\dagger\rangle_0)\exp(-R)E_l^a\exp(2R)E_l^a\exp(-R)(\Phi-\langle\Phi\rangle_0)\exp(R)|0\rangle$$
$$=-\frac{g^2}{2a}\langle[E_l^a,\Phi^\dagger][E_l^a,\Phi]\rangle_0\ .$$

Minimizing E in Eq. (3.3) for a set of trial functions, we obtain an upper bound of the mass gap.

Variation with respect to Φ is carried out as follows. Let us write Φ as the linear function of variational states φ_i, that is,

$$\Phi=\frac{1}{\sqrt{N_p}}\sum_{i=1}^{N}\sum_{\mathbf{x}}a_i\varphi_i(\mathbf{x})\ , \tag{3.4}$$

where $\varphi_n(\mathbf{x})$'s $(n=1,2,\ldots,N)$ are gauge-invariant operators with the given quantum numbers on position $\mathbf{x}$ and where the a_n's are variational parameters, N_p is the number of plaquettes ($N_p=\sum_{\mathbf{x}}1$), and N is the total number of trial states φ_n.

Then we obtain

$$E=\frac{g^2}{2a}a_i^*c_{ij}a_j/(a_k^*D_{kl}a_l)\ , \tag{3.5}$$

where

$$D_{ij}=\sum_{\mathbf{x}}[\langle\varphi_i(0)\varphi_j(\mathbf{x})\rangle_0-\langle\varphi_i(0)\rangle_0\langle\varphi_j(\mathbf{x})\rangle_0]\ , \tag{3.6a}$$

$$C_{ij}=-\left\langle[E_l^a,\varphi_i(0)]E_l^a,\sum_{\mathbf{x}}\varphi_j(\mathbf{x})\right\rangle_0\ . \tag{3.6b}$$

The extreme of E is obtained by

$$\frac{\partial E}{\partial a_i^*}=\frac{g^2}{2a}\frac{1}{a_k^*D_{kl}a_l}(C-2\beta aED)_{ij}a_j=0\ .$$

Therefore,

$$\det|C-2\beta aED|=0\ . \tag{3.7}$$

The solutions E^i $(i=1,2,\ldots,N)$ to Eq. (3.7) are the energy eigenvalues of the excited states. Owing to the fact that matrices C and D are real, symmetric and positive definite, all eigenvalues E are real and positive and the eigenstates are orthogonal to each other ($a_i^{\alpha *}D_{ij}a_j^\beta=\delta^{\alpha\beta}$) if their quantum numbers are completely specified.

Now, we examine the lowest excited state with zero momentum using the general procedure explained above. The lowest excited state must be rotationally and reflectionally invariant. In the strong-coupling limit Φ is dominated by the one-plaquette loop variable $\varphi_1(\mathbf{x})=\mathrm{tr}[U_p(\mathbf{x})+U_p^\dagger(\mathbf{x})]$. When we go into the weak-coupling region, larger loop variables are expected to contribute. After making many trials, we found that in the weak-coupling region, the lowest excited state Φ is dominated by larger square Wilson loop variables.

Now, let us discuss the calculation of the string tension using the exact vacuum wave function obtained in the preceding section.

In the Hamiltonian formalism, the string tension is usually computed from the energy of the q-$\bar{q}$ state.[19] However, we will compute it by another method, i.e., from the vacuum expectation value of Wilson loops. The string tension derived from the timelike Wilson loops has the exact meaning of energy (per unit length) of the q-$\bar{q}$ state in the Hamiltonian formalism. However, in this formalism it is somewhat cumbersome to evaluate the timelike Wilson loops. Therefore we evaluate the spacelike (or fixed time) Wilson loops. It is not well known that the string tension obtained from it is identical with the one obtained from the timelike Wilson loops in the continuum limit. In the space-time symmetric Euclidean formalism, there is no difference between the timelike and spacelike Wilson loops. The Hamiltonian formalism is obtained from the space-time symmetric theory by taking the lattice spacing of the time direction to vanish. The scaling behavior of the string tension obtained from both methods must be the same in the weak-coupling limit, although there might be a difference between the absolute value of them by a numerical factor.

IV. (2+1)-DIMENSIONAL U(1) LATTICE THEORY

Now we turn to study the mass gap and the string tension of (2+1)-dimensional U(1) LGT. In Sec. II we have found two Hamiltonians H_1 and H_2 which possess exact ground states. In this section we only study the mass gap and the string tension of H_1.

For U(1) LGT, we write H_1 as

$$H_1=\frac{g^2}{2a}\left\{\sum_l E_l^2-\frac{1}{g^4}\sum_p(U_p+U_p^\dagger)-\frac{1}{16g^8}\left[E_l^a,\sum_p(U_p+U_p^\dagger)\right]\times\left[E_l^a,\sum_{p'}(U_{p'}+U_{p'}^\dagger)\right]\right\}\ . \tag{4.1}$$

The exact ground state of H_1 is

$$|\Psi_0\rangle = \exp\left[\frac{1}{2g^4}\sum_p (U_p + U_p^\dagger)\right]|0\rangle . \qquad (4.2)$$

In (2+1)-dimensional LGT, it can be shown[20] that $[dU_l]=[dU_p]$, and the vacuum expectation of all gauge-invariant operators can be evaluated exactly. The norm of the ground state is

$$\langle\Psi_0|\Psi_0\rangle = \prod_p I_0(x) ,$$

where $x=1/g^4$ and $I_i(x)$ is the ith modified Bessel function. The vacuum expectation of U_p^n is

$$\langle U_p^n\rangle_0 = Y_n(x) , \qquad (4.3)$$

where $Y_n(x)=I_n(x)/I_0(x)$.

