Chiral anomalies and point-splitting regularization

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(Received 22 April 1988)

The anomalies of chiral gauge theories are discussed from the point of view of point-splitting regularization. The integrability of the regularized current is examined. Its relations with the Wess-Zumino consistency condition and Bose symmetry of the regularized Feynman diagrams are discussed. A point-splitting regularization scheme of the action is proposed as an alternative way to calculate the consistent anomalies.

I. INTRODUCTION

The interacting quantum field theories are plagued with ultraviolet divergences. To define a meaningful perturbative expansion one has to regularize certain products of operators. The renormalizability of the theory implies that the physical observables are independent of regularization schemes. Not all of the symmetries of the classical theory are preserved throughout the regularization process and some of them are anomalous even after the regulator is removed. A typical example is the Adler-Bell-Jackiw (ABJ) anomaly[1] which is related to a global axial-vector rotation of the fermion field. The form of this anomaly is regularization-scheme independent and it has many physical implications. Among them are the π decay amplitude and the solution of the U(1) problem in QCD. Another example is the anomaly associated with chiral gauge theories[2] which puts a severe restriction on the model building of the grand unification theory. This anomaly will be discussed in the present paper.

There have been clarifying studies[3] done on chiral anomalies up to now. In the case of the ABJ anomaly, the vector Ward identity due to gauge invariance fixes the axial-vector anomaly to a unique expression. In the case of chiral gauge theory, a gauge-invariant regularization is lacking. The associated anomaly depends on the finite part of the regularization. The typical forms are the covariant anomaly and the consistent anomaly, which have been clarified mathematically by Bardeen and Zumino[4] and also by Fujikawa.[5] It is demonstrated in Ref. 4 that it is always possible to add a local polynomial to the covariant current to get the consistent current and vice versa. There can be many other forms of anomalies, depending on the details of the regularization. All these different forms are proportional to the same Casimir operator of the gauge group and they are canceled simultaneously in an anomaly-free quantum field theory. In this paper, we will discuss the scheme dependence within the framework of point-splitting regularization. The integrability of the regularized current is investigated, which requires that the regularized current is a functional derivative of a quantum effective action and guarantees the consistency of the associated anomalies. For an anomaly-free theory, the covariance and the integrability can be preserved simultaneously. With the presence of anomalies, only one of the above conditions can be maintained by the appropriate choices of the operator insertion. Determining the correct operator insertion is trivial for the covariant current but is less trivial for the consistent one. However, instead of choosing carefully the operator insertion for the point-split current, one may regularize the classical action directly and then preserve manifestly the integrability of the related current. This gives a new method to compute the consistent anomalies. Our discussions are restricted in $D=2$ and 4. The conclusions are, however, generalizable to higher dimensions.

This paper is organized as follows. In Sec. II we will investigate the integrability condition and discuss its relation with the Wess-Zumino consistency condition.[6] In Secs. III and IV we shall discuss the point-splitting regularization in $D=2$ and 4, respectively. In Sec. V we shall propose a point-splitting regularization of the action which will give an explicit expression of the anomalous effective action upon the perturbative expansion. Some remarks then follow.

II. POINT-SPLITTING REGULARIZATION AND INTEGRABILITY

Ultraviolet divergence stems from the quantum fluctuation of the field amplitudes at a set of infinitely closed points in the spacetime continuum. The composite operators, e.g., the current, which contain the product of the field operators at the same point, are ill defined. To damp such a fluctuation, one may either regularize the definition of each composite operator or regularize the action to avoid the coincidence of the points where various fields interact. In this section and the next, we shall consider the first strategy.

Let $A_\mu = T^l A^l_\mu$ be the gauge potential with T^l the generator of the gauge group in a certain representation. ψ is the two-component spinor field in the representation generated by T^l. The Euclidean action of the system reads

$$S_E = \int d^D x \, [\tfrac{1}{2} \mathrm{Tr} F_{\mu\nu} F_{\mu\nu} + \bar{\psi} \tau_\mu (\partial_\mu + A_\mu) \psi] \,, \tag{1}$$

where we have adapted the chiral representation of γ matrices:

$$\gamma_5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \gamma_\mu = \begin{bmatrix} 0 & \tau_\mu^\dagger \\ \tau_\mu & 0 \end{bmatrix}, \tag{2}$$

