Feynman rules for the δ expansion

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In this paper we further develop a novel perturbation scheme for quantum field theory in which the form of the interaction is expanded about the free theory. A set of Feynman rules for the perturbation expansion is derived that does not require the introduction of the provisional Lagrangian used in earlier papers. The rules permit direct calculation to all orders in the δ expansion, and are almost identical to standard Feynman rules. Only the form of the propagator and the vertex function are changed.

I. INTRODUCTION

In a recent series of papers[1–4] we introduced and elaborated upon a new perturbative approach to quantum field theory. This approach relies upon the introduction of an artificial perturbation parameter that allows the interacting theory to be expanded about the free, noninteracting theory. For example, a scalar polynomial field theory with an interaction term of the form $\lambda(\phi^2)^P$ we rewrite as $\lambda(\phi^2)^{1+\delta}$ and consider δ as the perturbation parameter. In the resulting δ expansion, the Green's functions of the interacting field theory are obtained as power series in δ with coefficients that can be calculated by ordinary diagrammatic techniques. The n-point Green's function has an expansion of the form

$$G^{(n)}(x_1,x_2,\ldots,x_n;\delta)=\sum_{k=0}^{\infty}\delta^k g_k^{(n)}(x_1,x_2,\ldots,x_n) . \tag{1.1}$$

In earlier papers we discussed and illustrated a number of calculational advantages enjoyed by this new perturbation scheme. The results obtained in the δ expansion have a more complicated analytic dependence on the natural parameters of the theory (masses and coupling constants) than emerges from weak- and strong-coupling perturbation theory. In massless theories this technique can provide a natural way to remove the infrared divergences because, even in a massless theory, the lowest-order term in the δ expansion, corresponding to the free theory, typically has the form of a mass term. In field theories where there is no natural small perturbation parameter this technique offers a means for developing a perturbation series in powers of an artificial parameter. For example, even though a supersymmetric Lagrangian contains no natural perturbation parameter one can develop a δ-perturbation expansion for the ground-state energy.[3] Finally, in the sample problems that we have discussed, the δ expansion appears to produce series expansions for the Green's functions that are less divergent than are those in conventional weak-coupling perturbation expansions.

The apparent disadvantage of our technique is that one must use interaction Lagrangians containing powers of logarithms of the fields. To deal with these Lagrangians we developed a complicated scheme that allows us to calculate the coefficients of each term in the δ expansion by conventional weak-coupling techniques applied to a provisional Lagrangian. The calculation at each order in δ requires the introduction of a new provisional Lagrangian with polynomial interaction terms. The provisional Lagrangian for the δ^n calculation contains n interaction terms. After developing the weak-coupling expansion for the Green's functions of this provisional Lagrangian one extracts the coefficients in the δ expansion by application of an nth-order differential operator at the point $\delta=0$ (the free-theory limit).

In this paper we show how to calculate an arbitrary order in the δ expansion in a much simpler way that avoids altogether the introduction of the provisional Lagrangian. In this new method the number of diagrams that must be evaluated in each order is enormously reduced and the differential operator that is employed is trivial.

In Sec. II we will briefly review the details of the δ expansion technique as developed in previous papers, using the scalar self-interacting $\lambda(\phi^2)^P$ theory as an example. In Sec. III we will derive the new technique by calculating an arbitrary Green's function for the $\lambda(\phi^2)^P$ theory to fourth order in δ. We show how to generalize the result to any order in δ in Sec. IV.

The calculations in Secs. II and III are quite involved and require a very careful application of the δ-expansion techniques developed in Refs. 1 and 2. The purpose of these calculations is to establish a new and much simpler technique for developing the δ expansion. This result, which relies on a set of rules similar to standard Feynman rules, is summarized in Sec. IV. The reader who is unconcerned with following the fine details of establishing the equivalence between the two techniques may briefly review the δ expansion in Sec. II, follow the discussion of the one-vertex graphs in Sec. III A, and skip to the summary of the new technique in Sec. IV. The application of the technique to the calculation of the δ expansion for the two-point function for scalar field theory through second order, which is given in Sec. IV, will then be easy to follow.

II. REVIEW OF THE δ EXPANSION

For definiteness we shall review the δ expansion in the context of the Lagrangian[2]

$$L=\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}\mu^2\phi^2+\lambda M^2\phi^2(M^{2-d}\phi^2)^{\delta}\ , \tag{2.1}$$

where d is the space-time dimension, μ is the bare mass, and M is an arbitrary mass parameter introduced so that the coupling constant λ is dimensionless. We regard δ as a small positive parameter and expand (2.1) in powers of δ:

$$L=\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}(\mu^2+2\lambda M^2)\phi^2 +\lambda M^2\phi^2\sum_{k=1}^{\infty}\frac{\delta^k}{k!}[\ln(\phi^2M^{2-d})]^k\ . \tag{2.2}$$

The coefficient of δ^0 in (1.1) is simply the n-point Green's function of a free theory with bare mass $\mu^2+2\lambda M^2$. To calculate the other coefficients in the δ expansion of the Green's function $G^{(n)}(x_1,\ldots,x_n;\delta)$ in (1.1) through order δ^K we first introduce a provisional Lagrangian, specific to this order, that involves K separate polynomial interaction terms, namely,

$$\tilde{L}_K=\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}(\mu^2+2\lambda M^2)\phi^2 +2\lambda M^d\sum_{k=1}^{K}(\phi^2M^{2-d})^{\alpha_k+1}P_k\ , \tag{2.3}$$

where the coefficients P_k are polynomials in δ and $\{\alpha_k\}$, and the α_k are initially regarded as integers.