The lowest excited state of the system is a static $J^{PC}=0^{+-}$ state. For this state, we use the trial functions

$$\varphi_n(\mathbf{x}) = [U_p^\dagger(\mathbf{x})]^n - [U_p(\mathbf{x})]^n , \qquad (4.4)$$

where $U_p(\mathbf{x})$ is the plaquette located at the position $\mathbf{x}$, and $n=1, 2,3,\ldots,N$. N is the total number of trial functions.

The symmetric matrices C_{mn} and D_{mn} are easy to obtain for these trial functions ($n\geq m$):

$$C_{mn} = 8mn(Y_{n-m}+Y_{n+m}-2Y_nY_m) , \qquad (4.5a)$$

$$D_{mn} = 2(Y_{n-m}-Y_{n+m}) . \qquad (4.5b)$$

The resulting curves for the mass gap 0^{+-} are shown in Fig. 1. Inclusion of other trial functions does not alter the curve of $N=15$ significantly. Exponentially decreasing behavior ma vs $1/g^2$ is observed in the interval $2<1/g^2\leq 4.4$. However, (1) the slope in the weak-coupling region $1/g^2>2$ is much greater than that predicted in the periodic Gaussian model.[3] (2) The slope in the deep weak-coupling region is much greater than that in the crossover region $1/g^2\leq 2$ in previous Monte Carlo and analytical calculations. (3) The slope seems to be still increasing in the deep weak-coupling region, that is, ma may decrease with $1/g^2$ faster than exponentially.

For the $J^{PC}=0^{++}$ state, we use the trial functions

$$\varphi_n(\mathbf{x}) = U_{np}(\mathbf{x}) + U_{np}^\dagger(\mathbf{x}), \quad n=1,2,\ldots,N , \qquad (4.6)$$

where $U_{np}(\mathbf{x})$ is the $n\times n$ Wilson loop located at $\mathbf{x}$, and N is the total number of trial functions. For these trial functions ($n\geq m$),

$$C_{mn} = 16\sum_{i=1}^{m} iY_1^{n^2+m^2-2im}(1-Y_2^{im}) + 8m(n-m-1)Y_1^{n^2-m^2}(1-Y_2^{m^2}) , \qquad (4.7a)$$

$$D_{mn} = 2(n-m-1)^2Y_1^{n^2-m^2}(1+Y_2^{m^2}-2Y_1^{2m^2}) + 8\sum_{ij=1}^{m} Y_1^{n^2+m^2-2ij}(1+Y_2^{ij}-2Y_1^{2ij}) + 8(n-m-1)\sum_{i=1}^{m} Y_1^{n^2+m^2-2im}(1+Y_2^{im}-2Y_1^{2im}) . \qquad (4.7b)$$

The $\beta am(0^{++})$ vs $1/g^2$ curves corresponding to $N=1$, 2, 3, 4, 5, 8, 15, and 30, respectively, are given in Fig. 2. The scaling behavior $am=2.77g^2$ in weak-coupling region $1.6<1/g^2\leq 7.0$ is observed. $am(0^{++})$ in weak-coupling region $1.6<1/g^2\leq 7.0$ is observed. $am(0^{++})$ decreases with $1/g^2$ much more slowly than $am(0^{+-})$. It is conceivable that $m(0^{++})/m(0^{+-})\to\infty$ when $g^2\to 0$, similar result for the string tension compared with the mass gap is also noted in Ref. 4.

Now we are to compute the string tension by computing the vacuum expectation value of Wilson loops. The vacuum expectations of Wilson loops are easily obtained:

$$\langle W\rangle_0 = 2[Y_1(x)]^A , \qquad (4.8)$$

where A is the area of the loop in units of a^2.

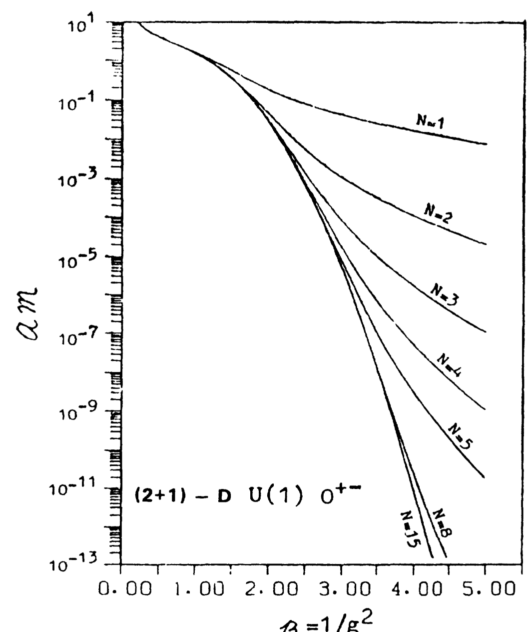

FIG. 1. Graph of mass gap $am(0^{+-})$ against $1/g^2$ with Hamiltonian H_1 in (2+1)-dimensional U(1) theory.

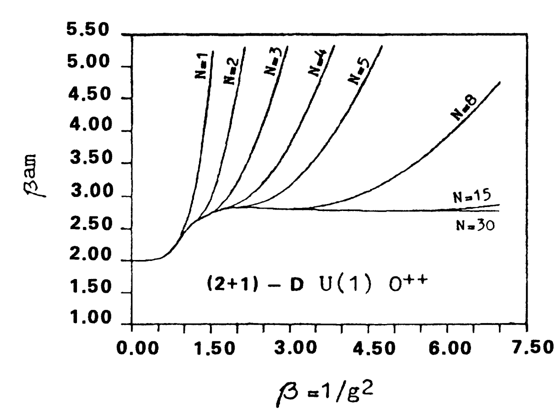

FIG. 2. Graph of mass gap $\beta am(0^{++})$ against $1/g^2$ with Hamiltonian H_1 in (2+1)-dimensional U(1) theory.