with

$$\tau_\mu = \begin{cases} (1, i), & D = 2 \,, \\ (1, i\sigma_i), & D = 4 \,, \end{cases} \tag{3}$$

and σ_i are the Pauli matrices. The quantized fermionic effective action is given by the formal functional integration

$$e^{-W(A)} = \int D\psi D\bar{\psi} \, e^{-S_f(A, \psi, \bar{\psi})} \,, \tag{4}$$

where

$$S_f = \int d^D x \, \bar{\psi} \tau_\mu (\partial_\mu + A_\mu) \psi \tag{5}$$

is the fermionic part of the action (1). To examine the gauge invariance of the quantized action, one calculates the gauge variation of $W(A)$. Consider a gauge transformation generated by infinitesimal parameters θ, i.e.,

$$\delta_\theta A_\mu = -D_\mu \theta = -(\partial_\mu \theta + [A_\mu, \theta]) \,.$$

The corresponding variation of the effective action is

$$\delta_\theta W(A) = \int d^D x \, \mathrm{Tr} \theta(x) D_\mu \langle J_\mu \rangle_A \,, \tag{6}$$

where J_μ is the Noether current associated with the gauge transformation and $\langle \; \rangle_A$ is the functional average with the action (5). Formally

$$\langle J^l_\mu(x) \rangle_A = -\frac{\delta}{\delta A^l_\mu} W(A) = \langle \bar{\psi}(x) T^l \tau_\mu \psi(x) \rangle_A \tag{7}$$

and

$$D_\mu \langle J^l_\mu(x) \rangle_A = 0 \,. \tag{8}$$

However J_μ contains the product of two operators at the same points and is therefore ill defined. A regularization scheme has to be introduced. It has been established that there is no satisfactory regularization scheme such that the right-hand side of (6) vanishes in general. The quantized theories are therefore anomalous. To calculate the anomaly, one has to specify the regularization scheme at first. In what follows we shall adapt the point-splitting regularization. The general structure of the regularized current reads

$$[J^l_\mu(x)]^{\mathrm{reg}} = \left\langle \bar{\psi} \left[x + \frac{\epsilon}{2} \right] T^l \Omega \left[x + \frac{\epsilon}{2}, x - \frac{\epsilon}{2} \right] \times \tau_\mu \psi \left[x - \frac{\epsilon}{2} \right] \right\rangle_A \,. \tag{9}$$

The average over all orientation of ϵ is understood. It follows from the formal expression (7) that J^l_μ are both covariant and integrable (i.e., it can be written as the functional derivative of something). This is, however, not the case for the regularized current (9). For an arbitrary operator insertion Ω, the right-hand side of (9) may not be covariant and the right-hand side of (6) with J_μ replaced by J^{reg}_μ may not be able to be written as a gauge variation of something. The form of the anomaly depends on the choice of $\Omega(x, x')$ even after the regulator is removed.

The covariance of J^{reg}_μ can be easily implemented by choosing $\Omega(x, x')$ to be the parallel transporter from $x - \epsilon/2$ to $x + \epsilon/2$ in the group manifold,[7] i.e.,

$$\Omega \left[x + \frac{\epsilon}{2}, x - \frac{\epsilon}{2} \right] = P \exp \left[-\int_{x - \epsilon/2}^{x + \epsilon/2} d\xi_p \, A_p(\xi) \right] . \tag{10}$$

The order of Ω and T^l does not matter when $\epsilon \to 0$. A straighforward calculation of (10) gives the standard covariant anomaly. For example,

$$D_\mu \langle J^l_\mu(x)^{\mathrm{reg}} \rangle_A = \begin{cases} \dfrac{i}{4\pi} \epsilon_{\mu\nu} \mathrm{Tr} T^l F_{\mu\nu}, & D = 2 \,, \\ \dfrac{1}{32\pi^2} \epsilon_{\mu\nu\rho\lambda} \mathrm{Tr} T^l F_{\mu\nu} F_{\rho\lambda}, & D = 4 \,. \end{cases} \tag{11}$$