In Ref. 2 the specific forms of the polynomials P_k are given for $k\leq 4$. For example, for $K=1$, $P_1=\delta$, and for $K=4$ the four polynomials are

$$P_1=\delta+[\tfrac{1}{3}+\tfrac{1}{6}(\alpha)_S+\tfrac{1}{2}(\alpha^2)_U]\delta^2 +\tfrac{1}{9}(4+5\alpha_1)\delta^3+\delta^4\ , \tag{2.4a}$$

$$P_2=i\delta+\left[-\frac{1}{3}-\frac{i}{6}(\alpha)_S+\frac{i}{2}(\alpha^2)_U\right]\delta^2 +\tfrac{1}{9}(-4i+5\alpha_2)\delta^2+\delta^4\ , \tag{2.4b}$$

$$P_3=-\delta+[\tfrac{1}{3}-\tfrac{1}{6}(\alpha)_S-\tfrac{1}{2}(\alpha^2)_U]\delta^2 +\tfrac{1}{9}(-4+5\alpha_3)\delta^3+\delta^4\ , \tag{2.4c}$$

$$P_4=-i\delta+\left[-\frac{1}{3}+\frac{i}{6}(\alpha)_S-\frac{1}{2}(\alpha^2)_U\right]\delta^2 +\tfrac{1}{9}(4i+5\alpha_4)\delta^3+\delta^4\ . \tag{2.4d}$$

Here $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ are the four exponents that appear in the provisional Lagrangian (2.3) for the case $K=4$ and $(\ )_S$ and $(\ )_U$ are two of the four special functional forms involving the four roots of 1, which are defined in the Appendix.

Using the Lagrangian (2.3) one calculates all diagrams having at most K vertices that contribute to $\tilde{G}^{(n)}$, the n-point function for $\tilde{L}$. One now regards the parameters α_k as continuous, applies a specific Kth-order derivative operator to the Green's function $\tilde{G}^{(n)}$, and takes the limit $\alpha_k\to 0$, $k=1,2,\ldots,K$. The result is $G^{(n)}$ for the Lagrangian (2.1) exact to order δ^K.

For $K=1$ the derivative operator is $D_1=\partial/\partial\alpha_1$. For larger values of K the derivative operator is more complicated. It is convenient to use the notation employed in (2.4) and explained in the Appendix to write, $K=4$,

$$D_4=\frac{1}{4}\left[\frac{\partial}{\partial\alpha}\right]_U+\frac{1}{8}\left[\frac{\partial^2}{\partial\alpha^2}\right]_A+\frac{1}{24}\left[\frac{\partial^3}{\partial\alpha^3}\right]_T +\frac{1}{96}\left[\frac{\partial^4}{\partial\alpha^4}\right]_S\ . \tag{2.5}$$

The use of the rules given above to perform calculations in example field theories has been illustrated in Refs. 1–4.

III. GENERAL FOURTH-ORDER CALCULATION

We now calculate, through fourth order in the δ expansion, the $2n$-point Green's function $G^{(2n)}$ for the Lagrangian (2.1). In the course of this calculation we will discover a new and simplified set of rules for the δ expansion that avoids the introduction of the provisional Lagrangian.

Notice from (2.4) that each of the polynomials P_k contains terms of order δ and higher. Therefore each of the interaction terms in the provisional Lagrangian (2.3) is at least proportional to δ. Thus the expansion of the Green's functions through order δ^4 can involve diagrams with at most four vertices.

We now consider the calculation of the $2n$-point Green's function $\tilde{G}^{(2n)}$ for the Lagrangian $\tilde{L}_K$ in (2.3). For the case $K=4$ the provisional Lagrangian $\tilde{L}_K$ contains four distinct interaction terms of the form $2\lambda M^d(\phi^2M^{2-d})^{\alpha_k+1}P_k$, $k=1,\ldots,4$. Initially we will choose the α_k so that $n<\alpha_k+1$ (later we will define a specific analytic continuation of $\tilde{G}^{(2n)}$ in the α_k). With this assumption every graph with $2n$ external legs and with one, two, three, or four vertices will contribute to the Green's function $\tilde{G}^{(2n)}$. To organize the discussion that follows we will consider separately the contributions to the $2n$-point function from graphs with one, two, three, and four vertices.

A. One-vertex graphs

Because the polynomials P_k are fourth order in δ, the one-vertex graphs in Fig. 1 produce contributions to $\tilde{G}^{(2n)}$ of order δ, δ^2, δ^3, and δ^4. We treat δ as a small parameter and calculate the contribution of these graphs in weak-coupling perturbation theory. The Feynman rules are as follows.