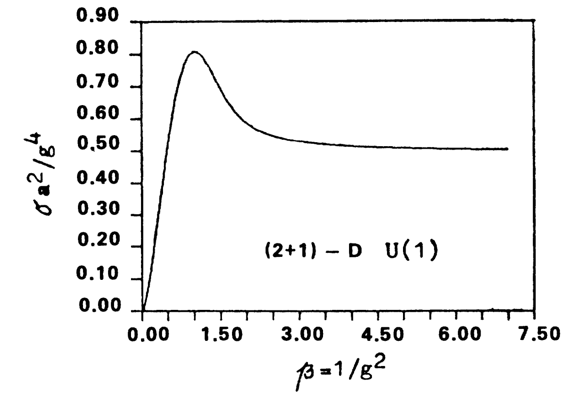

FIG. 3. Graph of string tension $\sigma a^2/g^4$ against $1/g^2$ with Hamiltonian H_1 in (2+1)-dimensional U(1) theory.

From this the string tension σ is obtained as

$$\sigma a^2 = -\ln Y_1(x) .$$

We show in Fig. 3 the curve $\sigma a^2/g^4$ vs $1/g^2$, the scaling behavior is $\sigma a^2 = 0.51 g^4$ in the interval $2.4 < 1/g^2 \le 7.0$. This scaling behavior is in agreement with that of $m(0^{++})$.

In Ref. 5, the mass gap and the string tension of (2+1)-dimensional U(1) LGT with Villain action is investigated by Monte Carlo method in the interval $1.5 < 1/g^2 < 2.0$. In this region, their results are in agreement with our results.

V. (2+1)-DIMENSIONAL SU(2) LATTICE GAUGE THEORY

Let us now study the mass gap and the string tension of (2+1)-dimensional SU(2) LGT. In Sec. II two Hamiltonians H_1 and H_2 which have different continuum limits and possess different exact ground states are found. In this section, in order to observe the difference of their property in the weak-coupling region, we study the mass gap and the string tension of both H_1 and H_2.

For the fundamental representation of the SU(2) group, the group element U_p can be parametrized by

$$U_p = \cos\psi_p + i\boldsymbol{\sigma}\cdot\mathbf{n}\sin\psi_p ,$$

where σ are Pauli matrices, and

$$\mathbf{n} = (\sin\theta_p\cos\phi_p, \sin\theta_p\sin\phi_p, \cos\theta_p) .$$

At first, let us study the property of H_1. The ground state of H_1 is

$$|\Psi_0\rangle = \exp(R_1)|0\rangle , \quad (5.1)$$

where $R_1 = (x/2)\sum_p \mathrm{tr} U_p$, and $x = 8/(3g^4)$.

The norm of the ground state is

$$Z = \langle \Psi_0 | \Psi_0 \rangle = \prod_p z = \prod_p I_1(2x)/(2x) . \quad (5.2)$$

The expectations of the following operators are useful to obtain the matrix elements C_{mn} and D_{mn}:

$$U = \langle \cos\psi_p \rangle_0 = Y_2(2x) , \quad (5.3a)$$

$$A = \langle \cos^2\psi_p \rangle_0 = [1 + 3Y_3(2x)]/4 , \quad (5.3b)$$

$$\langle n_i \rangle_0 = 0 , \quad (5.3c)$$

$$\langle n_i n_j \rangle_0 = \delta_{ij}/3 , \quad (5.3d)$$

where $Y_i(2x) = I_i(2x)/I_1(2x)$, and $I_i(2x)$ is the ith modified Bessel function.

Let

$$B_i = \langle \mathrm{tr}(U_{1p}U_{2p}\cdots U_{ip})\mathrm{tr}(U_{1p}U_{2p}\cdots U_{ip})\rangle_0 ,$$

where U_{ip} is the ith plaquette. Denote $U_{1p}U_{2p}\cdots U_{ip}$ by U_i, then, using the relations $\sigma^a_{ij}\sigma^a_{kl} = 2\delta_{il}\delta_{kj} - \delta_{ij}\delta_{kl}$, and $\mathrm{tr}U_p = \mathrm{tr}U_p^\dagger$, we can obtain the recurrence formula of B_i:

$$\begin{aligned} B_i &= \langle \mathrm{tr}(U_{i-1}U_{ip})\mathrm{tr}(U_{i-1}^\dagger U_{ip}^\dagger)\rangle_0 \\ &= \langle \mathrm{tr}(U_{i-1})\mathrm{tr}(U_{i-1}^\dagger)\cos^2\psi_{ip} + \mathrm{tr}(U_{i-1}\sigma^a)\mathrm{tr}(\sigma^b U_{i-1}^\dagger) n^a_{ip} n^b_{ip}\sin^2\psi_{ip}\rangle_0 \\ &= \langle \mathrm{tr}(U_{i-1})\mathrm{tr}(U_{i-1})(\cos^2\psi_{ip} - \tfrac{1}{3}\sin^2\psi_{ip}) + \tfrac{4}{3}\mathrm{tr}(U_{i-1}U_{i-1}^\dagger)\sin^2\psi_{ip}\rangle_0 \\ &= Y_3 B_{i-1} + 2Y_2/x . \end{aligned} \quad (5.4)$$

The initial value of this recurrence formula is

$$B_1 = \langle \mathrm{tr}(U_{1p})\mathrm{tr}(U_{1p})\rangle = 4[1 - 3Y_2(2x)/(2x)] .$$

Using the recurrence formula of B_i and its initial value, we can obtain B_i for all i.