The current regularized this way is, however, not integrable. Using the strategy outlined in the next section, one may check that

$$\lim_{\epsilon \to 0} \left[\frac{\delta}{\delta A^a_\mu(x)} \langle J^b_\nu(y)^{\mathrm{reg}} \rangle_A - \frac{\delta}{\delta A^b_\nu(y)} \langle J^a_\mu(x)^{\mathrm{reg}} \rangle_A \right] = \begin{cases} \dfrac{1}{2\pi} \delta^{ab} \epsilon_{\mu\nu} \delta^2(x - y), & D = 2 \,, \\ \dfrac{1}{16\pi^2} \epsilon_{\mu\nu\rho\lambda} \mathrm{Tr}(\{T^a, T^b\}) F_{\rho\lambda} \delta^4(x - y), & D = 4 \,. \end{cases} \tag{12}$$

In order for a quantum effective action to exist even in the presence of an anomaly, J^{reg}_μ should satisfy the integrability condition

$$\lim_{\epsilon \to 0} \left[\frac{\delta}{\delta A^a_\mu(x)} \langle J^b_\nu(y)^{\mathrm{reg}} \rangle_A - \frac{\delta}{\delta A^b_\nu(y)} \langle J^a_\mu(x)^{\mathrm{reg}} \rangle_A \right] = 0 \,, \tag{13}$$

which will determine the functional form of the operator insertion. The anomaly deduced from the integrability condition (12) will satisfy the Wess-Zumino consistency condition[6]

$$\delta_\theta G(\theta')-\delta_{\theta'} G(\theta)=G([\theta,\theta']) \tag{14}$$

with $G(\theta)\equiv -\int d^Dx\, \theta^l(x) D_\mu \langle J_\mu^l(x)^{\text{reg}}\rangle_A$. In the perturbation theory, the integrability condition (13) implies the Bose symmetry of the regularized Feynman diagrams. Indeed, the functional Taylor expansion of $\langle J_\mu^l(x)^{\text{reg}}\rangle_A$ in terms of A reads

$$\langle J_\mu^l(x)^{\text{reg}}\rangle_A=\sum_{N=1}^{\infty}\frac{1}{N!}\int d^Dy_1\cdots d^Dy_N A_{\mu_1}^{l_1}(y_1)\cdots A_{\mu_N}^{l_1}(y_N) K_{\mu,\mu_1,\ldots,\mu_N}^{l,l_1,\ldots,l_n}(x;y_1,\ldots,y_N)\,, \tag{15}$$

where $K_{\mu,\mu_1,\ldots,\mu_N}^{l,l_1,\ldots,l_N}(x;y_1,\ldots,y_N)$ corresponds to the amputated part of the one-loop diagram of $N+1$ external legs. The first $D-1$ of them are regularized. Clearly

$$K_{\mu,\mu_1,\ldots,\mu_i,\ldots,\mu_j,\ldots,\mu_N}^{l,l_1,\ldots,l_i,\ldots,l_j,\ldots,l_N}(x;y_1,\ldots,y_i,\ldots,y_j,\ldots,y_N)=K_{\mu,\mu_1,\ldots,\mu_j,\ldots,\mu_i,\ldots,\mu_N}^{l,l_1,\ldots,l_j,\ldots,l_i,\ldots,l_N}(x;y_1,\ldots,y_j,\ldots,y_i,\ldots,y_N)\,. \tag{16}$$

The integrability condition (13) requires that

$$K_{\mu,\mu_1,\ldots,\mu_i,\ldots,\mu_N}^{l,l_1,\ldots,l_i,\ldots,l_N}(x;y_1,\ldots,y_i,\ldots,y_N)=K_{\mu_i,\mu_1,\ldots,\mu,\ldots,\mu_N}^{l_i,l_1,\ldots,l,\ldots,l_N}(y_i,y_1,\ldots,x,\ldots,y_N)\,, \tag{17}$$

which together with (16) implies the Bose symmetry.

III. THE INTEGRABILITY CONDITION IN $D=2$ AND IN $D=4$

In this section, we shall examine in detail the integrability condition and shall determine the functional form of Ω in $D=2$ and 4. It follows from (9) and (13) that the integrability condition implies the following equations of the vacuum expectation values:

$$\begin{aligned}
&\mathrm{Tr}T^a\tau_\mu\Omega\left[x+\frac{\epsilon}{2},x-\frac{\epsilon}{2}\right]\left\langle\psi\left[x-\frac{\epsilon}{2}\right]\bar\psi(y)\right\rangle_A T^b\tau_\nu\left\langle\psi(y)\bar\psi\left[x+\frac{\epsilon}{2}\right]\right\rangle_A\\
&\quad-\mathrm{Tr}T^b\tau_\nu\Omega\left[y+\frac{\epsilon}{2},y-\frac{\epsilon}{2}\right]\left\langle\psi\left[y-\frac{\epsilon}{2}\right]\bar\psi(x)\right\rangle_A T^a\tau_\mu\left\langle\psi(x)\bar\psi\left[y+\frac{\epsilon}{2}\right]\right\rangle_A\\
&\quad+\mathrm{Tr}T^a\tau_\mu\frac{\delta\Omega\left[x+\frac{\epsilon}{2},x-\frac{\epsilon}{2}\right]}{\delta A_\nu^b(y)}\left\langle\psi\left[x-\frac{\epsilon}{2}\right]\bar\psi\left[x+\frac{\epsilon}{2}\right]\right\rangle_A\\
&\quad-\mathrm{Tr}T^b\tau_\nu\frac{\delta\Omega\left[y+\frac{\epsilon}{2},y-\frac{\epsilon}{2}\right]}{\delta A_\mu^a(x)}\left\langle\psi\left[y-\frac{\epsilon}{2}\right]\bar\psi\left[y+\frac{\epsilon}{2}\right]\right\rangle_A=0\,.
\end{aligned} \tag{18}$$

The right-hand side (RHS) vanishes only formally as $\epsilon\to 0$. A few lower-order terms in A contribute. The expectation value of the regularized current (9) can be written as

$$\langle J_\mu^l(x)^{\text{reg}}\rangle_A=-\mathrm{Tr}T^l\Omega\left[x+\frac{\epsilon}{2},x-\frac{\epsilon}{2}\right]\tau_\mu S_A\left[x-\frac{\epsilon}{2},x+\frac{\epsilon}{2}\right] \tag{19}$$

and its covariant divergence is

$$\begin{aligned}
D_\mu\langle J_\mu^l(x)^{\text{reg}}\rangle_A=-\mathrm{Tr}\Bigg\{&\left[A_\mu\left[x+\frac{\epsilon}{2}\right]-A_\mu(x)\right]T^l\Omega\left[x+\frac{\epsilon}{2},x-\frac{\epsilon}{2}\right]\\
&+T^l\left[A_\mu(x)-A_\mu\left[x-\frac{\epsilon}{2}\right]\right]\Omega\left[x+\frac{\epsilon}{2},x-\frac{\epsilon}{2}\right]\\
&+T^l\left\{\frac{\partial}{\partial x_\mu}\Omega\left[x+\frac{\epsilon}{2},x-\frac{\epsilon}{2}\right]+\left[A_\mu\left[x-\frac{\epsilon}{2}\right]\Omega\left[x+\frac{\epsilon}{2},x-\frac{\epsilon}{2}\right]\right]\right\}\Bigg\}\tau_\mu S_A\left[x-\frac{\epsilon}{2},x+\frac{\epsilon}{2}\right],
\end{aligned} \tag{20}$$

where $S_A(x,x')=\langle \psi(x)\bar{\psi}(x')\rangle_A$ is the Green's function of the Dirac operator in a background gauge field and is given formally by

$$S_A(x,x')=\left\langle x \left| \frac{1}{\tau_\mu(\partial_\mu+A_\mu)} \right| x' \right\rangle . \tag{21}$$

The traces in (19) and (20) are taken over both the group and the spinor indices. The object inside { } is of the order of ϵ and the first $D-1$ terms of the perturbative series of (21) in terms of A contribute to the anomaly.

A. $D=2$

The most general operator insertion can be put in the form

$$\Omega\left[x+\frac{\epsilon}{2},x-\frac{\epsilon}{2}\right]=P\exp\left[-\int_{x-\epsilon/2}^{x+\epsilon/2} d\xi_\mu X_\mu(A)\right] \tag{22}$$

with $X_\mu(A)$ a local linear polynomial of A_μ. $S_A(x-\epsilon/2,x+\epsilon/2)$ can be expanded perturbatively according to the power of A:

$$S_A\left[x-\frac{\epsilon}{2},x+\frac{\epsilon}{2}\right]=S_F\left[x-\frac{\epsilon}{2},x+\frac{\epsilon}{2}\right]-\int d^2y\, S_F\left[x-\frac{\epsilon}{2},y\right]A(y)S_F\left[y,x+\frac{\epsilon}{2}\right]+\cdots , \tag{23}$$

where $S_F(x,x')$ is the Green's function of the free Dirac operator and it diverges as $O(1/\epsilon)$ as $\epsilon\to 0$. The subsequent terms of (23) are finite in the limit $\epsilon\to 0$.