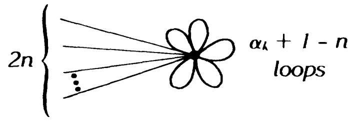

FIG. 1. The one-vertex graphs that contribute to $\tilde{G}^{(2n)}$.

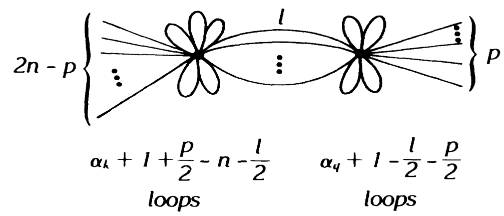

FIG. 2. The two-vertex graphs that contribute to $\tilde{G}^{(2n)}$.

For a boson propagator:

$$\frac{1}{p^2+\mu^2+2\lambda M^2} . \tag{3.1a}$$

For the α_k vertex:

$$2\lambda M^d P_k(2\alpha_k+2)!M^{(2-d)(\alpha_k+1)} . \tag{3.1b}$$

We first consider the order-δ terms in this calculation. The α_1 vertex contribution in order δ is

$$-\frac{\delta 2\lambda(2\alpha_1+2)!I^{\alpha_1+1-n}M^{d+(2-d)(\alpha_1+1)}}{(\alpha_1+1-n)!2^{\alpha_1+1-n}} \equiv -\delta v_n(\alpha_1) . \tag{3.2}$$

Here I is the boson loop integral given by

$$I=\frac{1}{(4\pi)^{d/2}}\int\frac{d^dp}{p^2+\mu^2+2\lambda M^2} . \tag{3.3}$$

(For $d \geq 2$ the loop integral must be suitably regulated, but the following arguments are not sensitive to how this is done.)

The α_2 vertex contribution in order δ is $-i\delta v_n(\alpha_2)$, the α_3 contribution is $+\delta v_n(\alpha_3)$, and the α_4 contribution is $+i\delta v_n(\alpha_4)$.

The sum of the order-δ contributions of the four graphs in Fig. 1 is

$$-\delta H_{2n}^{(1)} \equiv -\delta[v_n(\alpha_1)+iv_n(\alpha_2)-v_n(\alpha_3)-iv_n(\alpha_4)] .$$

Comparing to the forms defined in the Appendix we see that this can be written

$$-\delta H_{2n}^{(1)} = -\delta(v_n(\alpha))_T . \tag{3.4a}$$

Similarly, one finds that the order-δ^2 contributions from the one-vertex graphs, which we label $-\delta^2 H_{2n}^{(2)}$, can be written as

$$-\delta^2 H_{2n}^{(2)} = -\delta^2[\tfrac{1}{3}(v_n(\alpha))_A+\tfrac{1}{6}(\alpha)_S(v_n(\alpha))_U + \tfrac{1}{2}(\alpha^2)_U(v_n(\alpha))_T] . \tag{3.4b}$$

The δ^3 contributions are

$$-\delta^3 H_{2n}^{(3)} = -\delta^3[\tfrac{4}{9}(v_n(\alpha))_U+\tfrac{5}{9}(\alpha v_n(\alpha))_S] , \tag{3.4c}$$

and the δ^4 contributions are

$$-\delta^4 H_{2n}^{(4)} = -\delta^4(v_n(\alpha))_S . \tag{3.4d}$$

We now apply the derivative operator D_4 in (2.5) to the expressions (3.4) and use the rules given in the Appendix for the effect of the derivative forms $(\partial^l/\partial\alpha^l)_{S,T,U,A}$ on the functional forms $(f(\alpha))_{S,T,U,A}$. The result is

$$\text{one-vertex contribution} = -\delta v_n'(0) - \frac{\delta^2}{2!}v_n''(0) - \frac{\delta^3}{3!}v_n'''(0) - \frac{\delta^4}{4!}v_n^{(4)}(0) , \tag{3.5}$$

where $v_n(\alpha)$ is defined in (3.2) and

$$v^{(l)}(0) = \left.\frac{d^l}{d\alpha^l}v(\alpha)\right|_{\alpha=0} . \tag{3.6}$$

It is clear from the calculation given above that in order δ^K the one-vertex graphs will contribute to the n-point Green's function as follows:

$$G^{(n)} = -\sum_{k=1}^{K}\frac{\delta^k}{k!}v_n^{(k)}(0) . \tag{3.7}$$

B. Two-vertex graphs

Because the polynomials P_k in (2.4) are at least of order δ, the two-vertex graphs produce contributions to $\tilde{G}^{(2n)}$ of order δ^2, δ^3, δ^4, and higher. The general two-vertex graph is given in Fig. 2. Note that both the left and right vertices may be labeled by any of the four α_k corresponding to the four interaction terms in the Lagrangian (2.3). There are $2n$ external lines. The left vertex, labeled by α_k, has a total of $2\alpha_k+2$ lines, $2n-p$ of which are external and l of which are connecting lines, leaving $\alpha_k+1+p/2-l/2-n$ boson loops. The right vertex, labeled by α_q, has a total of $2\alpha_q+2$ lines, p of which are external, and l of which are connecting lines, leaving $\alpha_q+1-l/2-p/2$ boson loops. To obtain the contribution of all two-vertex graphs to $\tilde{G}^{(2n)}$ we must sum over all values of $l \geq 2$; $2n \geq p \geq 0$; $1 \leq k,q \leq 4$, subject to the constraint $p+l=$even.