The lowest excited state of SU(2) theory is a static $J^{PC} = 0^{++}$ state. We chose trial functions

$$\varphi_n(\mathbf{x}) = \mathrm{tr} U_{np}(\mathbf{x}) , \quad (5.5)$$

where $U_{np}(\mathbf{x})$ is the $n\times n$ Wilson loop whose lower-left corner is located at $\mathbf{x}$ and $n = 1, 2, 3, \ldots, N$. N is the total number of trial functions.

The symmetric matrix elements C_{mn} and D_{mn} can be obtained from the expectation of operators given above $(n \ge m)$:

$$C_{mn} = m(n - m - 1)Y_2^{n^2-m^2}(4 - B_{m^2}) + 2\sum_{k=1}^{m} k Y_2^{n^2+m^2-2km}(4 - B_{km}) , \quad (5.6a)$$

$$D_{mn}=4\sum_{kj=1}^{m}(Y_2^{n^2+m^2-2kj}B_{jk}-4Y_2^{n^2+m^2}) +(n-m-1)^2(Y_2^{n^2-m^2}B_{m^2}-4Y_2^{n^2+m^2}) +4(n-m-1)\sum_{k=1}^{m}(Y_2^{n^2+m^2-2km}B_{km} -4Y_2^{n^2+m^2})\ . \quad (5.6b)$$

By solving the eigenvalue equation (3.7) for $N=1$, 2,3,4,5,8,15, and 30, the curves βam vs $1/g^2$ are obtained in Fig. 4. Good scaling behavior $am=2.28g^2$ in the interval $1<1/g^2\leq 7$ is observed.

The string tension in Fig. 5 can be obtained from the vacuum expectation of the Wilson loop, its scaling behavior is $\sigma a^2=0.28g^4$ in the interval $2\leq 1/g^2\leq 7.0$. Therefore, we can obtain $m(0^{++})=4.3\sqrt{\sigma}$. The scaling behaviors $am\propto g^2$ and $\sigma a^2\propto g^4$ given above are in agreement with that predicted by the weak-coupling perturbation theory despite the fact that this Hamiltonian does not have the correct continuum limit.

We now consider the Hamiltonian H_2 which possesses the correct continuum limit. The exact ground state of H_2 is

$$|\Psi_0\rangle=\exp(R_2)|0\rangle\ , \quad (5.7)$$

where

$$R_2=\frac{x}{2}\sum_p \mathrm{tr}U_p+\frac{y}{2}\sum_p(\mathrm{tr}U_p)^2\ ,$$

$x=-8/(5g^4)$, and $y=2/(5g^4)$.

The norm of the ground state is

$$Z=\langle\Psi_0|\Psi_0\rangle=\prod_p z\ ,$$

where

$$z=\sum_{n=0}^{\infty}\frac{y^n}{n!}\frac{d^{2n}}{dx^{2n}}\frac{I_1(2x)}{x}=\sum_{n=0}^{\infty}\sum_{k\geq n}^{\infty}z(k,n) \quad (5.8)$$

and

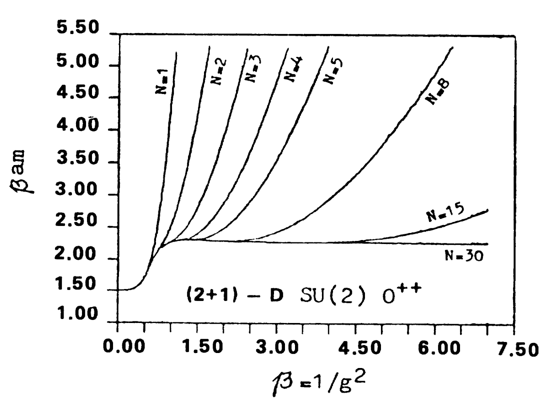

FIG. 4. Graph of mass gap $\beta am(0^{++})$ against $1/g^2$ with Hamiltonian H_1 in (2+1)-dimensional SU(2) theory.

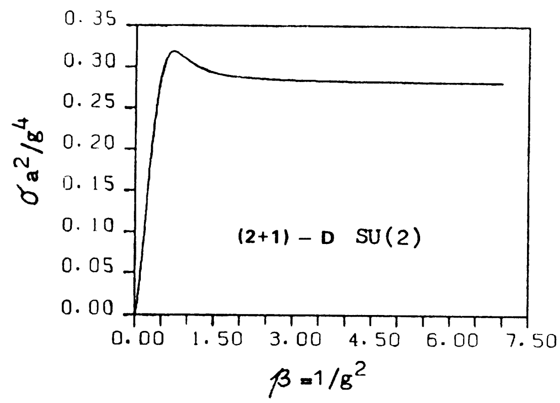

FIG. 5. Graph of string tension $\sigma a^2/g^4$ against $1/g^2$ with Hamiltonian H_1 in (2+1)-dimensional SU(2) theory.

