Grassmannian Kaluza-Klein theory and the standard model

R. Delbourgo and R. B. Zhang
Physics Department, University of Tasmania, Hobart, 7005 Australia
(Received 7 April 1988)

By appending anticommuting coordinates to the four commuting space-time coordinates it is possible to amalgamate gravity and internal forces without introducing infinitely many massive modes. We carry out this synthesis for the standard model.

I. INTRODUCTION

In an earlier Letter[1] we outlined a Grassmannian framework for unifying general covariance and gauge symmetries in the spirit of Kaluza and Klein, but without the problems related to Planck-mass excitations. In this paper we shall give a detailed analysis of the method and apply it to the amalgamation of gravity with chromodynamics and flavor dynamics.

An "ordinary" Kaluza-Klein model is a generally covariant theory in a $(4+K)$-dimensional bosonic spacetime: the symmetry motions in the extra K dimensions manifest themselves as gauge symmetries in the four-dimensional world. In order to make contact with low-energy physics, one appeals to the idea of spontaneous compactification, although neither the evidence nor the mechanism for the process is particularly clear.[2] At the end the geometry of the enlarged spacetime factorizes into a direct product of a four-dimensional Riemannian manifold and the internal space which is a k-dimensional compact manifold of Planckian size and thus is presumed "invisible" at normal energies. However, in dealing with the fields, it becomes necessary for consistency[3] to expand them into normal modes on the compact manifold and in this way an infinite tower of very massive states is conjured up. For some models there is a real danger that the integrated effect of these massive modes can produce an unhealthy spectrum such as ghostly and/or tachyonic states.[3,4] Such problems have to be solved and the whole task can become extremely unwieldy not to mention unlikely.

The fermionic extension which we have proposed, while retaining the basic concept that general coordinate transformations in the internal space correspond to gauge transformations in the four-dimensional world modifies the Kaluza-Klein mechanism by taking the internal space to be a $2N$-dimensional Grassmannian manifold. Here, the extra coordinates, being anticommuting, are not detectable in the ordinary way, regardless of the energy scale of observation: hence the concept of compactification becomes irrelevant. Also any field, when expanded in powers of the internal space coordinates can at most contain 2^{2N} terms and at worst leads to a finite mass spectrum. Thus all the problems associated with the infinite number of massive modes in the conventional treatment are bypassed.

This paper reports on some investigations on Grassmannian Kaluza-Klein theories. In Sec. II we explain the basic ideas involved in constructing such theories: in particular we develop an ansatz for the superbein for the enlarged spacetime and explicitly demonstrate the connection between the gauge symmetries and the internal space transformations. Section III applies these principles to the construction of SU(3) and $SU(2)\times U(1)$ models coupled to matter fields, thereby achieving the synthesis of gravity with the standard model. Section IV concludes with some discussion of directions for future work.

II. FERMIONIC INTERNAL SPACE

In this section we shall explain the general procedure for constructing Grassmannian Kaluza-Klein theories, providing details which were necessarily omitted in our Letter.[1] First we give a brief account of the geometry of the extended spacetime—a superspace—and then we choose an ansatz for the superbein which concretely realizes the behavior of the system under general coordinate transformations, generalized Lorentz invariance, and gauge symmetries. Finally we formulate actions for matter fields coupled to the gravity/Yang-Mills metric which embody the (reduced) supersymmetries.

Begin by enlarging spacetime to a superspace $\mathcal{M}$ by attaching an extra $2N$ anticommuting real coordinate. The superspace can be parametrized, at least locally, by

$$X^M=(x^\mu,\xi^m)\ ,$$

where x^μ are the usual real four bosonic coordinates and the ξ^m plus their conjugates $(m=1,2,\ldots,2N)$ make up N complex Grassmann variables. In this context conjugation is consistently defined as the operation

$$(\xi^m)^*=-\xi_m=\xi^n\eta_{nm}\ ,$$

where η_{mn} is a $2N\times 2N$ antisymmetric matrix. To be concrete we put it in the form

$$\eta_{nm}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}.$$

Together with the Minkowski metric $\eta_{\alpha\beta}$, we can specify the flat-space metric to be

$$I_{AB}=\begin{bmatrix}\eta_{\alpha\beta} & 0\\ 0 & \eta_{ab}\end{bmatrix} \tag{1}$$

and this preserves OSp(4/2N) and all its subgroups. Naturally we shall identify I as the metric for the tangent space of $\mathcal{M}$.