The graph in Fig. 2, for given values of k,q,p,l, contributes

$$\left[\frac{(2n)!}{(2n-p)!p!}\right]\left[\frac{2\lambda P_k M^{d+(d-2)(\alpha_k+1)}}{\left[\alpha_k+1+\frac{p}{2}-n-\frac{l}{2}\right]!}\frac{(2\alpha_k+2)!}{2^{\alpha_k+1+p/2-n-l/2}}I^{\alpha_k+1+p/2-n-l/2}\right]\frac{G_l\epsilon_{qk}}{l!}$$
$$\times\left[\frac{2\lambda P_q M^{d+(d-2)(\alpha_q+1)}(2\alpha_q+2)!}{\left[\alpha_q+1-\frac{l}{2}-\frac{p}{2}\right]2^{\alpha_q+1-l/2-p/2}}\right]I^{\alpha_q+1-l/2-p/2}, \tag{3.8}$$

where $\epsilon_{kq}=1$ if $k\neq q$ and $\epsilon_{kq}=\frac{1}{2}$ if $k=q$. The first factor in (3.8) is a symmetry factor associated with the external lines. The second factor is associated with the α_k vertex and includes the vertex strength, the symmetry factor, and boson self-loop integrals. Similarly the last factor is associated with the α_q vertex. The factor $G_l/l!$ is the internal loop integral and symmetry factor associated with the l connecting lines, where

$$G_l=\int d^dx\,\Delta^l(x)\ . \tag{3.9}$$

The factor ϵ_{qk} accounts for the difference in the symmetry factors when $q=k$.

It is important in performing the summation over k,q,p,l to first do the k and q sums holding the variables p and l fixed. Performing the k and q sums and using the factor ϵ_{qk} one obtains

$$(\delta H^{(1)}_{2n-p+l}+\delta^2H^{(2)}_{2n-p+l}+\delta^3H^{(3)}_{2n-p+l}+\delta^4H^{(4)}_{2n-p+l})(\delta H^{(1)}_{p+l}+\delta^2H^{(2)}_{p+l}+\delta^3H^{(3)}_{p+l}+\delta^4H^{(4)}_{p+l})\ , \tag{3.10}$$

where the $H^{(i)}_m$ are defined in (3.4) and where we have suppressed the factor $[(2n)!/(2n-p)!p!](G_l/l!)$. Expanding (3.10) and retaining only terms of order δ^4 or lower we get

$$\delta^2H^{(1)}_{2n-p+l}H^{(1)}_{p+l}+\delta^3(H^{(1)}_{2n-p+l}H^{(2)}_{p+l}+H^{(1)}_{p+l}H^{(2)}_{2n-p+l})+\delta^4(H^{(2)}_{2n-p+l}H^{(2)}_{p+l}+H^{(1)}_{2n-p+l}H^{(3)}_{p+l}+H^{(1)}_{p+l}H^{(3)}_{2n-p+l})\ . \tag{3.11}$$

Next, apply the operator D_4 to (3.11) and again use the rules given in the Appendix. The result, after setting the $\alpha_i=0$, is

$$\text{two-vertex contribution}=\delta^2[v'_{2n-p+l}(0)v'_{p+l}(0)]+\delta^3\left[\frac{v''_{2n-p+l}(0)v'_{p+l}(0)}{2!}+\frac{v''_{p+l}(0)v'_{2n-p+l}(0)}{2!}\right]$$
$$+\delta^4\left[\left[\frac{v'''_{2n-p+l}(0)v'_{p+l}(0)}{3!}+\frac{v'''_{p+l}(0)v'_{2n-p+l}(0)}{3!}\right]+\frac{v''_{2n-p+l}(0)}{2!}\frac{v''_{p+l}(0)}{2!}\right]. \tag{3.12}$$

The result in (3.12) must be multiplied by the symmetry factor and loop integral factor $[(2n)!G_l/l!]/(2n-p)!p!$ and the result summed over p and l to obtain the full two-vertex contribution to the δ expansion of the $2n$-point Green's function $G^{(2n)}$.

Note that (3.12) can be rewritten in a factored form whose relation to (3.7) and whose generalization to arbitrary order is obvious:

$$\left[\delta v'_{2n-p+l}(0)+\frac{\delta^2}{2!}v''_{2n-p+l}(0)+\frac{\delta^3}{3!}v'''_{2n-p+l}(0)\right]\left[\delta v'_{p+l}(0)+\frac{\delta^2}{2!}v''_{p+l}(0)+\frac{\delta^3}{3!}v'''_{p+l}(0)\right]. \tag{3.13}$$

C. Three-vertex graphs

These graphs produce contributions to $\tilde{G}^{(2n)}$ of order δ^3, δ^4, and higher. The general three-vertex graph is given in Fig. 3, where any of the three vertices may be labeled by any of the α_k, $k=1,\ldots,4$. As in the two-vertex graph of Fig. 2, this general graph is labeled by the total number of external lines $2n$ and a set of additional external and internal line numbers l,m,p,q,r which must be constrained so that the total number of lines at each vertex is an even number: $2\alpha+2$. The graph must also be one-particle irreducible. Thus there are constraints that $rlm\neq0$, $q+k+m=$even, and $p+r+l=$even.