$$z(k,n)=\frac{(2k)!}{n!k!(k+1)!(2k-2n)!}4^{2k-2n}y^{2k-n}\ .$$

The expectation value of operators given in (5.3) are changed into

$$U=\langle\cos\psi_p\rangle_0 =z^{-1}\sum_{n=0}^{\infty}\sum_{k=n+1}^{\infty}z(k,n)(k-n)/(4y)\ , \quad (5.9a)$$

$$A=\langle\cos^2\psi_p\rangle_0 =z^{-1}\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}z(k,n)n/(4y)\ , \quad (5.9b)$$

$$\langle n_i\rangle_0=0\ , \quad (5.9c)$$

$$\langle n_in_j\rangle_0=\delta_{ij}/3\ . \quad (5.9d)$$

The recurrence formula in (5.4) is rewritten as

$$B_i=\langle\mathrm{tr}(U_{1p}U_{2p}\cdots U_{ip})\mathrm{tr}(U_{1p}U_{2p}\cdots U_{ip})\rangle_0 =B_{i-1}(4A-1)/3+4(1-A)/3\ , \quad (5.10)$$

and its initial value is

$$B_1=\langle\mathrm{tr}(U_{1p})\mathrm{tr}(U_{1p})\rangle_0=4A\ .$$

Choosing the same trial functions as that of H_1 for a static $J^{PC}=0^{++}$, the symmetric matrix elements C_{mn} and D_{mn} can be obtained as $(n\geq m)$

$$C_{mn}=m(n-m-1)U^{n^2-m^2}(4-B_{m^2}) +2\sum_{i=1}^{m}iU^{n^2+m^2-2im}(4-B_{im})\ , \quad (5.11a)$$

$$D_{mn}=4\sum_{ij=1}^{m}(U^{n^2+m^2-2ij}B_{ij}-4U^{n^2+m^2}) +(n-m-1)^2(U^{n^2-m^2}B_{m^2}-4U^{n^2+m^2}) +4(n-m-1)\sum_{i=1}^{m}(U^{n^2+m^2-2im}B_{im}-4U^{n^2+m^2})\ . \quad (5.11b)$$

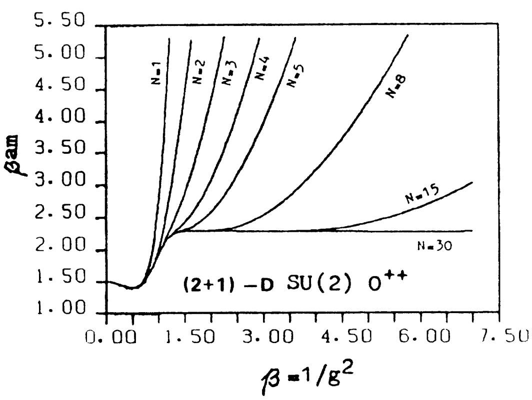

FIG. 6. Graph of mass gap $\beta am(0^{++})$ against $1/g^2$ with Hamiltonian H_2 in $(2+1)$-dimensional SU(2) theory.

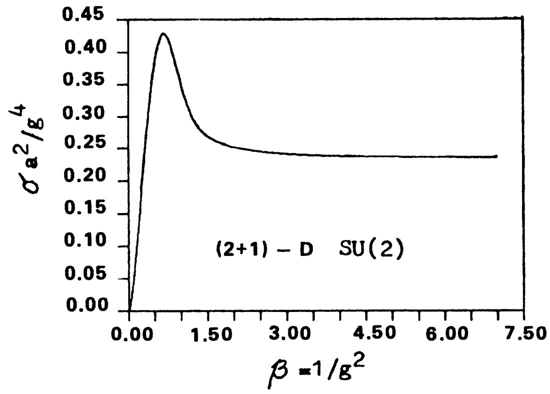

FIG 7. Graph of string tension $\sigma a^2/g^4$ against $1/g^2$ with Hamiltonian H_2 in $(2+1)$-dimensional SU(2) theory.

In the same way, the curves βam vs $1/g^2$ are given in Fig. 6. Good scaling behavior $am=2.28g^2$ in the interval $1.2<1/g^2\leq 7$ is obtained, it is the same as that for H_1. The difference between them only appears in the intermediate coupling region. Why H_1 and H_2, which possess different continuum limits, have the same value for am, is not known to us at present.

Figure 7 shows that the scaling behavior of string tension for H_2 is $\sigma a^2=0.228g^4$, the absolute value of which is slightly smaller than that for H_1, although both of them have the same scaling behavior. Therefore, for Hamiltonian H_2, we can obtain $m(0^{++})=4.8\sqrt{\sigma}$. These results above should be compared with those obtained from Monte Carlo calculations and other analytic approximations.

The first Monte Carlo calculation for the Euclidean version of the model was made by D'Hoker.[17] His results were in agreement with the theoretical expectations in (1.2), and he estimated that in the continuum limit $\sigma a^2=(0.26\pm0.02)g^4$, $M/\sqrt{\sigma}=4.5\pm0.5$. Ambjorn, Hey, and Otto[10] performed a more careful Monte Carlo analysis, giving $\sigma a^2=0.2g^4$. Irback and Peterson[18] obtained $m(0^{++})=(4.7\pm1.2)\sqrt{\sigma}$ in the range $4<4/g^2\leq 6.5$ by using a long-distance correlation Monte Carlo method. The latest Monte Carlo result[14] is $ma=(2.15\pm0.2)g^2$ in the range $4.5<4/g^2\leq 5.5$. The cluster expansion methods[8] show the results $\sigma a^2=(0.14\pm0.01)g^4$ and $ma=(2.2\pm0.25)g^2$ in the continuum limit. Finally, the scalar mass gap has been evaluated in weak-coupling perturbation theory by Muller and Ruhl,[12] they obtained a result $ma=0.2637g^2$ which is much smaller than the results quoted above.