Upon substituting (23) into (20) one obtains

$$\langle J_\mu^l(x)^{\text{reg}}\rangle_A=\frac{1}{4\pi}\text{Tr}T^l[X_\mu(A)-i\epsilon_{\mu\nu}X_\nu(A)]+\text{regulator-independent terms} . \tag{24}$$

The regulator-independent terms satisfy the integrability condition automatically. Thus

$$\frac{\delta}{\delta A_\nu^b(y)}\langle J_\mu^{\text{reg}}(x)^a\rangle_A-\frac{\delta}{\delta A_\mu^a(x)}\langle J_\nu^{\text{reg}}(y)^b\rangle_A=\frac{1}{4\pi}\left[\frac{\delta}{\delta A_\nu^b(y)}\text{Tr}T^a[X_\mu(A)-i\epsilon_{\mu\rho}X_\rho(A)]\right.$$
$$\left.-\frac{\delta}{\delta A_\mu^a(x)}\text{Tr}T^b[X_\nu(A)-i\epsilon_{\nu\rho}X_\rho(A)]\right] \tag{25}$$

as can be derived directly from (18).

For $X_\mu(A_\mu)=A_\mu$, which corresponds to the covariant regularization, the right-hand side of (25) does not vanish and is given by (12). The integrability can be satisfied by taking

$$X_\mu(A)=c(A_\mu+i\epsilon_{\mu\nu}A_\nu) \tag{26}$$

with c an arbitrary constant. Substituting (26) into (20), one ends up with the consistent chiral anomaly

$$D_\mu J_\mu^l=-\frac{i}{4\pi}\text{Tr}T^l(\partial_\mu A_\mu+i\epsilon_{\mu\nu}\partial_\mu A_\nu) . \tag{27}$$

The first term of (27) is trivial since

$$\int d^2x\, \theta^l\text{Tr}T^l\partial_\mu A_\mu=\delta_\theta\int d^2x\, \text{Tr}A_\mu A_\mu \tag{28}$$

and (28) can be absorbed into the effective action. One observation is that both the insertion (25) and the anomaly are functionals of $A_1+iA_2=\tau_\mu A_\mu$. As was pointed out by Jackiw[8] this dependence is more natural since the chiral Dirac operator depends only on $\tau_\mu A_\mu$. In fact, the case of $D=2$ is rather special. The chiral spinor has only one component and $J_2=iJ_1$ for the point-splitting regularization. It follows from the integrability condition (13) that

$$\left[\frac{\delta}{\delta A_1^l}+i\frac{\delta}{\delta A_2^l}\right](J_\mu^l)^{\text{reg}}=0 . \tag{29}$$

This implies that $(J_\mu^l)^{\text{reg}}$ cannot depend on $A_1^l-iA_2^l$. Since this should be true for any group, J^{reg} has to be a functional of A_1+iA_2 only. Furthermore, the current without any operator insertions, i.e., $c=0$, is already integrable. This is, however, not the case in higher dimensions.

B. $D=4$

Instructed by the result in $D=2$ one may try the point-split current without operator insertions, i.e.,

$$J_\mu^l(x)^{\text{reg}}=\bar{\psi}\left[x+\frac{\epsilon}{2}\right]T^l\tau_\mu\psi\left[x-\frac{\epsilon}{2}\right] . \tag{30}$$

The integrability condition reduces to

$$\left[\frac{\delta}{\delta A_\nu^b(y)}\langle J_\mu^a(x)^{\rm reg}\rangle-\frac{\delta}{\delta A_\mu^a(x)}\langle J_\nu^b(y)^{\rm reg}\rangle\right]=-{\rm Tr}T^a\tau_\mu S_A\left[x-\frac{\epsilon}{2},y\right]T^b\tau_\nu S_A\left[y,x+\frac{\epsilon}{2}\right]$$
$$+{\rm Tr}T^a\tau_\mu S_A\left[x,y+\frac{\epsilon}{2}\right]T^b\tau_\nu S_A\left[y-\frac{\epsilon}{2},x\right]. \tag{31}$$