The value of the graph in Fig. 3 is calculated by the same technique as above. Its general structure is as follows:

$$(\text{external symmetry factor})\times(\text{connecting-loop integrals and symmetry factors})$$
$$\times(\text{product of three-vertex contributions, boson self-loop integrals ,}$$
$$\text{and symmetry factors})\ . \tag{3.14}$$

The first two factors are standard to any perturbation theory calculation. Only the last factor depends upon the α_i. It is a straightforward but tedious calculation to show that, after performing the i,j,k sums, this factor reduces to

$$-(\delta H^{(1)}_{p+r+l}+\delta^2 H^{(2)}_{p+r+l}+\delta^3 H^{(3)}_{p+r+l}+\delta^4 H^{(4)}_{p+r+l})(\delta H^{(1)}_{q+r+m}+\delta^2 H^{(2)}_{q+r+m}+\delta^3 H^{(3)}_{q+r+m}+\delta^4 H^{(4)}_{q+r+m})$$
$$\times(\delta H^{(1)}_{2n-p-q+l+m}+\delta^2 H^{(2)}_{2n-p-q+l+m}+\delta^3 H^{(3)}_{2n-p-q+l+m}+\delta^4 H^{(4)}_{2n-p-q+l+m}) . \quad (3.15)$$

Expanding (3.15) and retaining only terms of order δ^4 or lower yields

$$-\delta^3 H^{(1)}_{p+r+l}H^{(1)}_{q+r+m}H^{(1)}_{2n-p-q+l+m}$$
$$-\delta^4(H^{(2)}_{p+r+l}H^{(1)}_{q+r+m}H^{(1)}_{2n-p-q+l+m}+H^{(2)}_{q+r+m}H^{(1)}_{p+r+l}H^{(1)}_{2n-p-q+l+m}+H^{(2)}_{2n-p-q+l+m}H^{(1)}_{p+r+l}H^{(1)}_{q+r+m}) . \quad (3.16)$$

Applying the operator D_4 to (3.16), using the rules given in the Appendix, and setting the α's$=0$ gives

$$\text{three-vertex contribution}=-\delta^3[v'_{2n-p-q+l+m}(0)v'_{q+r+m}(0)v'_{p+r+l}(0)]$$
$$-\delta^4\frac{1}{2!}[v'_{2n-p-q+l+m}(0)v'_{q+r+m}(0)v''_{p+r+l}(0)+v'_{2n-p-q+l+m}(0)v''_{q+r+m}(0)v'_{p+r+l}(0)$$
$$+v''_{2n-p-q+l+m}(0)v'_{q+r+m}(0)v'_{p+r+l}(0)] . \quad (3.17)$$

This result must be multiplied by the symmetry factors and connecting-loop integral factors mentioned above and the results summed over p,q,l,m (subject to the constraints discussed previously[5]) to obtain the full three-vertex contribution to the δ expansion of the $2n$-point Green's function $G^{(2n)}$. As expected, (3.17) can be rewritten as the product of three truncated Taylor series in a form analogous to (3.13) and (3.7).

D. Four-vertex graphs

These graphs produce contributions to $\tilde{G}^{(2n)}$ of order δ^4 and higher. The general four-vertex graph is given in Fig. 4. The vertices may be labeled by any of the α_k, $k=1,\ldots,4$. This general graph has $2n$ external lines. It is labeled by a set of external line numbers p,q,l which can take a variety of values and is further labeled by internal line numbers i,j,k,m,r,x. Note that some of these indices can assume the value zero, so long as the graph remains one-particle irreducible. The constraints on these line numbers are similar to those obtained in the two- and three-vertex cases but are irrelevant for the present discussion. The contribution of a general four-vertex graph has the same structure as in the expression (3.14). For the purpose of the δ expansion only the dependence on the α_i is important. The external symmetry factor, the symmetry factors associated with the connecting lines, and the connecting-line loop integrals are not important as they do not depend on the parameter α_i.

The calculation follows exactly the same procedure as in the other subsections above. Apart from external symmetry factors and loop-integral factors, the result is

$$\delta^4 v'_{p+i+j+x}(0)v'_{q+j+k+r}(0)v'_{2n-q-p-l+k+x+m}(0)$$
$$v'_{l+m+i+r}(0) . \quad (3.18)$$

After calculation of the symmetry and loop factors and summation, subject to the constraints[5] on i,j,k,l,m,p,q,r,x (3.18) gives the full four-vertex contribution to the δ expansion of the $2n$-point Green's function $G^{(2n)}$.