Next introduce the superbein $E_M{}^A$ and the connection $(\Phi_M)_A{}^B$. As usual E is related to the full metric tensor G via

$$G_{MN}=E_M{}^A E_N{}^B I_{BA}[AN] \qquad (2)$$

and Φ takes its values in the superalgebra OSp(4/2N). The bracket notation is as follows: When there is a single argument, as in Eq. (8), $[B]=+1$ for bosonic, -1 for fermionic indices; when there is a double argument, as in Eq. (7), $[AB]=-1$ if *both* indices are fermionic, $+1$ otherwise. In particular,

$$(\Phi_M)^{AB}=-[AB](\Phi_M)^{BA} . \qquad (3)$$

Under the transformation $X^M \to X'^M(X)$, the one-forms

$$E^A(X)=dX^M E_M{}^A(X), \quad \Phi_A{}^B(X)=dX^M(\Phi_M)_A{}^B$$

obey the transformation rules

$$\begin{aligned} &E^A(X)\to E'^A(X')=E^A(X) ,\\ &\Phi_A{}^B \to \Phi'_A{}^B(X')=\Phi_A{}^B(X) , \end{aligned} \qquad (4a)$$

while under the local frame rotation $L_A{}^B(X)$, $L\in$ OSp(4/2N), we have

$$\begin{aligned} &E^A(X)\to E'^A(X)=E^B(X)L_B{}^A(X) ,\\ &\Phi_A{}^B(X)\to \Phi'_A{}^B(X)\\ &\qquad =(L^{-1}\Phi L)_A{}^B(X)+(dL^{-1}\cdot L)_A{}^B(X) . \end{aligned} \qquad (4b)$$

As usual the torsion is defined through

$$T^A(X)=dE^A(X)+E^B(X)\Phi_B{}^A(X) \qquad (5)$$

and when it is specified we can determine $\Phi_A{}^B$ in terms of $E_M{}^A$ and its derivatives. In what follows we shall never treat Φ as an independent dynamical variable but instead as fields constructed out of E. The curvature is given in terms of the connection,

$$R_A{}^B=d\Phi_A{}^B+\Phi_A{}^C\Phi_C{}^B , \qquad (6)$$

and in component form reads

$$\begin{aligned} (R_{MN})_A{}^B=&\,\partial_M(\Phi_N)_A{}^B-[MN]\partial_N(\Phi_M)_A{}^B\\ &+[NA][NC](\Phi_M)_A{}^C(\Phi_N)_C{}^B\\ &-[MN][MA][MC](\Phi_N)_A{}^C(\Phi_M)_C{}^B . \end{aligned} \qquad (7)$$

The scalar curvature is, of course, obtained as

$$R=[B]E_B{}^N E_A{}^M[AN]I^{AC}(R_{MN})_C{}^B . \qquad (8)$$

This is needed to construct the generalized Einstein-Hilbert action

$$S=\int d^{2N+4}X\, ER=\int d^4x\, d^{2N}\xi\, ER , \qquad (9)$$

where $E=\text{sdet}(E_M{}^A)$. As we will see later, this action essentially reduces to that of a gravity/Yang-Mills system after the fermionic integrations are performed. It is also comparatively easy to incorporate a scalar matter field: one introduces the superfield $\Phi(X)$, a zero-form on $\mathcal{M}$, and an OSp(4/2N) scalar. It suffices to take the obvious expression

$$S=\int d^{4+2N}X\, E(G^{NM}\partial_M\Phi^\dagger\partial_N\Phi+\mathcal{M}^2\Phi^\dagger\Phi) . \qquad (10)$$

It is nontrivial to incorporate spinorial matter fields into the present framework since this requires investigations of OSp(4/2N) spinor representations,[5] which are rather complicated objects. Fortunately we may avoid the problem in the following discussion, where we will break the full orthosymplectic group into SO(4)$\times$Sp(2N) and merely consider fields which are Sp(2N) scalar but SO(4) spinors.