Substituting the perturbative expansion of S_A,

$$S_A\left[x+\frac{\epsilon}{2},x-\frac{\epsilon}{2}\right]=S_F\left[x+\frac{\epsilon}{2},x-\frac{\epsilon}{2}\right]-\int d^4y\,S_F\left[x+\frac{\epsilon}{2},y\right]\tau_\mu A_\mu(y)S_F\left[y,x-\frac{\epsilon}{2}\right]$$
$$+\int d^4y_1\int d^4y_2 S_F\left[x+\frac{\epsilon}{2},y\right]\tau_\mu A_\mu(y_1)S_F(y_1,y_2)\tau_\nu A_\nu(y_2)S_F\left[y_2,x-\frac{\epsilon}{2}\right]+\cdots \tag{32}$$

and working in the momentum representation, one ends up with

$$\text{RHS of (31)}=-\tfrac{1}{2}\delta^4(x-y)[{\rm Tr}T^aT^b\partial_\lambda A_\rho\epsilon_\lambda S_{\mu\nu\rho}(\epsilon)-{\rm Tr}T^a\partial_\lambda A_\rho T^b\epsilon_\lambda S_{\mu\rho\nu}(\epsilon)]\,, \tag{33}$$

where

$$S_{\mu\nu\rho}(\epsilon)=\int\frac{d^4p}{(2\pi)^4}e^{-ip\epsilon}{\rm Tr}\tau_\mu S_F(p)\tau_\nu S_F(p)\tau_\rho S_F(p) \tag{34}$$

the trace in (33) is taken over the group indices and that in (34) over the spinor indices. In deriving (33), we have dropped the terms which vanish as $\epsilon\to 0$. Upon average over all directions of ϵ one ends up with

$$\epsilon_\lambda S_{\mu\nu\rho}(\epsilon)=-\frac{1}{48\pi^2}(\delta_{\mu\nu}\delta_{\rho\lambda}+\delta_{\mu\rho}\delta_{\nu\lambda}+\delta_{\mu\lambda}\delta_{\nu\rho})+\frac{1}{16\pi^2}\epsilon_{\mu\nu\rho\lambda} \tag{35}$$

and

$$\frac{\delta}{\delta A_\nu^b(y)}\langle J_\mu^a(x)^{\rm reg}\rangle-\frac{\delta}{\delta A_\mu^a(x)}\langle J_\nu^b(y)^{\rm reg}\rangle=-\frac{\delta^4(x-y)}{96\pi^2}{\rm Tr}[[T^a,T^b](\partial_\mu A_\nu+\partial_\nu A_\mu+\delta_{\mu\nu}\partial,A)+3\epsilon_{\mu\nu\rho\lambda}\{T^a,T^b\}\partial_\rho A_\lambda]\,. \tag{36}$$

The regularized current can be easily made integrable by adding a term

$$X_\mu^l(x)=-\frac{1}{288\pi^2}{\rm Tr}T^l([\partial_\lambda A_\mu,A_\lambda]+[\partial_\lambda A_\lambda,A_\mu]+[\partial_\mu A_\lambda,A_\lambda]+3\epsilon_{\mu\lambda\alpha\beta}\{A_\alpha,\partial_\lambda A_\beta\}+3\epsilon_{\mu\lambda\alpha\beta}\{A_\alpha,\partial_\lambda A_\beta\})\,. \tag{37}$$

This is equivalent with the following regularized current with the operator insertion:

$$J^{\rm reg}(x,\epsilon)_\mu^l=\left\langle\bar\psi\left[x+\frac{\epsilon}{2}\right]T^l\tau_\mu\exp[-\tfrac{1}{24}\epsilon_\mu\epsilon_\nu\epsilon_\rho(\partial_\rho\mathcal{A}_\mu\mathcal{A}_\nu-\mathcal{A}_\mu\partial_\rho\mathcal{A}_\nu)]\psi\left[x-\frac{\epsilon}{2}\right]\right\rangle_A\,, \tag{38}$$

where

$$\mathcal{A}_\mu=\tau_\mu^\dagger\tau_\nu A_\nu \tag{39}$$

the chiral anomaly deduced from (38) reads

$$D_\mu J_\mu^l(x)^{\rm reg}=\frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\lambda}{\rm Tr}T^l\partial_\mu(A_\nu\partial_\rho A_\lambda+\tfrac{1}{2}A_\nu A_\rho A_\lambda)+{\rm Tr}T^l\Gamma\,, \tag{40}$$

where

$$\Gamma=-\frac{1}{32\pi^2\epsilon^2}\partial_\mu A_\mu-\frac{1}{192\pi^2}\partial^2\partial_\mu A_\mu-\frac{1}{576\pi^2}[A_\mu,2\partial_\mu\partial_\nu A_\nu+\partial^2A_\mu]$$
$$-\frac{1}{48\pi^2}\{A_\mu A_\nu,\partial_\mu A_\nu+\partial_\nu A_\mu\}-\frac{1}{48\pi^2}\{A^2,\partial_\mu A_\mu\}\,. \tag{41}$$