VI. $(2+1)$-DIMENSIONAL SU(3) LATTICE GAUGE THEORY

The mass gap and the string tension of $(2+1)$-dimensional SU(3) LGT is investigated in this section. In Sec. II, we have found four Hamiltonians which possess the exact ground states. H_2, H_3, and H_4 possess the correct continuum limit and H_1 does not. But the preceding section shows that, although H_1 does not possesses the correct continuum limit, it gives the mass gap with the same scaling behavior as that of H_2, which possesses the correct continuum limit.

It seems difficult to discuss the mass gap of H_2, H_3, and H_4 in SU(3) theory. Therefore, before we find an effective method to calculate the property of H_2, H_3, and H_4, we only discuss the mass gap and the string tension of H_1 in this section. Perhaps it can give us some useful information. The exact ground state of H_1 is

$$|\Psi_0\rangle=\exp\left[\frac{x}{2}\sum_p \mathrm{tr}(U_p+U_p^\dagger)\right]|0\rangle\ , \tag{6.1}$$

where $x=3/(4g^4)$.

The SU(3) one-link-invariant group integral in lattice gauge theory is derived by Eriksson and Svartholm:[21]

$$\int_{\mathrm{SU}(3)} dU\exp[\mathrm{tr}(U^\dagger J+J^\dagger U)]=2\sum_{jkln=0}^{\infty}\frac{1}{(j+2k+3l+n+2)!(k+2l+n+1)!}\frac{X^j}{j!}\frac{Y^k}{k!}\frac{D^l}{l!}\frac{\Delta^n}{n!}\ , \tag{6.2}$$

where

$$X=\mathrm{tr}(JJ^\dagger)\ ,$$
$$Y=\tfrac{1}{2}\{[\mathrm{tr}(JJ^\dagger)]^2-\mathrm{tr}(JJ^\dagger)^2\}\ ,$$
$$D=\det(JJ^\dagger)\ ,$$
$$\Delta=\det J+\det J^\dagger\ .$$

Letting J be equal to the constant x, we obtain

$$z=\int_{\mathrm{SU}(3)} dU_p\exp[x\,\mathrm{tr}(U_p+U_p^\dagger)]$$
$$=\sum_{jkln=0}^{\infty} z(j,k,l,n)=\sum_{i=0}^{\infty} z_i x^i\ , \tag{6.3}$$

where

$$z(j,k,l,n)=2\frac{3^{j+k}2^n}{j!k!l!n!}\times\frac{x^{2j+4k+6l+3n}}{(j+2k+3l+n+2)!(k+2l+n+1)!},$$

and z_i can be obtained by computer.

The derivatives of Eq. (6.2) with respect to J_{ij} and J_{ij}^* give us vacuum expectation of the following operators:

$$\langle (U_p)_{ij}\rangle_0=z^{-1}\int dU_p(U_p)_{ij}\exp[x\,\mathrm{tr}(U_p+U_p^\dagger)]$$
$$=z^{-1}\sum_i iz_ix^{i-1}/6\delta_{ij}=U\delta_{ij}, \quad (6.4a)$$

$$z^{-1}\int dU_p(U_p)_{ij}(U_p)_{kl}\exp[x\,\mathrm{tr}(U_p+U_p^\dagger)]$$
$$=z^{-1}\int dU_p(U_p^*)_{ij}(U_p^*)_{kl}\exp[x\,\mathrm{tr}(U_p+U_p^\dagger)]$$
$$=a_1\delta_{ij}\delta_{kl}+a_2\delta_{jk}\delta_{il}, \quad (6.4b)$$

$$z^{-1}\int dU_p(U_p)_{ij}(U_p^*)_{kl}\exp[x\,\mathrm{tr}(U_p+U_p^\dagger)]$$
$$=b_1\delta_{ij}\delta_{kl}+b_2\delta_{ik}\delta_{jl}, \quad (6.4c)$$

where

$$a_1=z^{-1}\sum_{jkln}\frac{1}{36x^2}[(2j+4k+6l+3n)^2+9n-4k-4j]z(j,k,l,n), \quad (6.5a)$$

$$a_2=z^{-1}\sum_{jkln}-\frac{1}{6x^2}(2k+6l+3n)z(j,k,l,n), \quad (6.5b)$$

$$b_1=z^{-1}\sum_{jkln}\frac{1}{36x^2}[(2j+4k+6l+3n)^2-(4k+4j+9n)]z(j,k,l,n), \quad (6.5c)$$

$$b_2=z^{-1}\sum_{jkln}\frac{1}{3x^2}(j+k)z(j,k,l,n). \quad (6.5d)$$

Note that

$$3=z^{-1}\int dU_p\mathrm{tr}(U_pU_p^\dagger)\exp[x\,\mathrm{tr}(U_p+U_p^\dagger)]$$
$$=z^{-1}\int dU_p(U_p)_{ij}(U_p^*)_{ij}\exp[x\,\mathrm{tr}(U_p+U_p^\dagger)]$$
$$=3b_1+9b_2.$$

Using the computer techniques we can show that b_1 and b_2 in Eq. (6.5) satisfy this relation. Therefore, the correctness of Eq. (6.4) is confirmed.