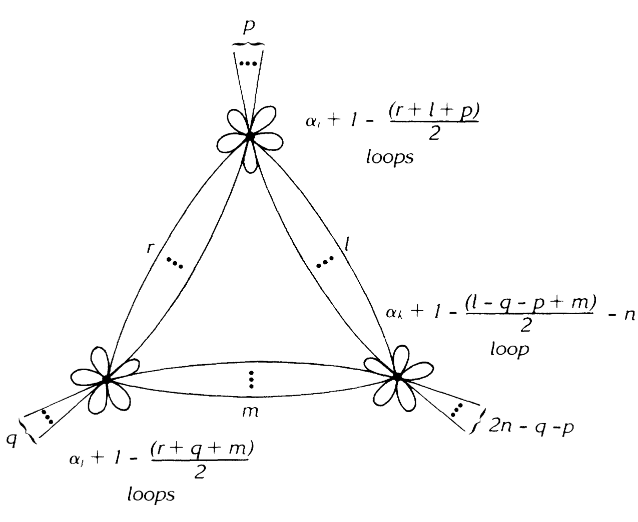

FIG. 3. The three-vertex graphs that contribute to $\tilde{G}^{(2n)}$.

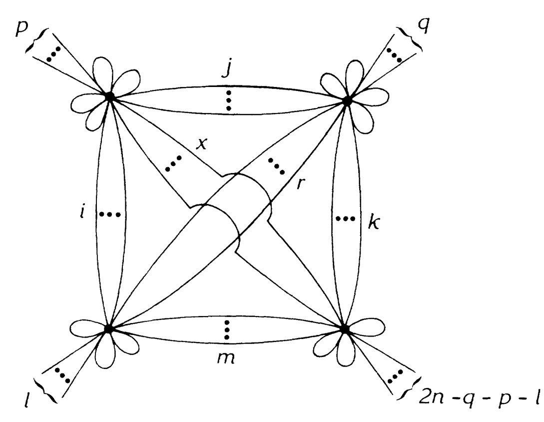

FIG. 4. The four-vertex graphs that contribute to $\tilde{G}^{(2n)}$.

IV. SUMMARY OF FEYNMAN RULES FOR THE δ EXPANSION

We now summarize the rules established in the previous section for developing the δ expansion for the Lagrangian (2.1). The rules for calculating any given graph are identical to the standard rules for a scalar field theory, except for the vertex factor. In the following rules, what differs from standard perturbation theory is the prescription for which graphs are to be included, the specific form of the vertex factors, and the mass term in the propagator.

To determine what graphs contribute to the $2n$-point function $G^{(2n)}$ we first note that δ, in the interaction term $(\phi^2)^{1+\delta}$, can be an arbitrarily large integer. To obtain the expansion of $G^{(2n)}$ to order δ^k we include all one-particle-irreducible diagrams with up to k vertices, $2n$ external lines, and *any number of internal lines.* Each of the diagrams with j vertices contributes to order δ^j and higher in the δ expansion. To obtain the contribution of a given j-vertex diagram to the order δ^k in the δ-expansion one must use the power-series expansion of each vertex factor $v_{2l}(\delta)$ in the diagram:

$$v_{2l}(\delta)=\sum_{m=1}\frac{\delta^m}{m!}v_{2l}^{(m)}(0)\ .$$

One then retains all terms of order δ^k or lower in the expansion of the full diagram.

Finally, recall that the expansion of the interaction term $\lambda M^2\phi^2(M^{2-d}\phi^2)^\delta$ in (2.1) about the point $\delta=0$ introduces a term $\lambda\phi^2M^2$ in the free Lagrangian. This term shifts the bare mass μ^2 so that the propagator that enters the δ expansion calculations is $(p^2+\mu^2+2\lambda M^2)^{-1}$.

To illustrate this procedure we give here the expressions, obtained from the Feynman rules, that contribute to the δ expansion for the two-point function to second order. The three classes of diagrams are given in Fig. 5. The value of the vertex with $2n$ external lines is

$$\begin{aligned}v_{2n}(\delta)&=\frac{2\lambda(2\delta+2)!M^2}{(\delta+1-n)!2^{\delta+1-n}I^{n-1}}(IM^{2-d})^\delta\\&=\frac{2^{n+1}\lambda M^2}{\sqrt{\pi}I^{n-1}}(\delta+1)\delta(\delta-1)\cdots(\delta+2-n)\\&\quad\times\Gamma(\delta+\tfrac{3}{2})(2IM^{2-d})^\delta\ ,\end{aligned}\tag{4.1}$$

where I is given by (3.3).

The graph in Fig. 5(a) contributes

$$-\left[\delta v_2'(0)+\frac{\delta^2}{2!}v_2''(0)\right]\ .\tag{4.2}$$

The graphs in Fig. 5(b) contribute

$$\delta^2\sum_{l=1}^{\infty}\frac{G(2l+1)}{2(2l+1)!}[v_{2l+2}'(0)]^2\ ,\tag{4.3}$$

and the graphs in Fig. 5(c) contribute

$$\delta^2\sum_{l=1}^{\infty}\frac{G(2l)}{(2l)!}v_{2l+2}'(0)v_{2l}'(0)\ ,\tag{4.4}$$

where

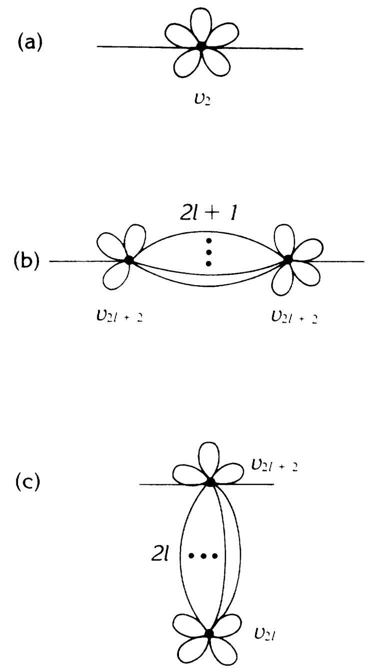

FIG. 5. The graphs that contribute to the two-point function through second order.