The contribution of (41) to the anomaly can be written as the gauge variation of the following local functional:

$$S=-\int dx^4{\rm Tr}\left[\frac{1}{16\pi^2\epsilon^2}A^2+\frac{1}{1152\pi^2}A_\mu\partial^2A_\mu+\frac{1}{567\pi^2}A_\mu\partial_\mu\partial_\nu A_\nu+\frac{1}{36\pi^2}(A_\mu A_\nu A_\mu A_\nu+2A^4)\right], \tag{42}$$

i.e.,

$$\mathrm{Tr}\int d^4x\,\theta\Gamma=\delta_\theta S\ . \tag{43}$$

Therefore, Γ is a cohomologically trivial term and can be absorbed into the effective action.

IV. THE POINT-SPLITTING REGULARIZATION OF THE ACTION

The point-splitting regularization discussed above is restricted to the definition of the composite operator alone. The radiative corrections are still calculated by the unregularized Feynman rules. Although the integrability can be established by a careful choice of the operator insertion into the point-split current, the integrability condition will be satisfied automatically if one regularizes the classical action and defines the current as the functional derivative of the regularized action. In this way all the Feynman rules are modified systematically. One example of such a regularization is the lattice formulation,[9] in which the fermion fields in the bilinear products of the kinetic energy and the interaction part are defined at different lattice sites. Such a regularization preserves the gauge invariance but causes the species doubling due to the point splitting of the kinetic energy. A possible point-split action in the continuum reads

$$S_f^{\mathrm{reg}}=\int d^Dx\left[\bar\psi(x)\tau_\mu\partial_\mu\psi(x)+\bar\psi\left[x+\frac{\epsilon}{2}\right]\tau_\mu A_\mu(x)\psi\left[x-\frac{\epsilon}{2}\right]\right], \tag{44}$$

where the kinetic energy term remains unregularized. It is easy to show that this regularization is sufficient to remove all the ultraviolet divergences due to the fermion loops with a finite ϵ. Indeed, for a fermion loop with l external legs, there will be at most $l-1$ fermion propagators which diverge like x^{-D+1} (i.e., the two ends of the propagator coincide). This singular product will be integrated over $l-1$ relative coordinates and the resulting degrees of divergence is $(l-1)D-(l-1)(D-1)=l-1<0$. Therefore all fermion loops are finite after integration.

The current operator associated with the action (41) is

$$J_\mu^l(x,\epsilon)=\bar\psi\left[x+\frac{\epsilon}{2}\right]T^l\tau_\mu\psi\left[x-\frac{\epsilon}{2}\right]. \tag{45}$$

It appears that the form of the current is the same as that defined in the previous section without any operator insertions. But the vacuum expectation value of (45) is quite different from that of (30) since the Feynman rules for the perturbative expansion are modified. The fermion propagator from the regulated action is

$$S_A(x,x')=\int\frac{d^Dp}{(2\pi)^D}\int\frac{d^Dp'}{(2\pi)^D}e^{ipx-ip'x'}S_A(p,p') \tag{46}$$

with the perturbative expansion

$$\begin{aligned}S_A(p,p')=&(2\pi)^D\delta^D(p-p')S_F(p)-\exp\left[-\frac{i}{2}(p+p')\epsilon\right]S_F(p)\tau_\mu A_\mu(p-p')S_F(p')\\&+\int\frac{d^Dq}{(2\pi)^D}\exp\left[-\frac{i}{2}p\epsilon-iq\epsilon-\frac{i}{2}p'\epsilon\right]S_F(p)\tau_\mu A_\mu(p-q)S_F(q)\tau_\nu A_\nu(q-p')S_F(p')-\cdots\ .\end{aligned} \tag{47}$$