Let $U_i=U_{1p}U_{2p}\cdots U_{ip}$, and U_{ip} is the ith plaquette, then using the relation in Eq. (6.4), we can obtain the recurrence formulas:

$$A_n=\langle \mathrm{tr}(U_{1p}U_{2p}\cdots U_{np}U_{1p}U_{2p}\cdots U_{np})\rangle_0$$
$$=\langle \mathrm{tr}(U_{n-1}U_{np}U_{n-1}U_{np})\rangle_0$$
$$=\langle (U_{n-1})_{ij}(U_{n-1})_{kl}(U_{np})_{jk}(U_{np})_{li}\rangle_0$$
$$=a_1A_{n-1}+a_2B_{n-1}, \quad (6.6a)$$

$$B_n=\langle \mathrm{tr}(U_n)\mathrm{tr}(U_n)\rangle_0=a_1B_{n-1}+a_2A_{n-1}, \quad (6.6b)$$

$$C_n=\langle \mathrm{tr}(U_n)\mathrm{tr}(U_n^\dagger)\rangle=b_1C_{n-1}+3b_2. \quad (6.6c)$$

The initial values of these recurrence formulas are

$$A_1=\langle (U_p)_{ij}(U_p)_{ij}\rangle_0=3a_1+9a_1, \quad (6.7a)$$

$$B_1=\langle (U_p)_{ii}(U_p)_{jj}\rangle_0=9a_1+3a_2, \quad (6.7b)$$

$$C_1=\langle (U_p)_{ii}(U_p^*)_{jj}\rangle_0=9b_1+3b_2. \quad (6.7c)$$

The lowest excited state of this system is a static $J^{PC}=0^{++}$ state. For this state, we choose trial functions as

$$\varphi_n(\mathbf{x})=\mathrm{tr}[U_{np}(\mathbf{x})+U_{np}^\dagger(\mathbf{x})],\quad n=1,2,\ldots,N, \quad (6.8)$$

where $U_{np}(\mathbf{x})$ is the $n\times n$ Wilson loop located at $\mathbf{x}$ and N is the total number of trial functions.

The symmetric matrix elements C_{mn} and D_{mn} for $n\geq m$ are

$$C_{mn}=\tfrac{4}{3}m(n-m-1)U^{n^2-m^2}(B_{m^2}-3A_{m^2}+9-C_{m^2})+\tfrac{8}{3}\sum_{i=1}^{m}iU^{n^2+m^2-2im}(B_{im}-3A_{im}+9-C_{im}), \quad (6.9a)$$

$$D_{mn}=8\sum_{ij=1}^{m}U^{n^2+m^2-2ij}(B_{ij}+C_{ij}-18U^{2ij})+8(n-m-1)\sum_{i=1}^{m}U^{n^2+m^2-2im}(B_{im}+C_{im}-18U^{2im})+2(n-m-1)^2U^{n^2-m^2}(B_{m^2}+C_{m^2}-18U^{2m^2}). \quad (6.9b)$$

The curves βam vs $1/g^2$ are given in Fig. 8, the curves for $N=15$ and for $N=30$ coincide with the range considered. The scaling behavior $am=3.62g^2$ is observed in the interval $2<1/g^2\leq 7$.

For the $J^{PC}=0^{+-}$ state, we use the trial functions

$$\varphi_n(\mathbf{x})=\mathrm{tr}[U_{np}(\mathbf{x})-U_{np}^\dagger(\mathbf{x})],\quad n=1,2,\ldots,N. \quad (6.10)$$

In the same way, the symmetric matrix elements C_{mn} and D_{mn} for $n\geq m$ are

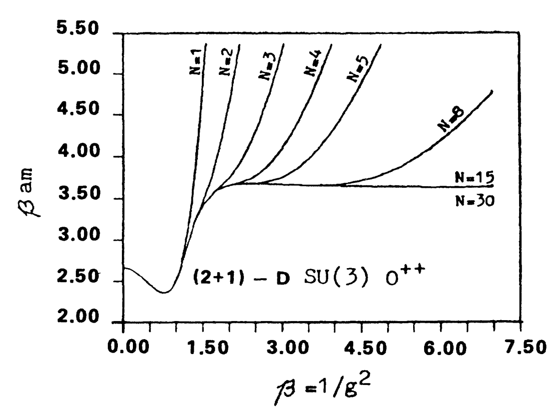

FIG. 8. Graph of mass gap $\beta am(0^{++})$ against $1/g^2$ with Hamiltonian H_1 in (2+1)-dimensional SU(3) theory.

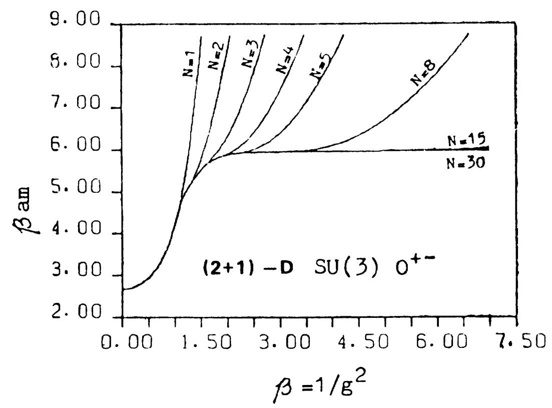

FIG. 9. Graph of mass gap $\beta am(0^{+-})$ against $1/g^2$ with Hamiltonian H_1 in $(2+1)$-dimensional SU(3) theory.

$$C_{mn}=\tfrac{4}{3}m(n-m-1)U^{n^2-m^2}(3A_{m^2}-B_{m^2}+9-C_{m^2})+\tfrac{8}{3}\sum_{i=1}^{m} iU^{n^2+m^2-2im}(3A_{im}-B_{im}+9-C_{im}) , \quad (6.11a)$$

$$D_{mn}=8\sum_{ij=1}^{m} U^{n^2+m^2-2ij}(C_{ij}-B_{ij})+2(n-m-1)^2U^{n^2-m^2}(C_{m^2}-B_{m^2})+8(n-m-1)\sum_{i=1}^{m} U^{n^2+m^2-2im}(C_{im}-B_{im}) . \quad (6.11b)$$

The curves βam vs $1/g^2$ are given in Fig. 9. Good scaling behavior $am=5.97g^2$ is observed in the range $2<1/g^2\leq 7$.