$$G(n)=\int d^dx\,\Delta^n(x)\ ,\tag{4.5}$$

and $\Delta(x)$ is the boson propagator in configuration space.

From (4.1) one finds

$$v_2'(0)=2\lambda M^2[\psi(\tfrac{3}{2})+1+\ln(2IM^{2-d})]\ ,$$

$$v_2''(0)=2\lambda M^2\{[\psi(\tfrac{3}{2})+1+\ln(2IM^{2-d})]^2+\psi'(\tfrac{3}{2})-1\}\ ,$$

$$v_{2l}'(0)=\lambda M^2\frac{2^l(-1)^l(l-2)!}{I^{l-1}},\quad l>1\ ,$$

$$\begin{aligned}v_{2l}''(0)=&\frac{\lambda M^2 2^{l+1}}{I^{l-1}}(-1)^l(l-2)!\\&\times\left[\psi(\tfrac{3}{2})+\ln(2IM^{2-d})-\tfrac{1}{2}-\tfrac{1}{3}-\cdots-\frac{1}{l-2}\right],\\&\qquad l>1\ .\end{aligned}$$

The sample calculation given above is one part of the calculation required for the renormalization of the δ expansion, which will be the subject of a subsequent paper in this series. Obviously the renormalization program in the δ expansion is very different from that in standard weak-coupling perturbation theory.

Finally we remark on the generalization of these rules to the case of Lagrangians with more than one field or with multiple interaction terms. The procedure is similar to that outlined above. A separate vertex function $v(\delta)$ analogous to (4.1) must be defined corresponding to each interaction vertex of the theory. Each must be expanded to an appropriate order in δ, consistent with the order of the δ expansion at hand. All possible conventional Feynman graphs in the theory must be calculated, using the standard Feynman rules except for the changes in the propagators and vertex functions described above.

Note added. After this work was completed we received a paper[6] from Nicholas Brown which contains some of these results, although the point of view and conclusion are rather different. We are grateful to Professor Brown for sending us an advance copy of this work and for correspondence on the subject. We have also received a recent paper[7] by I. Yotsuyanagi which presents a set of simplified Feynman rules for the δ expansion similar to those derived here.

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APPENDIX

In the fourth-order calculation described in this paper it is useful to consider $\alpha_1,\alpha_2,\alpha_3,\alpha_4$, to be the elements of a 4-tuple, $\alpha=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$. We then define forms, composed of the elements of any such 4-tuple, as follows:

$$(\alpha)_S=\alpha_1+\alpha_2+\alpha_3+\alpha_4\ , \tag{A1a}$$

$$(\alpha)_T=\alpha_1+i\alpha_2-\alpha_3-i\alpha_4\ , \tag{A1b}$$

$$(\alpha)_U=\alpha_1-i\alpha_2-\alpha_3+i\alpha_4\ , \tag{A1c}$$

$$(\alpha)_A=\alpha_1-\alpha_2+\alpha_3-i\alpha_4\ . \tag{A1d}$$

Note that the coefficients in these forms are the four roots of 1: 1, -1, i, $-i$. These forms can be composed for any four-component object, such as $\alpha^2=(\alpha_1^2,\alpha_2^2,\alpha_3^2,\alpha_4^2)$. Thus $(\alpha^2)_U$, which appears in (2.4), is given by $(\alpha^2)_U=\alpha_1^2-i\alpha_2^2-\alpha_3^2+i\alpha_4^2$. More generally, let $A(x)$ be any function. Then we define

$$(A(\alpha))_S\equiv A(\alpha_1)+A(\alpha_2)+A(\alpha_3)+A(\alpha_4)\ , \tag{A2a}$$

$$(A(\alpha))_U\equiv A(\alpha_1)-iA(\alpha_2)-A(\alpha_3)+iA(\alpha_4)\ , \tag{A2b}$$

and so on. It is useful to note that $(A(0))_S=4A(0)$ and $(A(0))_U=(A(0))_T=(A(0))_A=0$.