The divergence of the regularized current is

$$\begin{aligned}D_\mu J_\mu^l(x)^{\mathrm{reg}}=\mathrm{Tr}\tau_\mu\Bigg\{&[A_\mu(x+\epsilon)-A_\mu(x)]T^lS_A\left[x-\frac{\epsilon}{2},x+\frac{3}{2}\epsilon\right]-T_l[A_\mu(x+\epsilon)-A_\mu(x)]S_A\left[x-\frac{3}{2}\epsilon,x+\frac{\epsilon}{2}\right]\\&+A_\mu(x)T^l\left[S_A\left[x-\frac{\epsilon}{2},x+\frac{3}{2}\epsilon\right]-S_A\left[x-\frac{\epsilon}{2},x+\frac{\epsilon}{2}\right]\right]\\&-T^lA_\mu(x)\left[S_A\left[x-\frac{3}{2}\epsilon,x+\frac{\epsilon}{2}\right]-S_A\left[x-\frac{\epsilon}{2},x+\frac{\epsilon}{2}\right]\right]\Bigg\},\end{aligned} \tag{48}$$

where we have used the following Dirac equations:

$$\left\langle\left[-\frac{\partial}{\partial x_\mu}\bar\psi(x)\tau_\mu+\bar\psi(x+\epsilon)\tau_\mu A_\mu\left[x+\frac{\epsilon}{2}\right]\right]\cdots\right\rangle=0 \tag{49}$$

and

$$\left\langle\left[\tau_\mu\frac{\partial}{\partial x_\mu}\psi(x)+\tau_\mu A_\mu\left[x-\frac{\epsilon}{2}\right]\psi(x-\epsilon)\right]\cdots\right\rangle=0\ . \tag{50}$$

Substituting (46) and (47) into (48) one obtains the Wess-Zumino consistent anomaly

$$D_\mu \langle J^l_\mu(x)\rangle^{\text{reg}}_A = \frac{1}{4\pi}(\delta_{\mu\nu}+i\epsilon_{\mu\nu})\text{Tr}T^l\partial_\mu A_\nu \quad (51)$$

for $D=2$ and

$$D_\mu \langle J^l_\mu(x)\rangle^{\text{reg}}_A = \frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\lambda}\text{Tr}T^l\partial_\mu(A_\nu\partial_\rho A_\lambda+\tfrac{1}{2}A_\nu A_\rho A_\lambda) + \text{Tr}^l\left[\frac{1}{4\pi^2\epsilon^2}\frac{\partial A_\mu}{\partial x_\mu}+\frac{1}{192\pi^2}\partial^2\partial_\mu A_\mu+\frac{1}{576\pi^2}[A_\mu(2\partial_\mu\partial_\nu A_\nu+\partial^2 A_\mu)] + \frac{1}{48\pi^2}\{A_\mu A_\nu,\partial_\mu A_\nu+\partial_\nu A_\mu\}+\frac{1}{48\pi^2}\{A^2,\partial_\mu A_\mu\}\right] \quad (52)$$

for $D=4$. This expression is identical to that of (40) even for the cohomologically trivial part. An explicit expression of the effective action which will produce the above anomalies can be obtained by expanding the regularized path integration:

$$e^{-W(A)}=\int D\psi D\bar{\psi}\, e^{-S_f^{\text{reg}}(A,\psi,\bar{\psi})} \quad (53)$$

with S_f^{reg} given by (44).

The scheme dependence of the chiral anomaly is a reflection of the inconsistency of the ordinary renormalization method in an anomalous gauge theory. Different currents give rise to anomalies of distinct mathematical structures and they cannot be transformed to each other by adding counterterms to the action. The long-wavelength behavior is sensitive to the detail of the short-distance cutoff. The theory cannot be quantized by the usual procedure and new ingredients are needed. The covariant anomaly, due to its covariance, may be relevant to some physical process. The consistent anomaly is a direct implication of the noninvariance of the effective action under a gauge transformation and may indicate some structure of the quantized theory if it exists. The possibility of quantizing an anomalous gauge theory has been seriously considered by several authors[10] and it will detect the fundamental nature of quantum field theories.

ACKNOWLEDGMENTS

We are grateful to Professor Nair for his critical reading of the manuscript and for his valuable comments. One of the authors (H.C.R.) would like to thank Professor Fishman and Professor Y.S. Wu for conversations. The work of H.C.R. was supported in part by the U.S. Department of Energy under Contract No. DE-AC02-87ER40325.B000.

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