The string tension of this system is shown in Fig. 10, its scaling behavior is $\sigma a^2=0.89g^4$ in the range $3<1/g^2\leq 7$. The mass gap and string tension above show the same scaling behavior as that in $(2+1)$-dimensional SU(2) theory, but the scaling behavior only appears when $1/g^2\geq 2$.

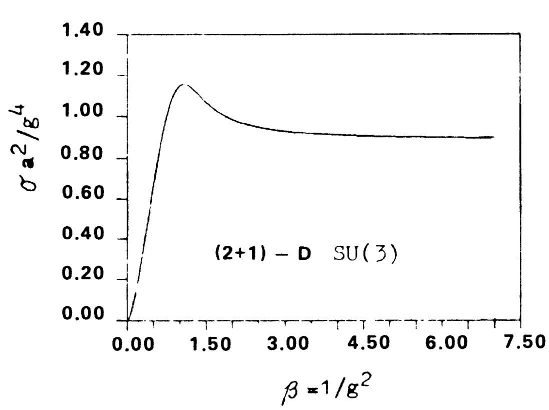

FIG. 10. Graph of string tension $\sigma a^2/g^4$ against $1/g^2$ with Hamiltonian H_1 in $(2+1)$-dimensional SU(3) theory.

VII. CONCLUSION AND DISCUSSION

In the preceding sections, we have made use of the arbitrariness in defining lattice gauge theory to propose new forms of Hamiltonians for $(d+1)$-dimensional $(d=2,3)$ SU(N) LGT, the ground state of which can be exactly obtained. Four such Hamiltonians are found. The form H_1 is the simplest, but H_1 does not possess the correct continuum limit in $(2+1)$-dimensional theory. On the other hand, H_2, H_3, and H_4 possess the correct continuum limit in both $(2+1)$- and $(3+1)$-dimensional SU(N) LGT.

By the same method, we can obtain more Hamiltonians with exact ground states, but perhaps they are too complicated to be useful. Since the ground state is exactly known, we can adopt the Rayleigh-Ritz variational method to obtain rigorous upper bounds of elementary excitation energies.

In this paper, we have studied the mass gap and string tension of $(2+1)$-dimensional LGT. For the U(1) group, the glueball states 0^{+-}, 0^{++} and string tension of H_1 were studied. For the SU(2) group, the glueball state 0^{++} and string tension of H_1 and H_2 were studied. For the SU(3) group, the glueball states 0^{++} and 0^{+-} and string tension of H_1 were studied. Except for the 0^{+-} state of the U(1) group, we chose the $n\times n$ Wilson loop variables enclosing up to 30×30 Wilson loop as trial functions, and found that other trial functions do not decrease the mass gap significantly. All the results mentioned above extend to the very deep weak-coupling region ($1/g^2\sim 7$), and show a nice scaling behavior $a\propto g^2$, which is in agreement with that predicted by the weak-coupling perturbation theory. The absolute values of mass gap and string tension are in agreement with that investigated in present Monte Carlo (MC) and other analytic calculations, but the results are extended to regions far beyond that investigated in the present MC and other analytic calculations.

In the case of non-Abelian theories, universality is confirmed. But for Abelian U(1) theory, the string tension and the glueball state 0^{++} show the scaling behavior $a\propto g^2$, but the mass gap for 0^{+-} decreases with $1/g^2$ faster than exponentially and its slope is much greater than that predicted in the periodic Gaussian model. At present we do not know the reason for this discrepancy, it may be a signal of nonuniversality for Abelian U(1) theory.

The results above are inspiring. The following are our proposals for further investigations.

(1) Investigation of other Hamiltonians and other glueball states. We have studied the mass gap of H_1 in $(2+1)$-dimensional U(1), SU(3) theory. We expect that the mass gap of H_2, H_3, and H_4 will be further investigated. We also expect the higher excitation energies will be calculated, so that universality can be further confirmed.

(2) Investigation of quark-antiquark potential. We have calculated the string tension from the vacuum expectation value of the spacelike Wilson loops in this paper. As we have mentioned before, however, the absolute value of the string tension derived from the spacelike

Wilson loops might be different from that of the energy of quark-antiquark state per unit length, although both of them should have the same scaling behavior. So we had better evaluate the quark-antiquark potential in order to compare numerically with the other physical qualities such as the mass gap.

(3) Investigation of the mass gap and string tension in $3+1$ dimensions. In $3+1$ dimensions, the expectation value of operators in Eq. (3.6) cannot be calculated exactly. A possible way is to calculate the expectation values of operators in Eq. (3.6) using the Monte Carlo method. In this respect, some attempts have been made to $(2+1)$-dimensional SU(2) LGT. We substitute the 120-element icosahedral subgroup $\tilde{Y}$ for the continuous SU(2) group, and worked on 18×18 lattice size to perform the Monte Carlo integration in Eq. (3.6). The results are in agreement with that in Sec. V in the range $1/g^2<2$. However, the extension to the $(3+1)$-dimensional continuous group needs a great deal of computer time.

(4) Investigation of Hamiltonian including fermions with exact ground state.

ACKNOWLEDGMENTS

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