Similarly, one can define derivative operators composed from $(\partial/\partial\alpha_1,\partial/\partial\alpha_2,\partial/\partial\alpha_3,\partial/\partial\alpha_4)$ as follows:

$$\left[\frac{\partial}{\partial\alpha}\right]_U=\frac{\partial}{\partial\alpha_1}-\frac{i\partial}{\partial\alpha_2}-\frac{\partial}{\partial\alpha_3}+\frac{i\partial}{\partial\alpha_4}\ , \tag{A3a}$$

$$\left[\frac{\partial^2}{\partial\alpha^2}\right]_A=\frac{\partial^2}{\partial\alpha_1^2}-\frac{\partial^2}{\partial\alpha_2^2}+\frac{\partial^2}{\partial\alpha_3^2}-\frac{\partial^2}{\partial\alpha_4^2}\ , \tag{A3b}$$

$$\left[\frac{\partial^3}{\partial\alpha^3}\right]_T=\frac{\partial^3}{\partial\alpha_1^3}+\frac{i\partial^3}{\partial\alpha_2^3}-\frac{\partial^3}{\partial\alpha_3^3}-\frac{i\partial^3}{\partial\alpha_4^3}\ , \tag{A3c}$$

$$\left[\frac{\partial^4}{\partial\alpha^4}\right]_S=\frac{\partial^4}{\partial\alpha_1^4}+\frac{\partial^4}{\partial\alpha_2^4}+\frac{\partial^4}{\partial\alpha_3^4}+\frac{\partial^4}{\partial\alpha_4^4}\ . \tag{A3d}$$

It is straightforward to establish the results below for the action of each of these derivative forms of the functional forms defined in (A2).

$$\left[\frac{\partial}{\partial\alpha}\right]_U(A(\alpha))_S=(A'(\alpha))_U=A'(\alpha_1)-iA'(\alpha_2)-A'(\alpha_3)+iA'(\alpha_4)\ , \tag{A4a}$$

$$\left[\frac{\partial}{\partial\alpha}\right]_U(A(\alpha))_T=(A'(\alpha))_S\ , \tag{A4b}$$

$$\left[\frac{\partial}{\partial\alpha}\right]_U(A(\alpha))_U=(A'(\alpha))_A\ , \tag{A4c}$$

$$\left[\frac{\partial}{\partial\alpha}\right]_U(A(\alpha))_A=(A'(\alpha))_T\ . \tag{A4d}$$

The result in (A4) can be conveniently summarized by the notation

$$\left[\frac{\partial}{\partial\alpha}\right]_U(A(\alpha))_X=(A'(\alpha))_{UX}\ , \tag{A5a}$$

where the subscript UX is determined from the multiplication Table I below, with $X=S$, T, U, or A.

Similarly, the action of the other derivative operators defined in (A3) is given by

$$\left[\frac{\partial^2}{\partial\alpha^2}\right]_A(A(\alpha))_X=(A''(\alpha))_{AX}\ , \tag{A5b}$$

$$\left[\frac{\partial^3}{\partial\alpha^3}\right]_T(A(\alpha))_X=(A'''(\alpha))_{TX}\ , \tag{A5c}$$

$$\left[\frac{\partial^4}{\partial\alpha^4}\right]_S(A(\alpha))_X=(A^{(4)}(\alpha))_{SX}\ . \tag{A5d}$$

For the calculations in Sec. III of this paper one also needs to evaluate the results of the application of the derivative operators (A3) to products of functional forms of the type defined in (A2). By repeated application of the chain rule and the multiplication rules given in Table I and used in (A5) one can establish

$$\left[\frac{\partial}{\partial\alpha}\right]_U[(A(\alpha))_X(B(\alpha))_Y]=(A'(\alpha))_{UX}(B(\alpha))_Y+(A(\alpha))_X(B'(\alpha))_{UY} \tag{A6a}$$

and

TABLE I. The table entry in row x, column y gives the correct subscript $(\ )_{XY}$ in (A5).

	U	S	T	A
U	A	U	S	T
S	U	S	T	A
T	S	T	A	U
A	T	A	U	S

$$\left[\frac{\partial^2}{\partial\alpha^2}\right]_A [(A(\alpha))_X(B(\alpha))_Y] = (A''(\alpha))_{AX}(B(\alpha))_Y + (A(x))_X(B''(\alpha))_{AY} + 2(A'(\alpha)B'(\alpha))_{AXY} , \tag{A6b}$$

where the subscript AXY refers to repeated application of the multiplication rules embodied in Table I (the order is immaterial) and $X, Y = A, S, T,$ or U.

Similarly,

$$\left[\frac{\partial^3}{\partial\alpha^3}\right]_T [(A(\alpha))_X(B(\alpha))_Y] = (A'''(\alpha))_{TX}(B(\alpha))_Y + (A(\alpha))_X(B'''(\alpha))_{TY} + 3(A''(\alpha))(B'(\alpha))_{TXY} + 3(A'(\alpha)B''(\alpha))_{TXY} , \tag{A6c}$$

$$\left[\frac{\partial^4}{\partial\alpha^4}\right]_S [(A(\alpha))_X(B(\alpha))_Y] = (A^{(4)}(\alpha))_X(B(\alpha))_Y + (A(\alpha))_X(B^{(4)}(\alpha))_Y + 4(A'''(\alpha)B'(\alpha))_{XY} + 4(A'(\alpha)B'''(\alpha))_{XY} + 6(A''(\alpha)B''(\alpha))_{XY} . \tag{A6d}$$

Similar rules can be derived for the result of applying the four-derivative forms to the product of three or four functions.

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