All the arguments so far have been quite formal and standard. In order to obtain a theory of gravity coupled to Yang-Mills theory out of (9), a more concrete realization is needed. Let us therefore consider the superbein first. When it is expanded into powers of the Grassmann ξ, some of the coefficients fields will obey the wrong spin statistics and it is essential that we eliminate them, at the classical level anyway. We do so by requiring that only even powers of ξ appear in the bosonic fields $E_\mu{}^\alpha$ and $E_m{}^a$, and only odd powers of ξ appear in the expansions of the fermionic components $E_\mu{}^a$ and $E_m{}^\alpha$. At this stage we also insist that general coordinate transformations and frame rotations be restricted to the subclass which does not involve any ghost fields as transformation parameters. Putting this another way, we impose the following gauge conditions on the superbein:

$$PE_M{}^A(X)=\begin{bmatrix} E_\mu{}^\alpha(X) & 0\\ 0 & E_m{}^a(X)\end{bmatrix} \qquad (11)$$

with $P=\frac{1}{2}[1+(-)^\Delta]$, $\Delta=\xi^m\partial_m$. They do not completely fix the gauge for E, since there remains the residual symmetry

$$\begin{aligned} &x^\mu\to x'^\mu=x'^\mu(x,\xi^k\cdot\xi^l) ,\\ &\xi^m\to\xi'^m=\xi^n\sigma_n{}^m(x,\xi^k\cdot\xi^l) ,\\ &L_A{}^B(X)=\begin{bmatrix} B_\alpha{}^\beta(x,\xi^k\cdot\xi^l) & \xi^m(F_m(x,\xi^k\cdot\xi^l))_\alpha{}^b\\ \xi^m(F_m(x,\xi^k\cdot\xi^l))_a{}^\beta & B_a{}^b(x,\xi^k\cdot\xi^l)\end{bmatrix} . \end{aligned} \qquad (12)$$

The only difference between the present case and the normal kind of gauge fixing is that the restriction (11) is required even at the classical level.

The remaining symmetry (12) can be used to simplify the superbein further. Indeed one can always move to a gauge where

$$E_M{}^A(X)=\begin{bmatrix} e_\mu{}^\alpha(x,\xi^k\cdot\xi^l) & \xi^m(A_\mu(x,\xi^k\cdot\xi^l))_m{}^a\\ 0 & \phi_m{}^a(x,\xi^k\cdot\xi^l)\end{bmatrix} . \qquad (13)$$

Now the most general transformations which leave the constrained form of E intact are

$$\begin{aligned} &x^\mu\to x'^\mu=f^\mu(x) ,\\ &\xi^m\to\xi'^m=\xi^n\sigma_n{}^m(x,\xi^k\xi^l) , \end{aligned} \qquad (14a)$$

$$L_A{}^B(X)=\begin{bmatrix} l_\alpha{}^\beta(x) & 0 \\ 0 & U_a{}^b(x,\xi^k\xi^l) \end{bmatrix} . \tag{14b}$$

We will take (13) and (14) as our starting point and require the Grassmannian Kaluza-Klein models to preserve the given symmetries. Note that the frame rotations defined by (14b) describe an SO(4)×Sp(2N) group only, the representations of which are well known. More specifically we will take the spinorial matter field to be an Sp(2N) scalar but SO(4) spinor. An action for this field will simply be required to be invariant under (14).

In principle we could take (13) as the complete set of dynamical variables and proceed to construct an action for them. However this would lead to an inordinately complicated system since, except for QED, large numbers of component fields could arise in the ξ^2 expansions. Here we shall examine the simpler ansatz, consistent with our self-imposed restrictions:[6]

$$\begin{aligned} E_M{}^A(X)&=\begin{bmatrix} e_\mu{}^\alpha(x) & \rho(\xi A_\mu)^a(x) \\ 0 & \rho\delta_m{}^a \end{bmatrix} ,\\ \rho&=\exp\left[\frac{c}{\kappa^2}\xi^2\right] ,\\ c&=\frac{1}{16}\left[\frac{N-1}{N}\right]^{N-1} , \end{aligned} \tag{15}$$

where $A_\mu(x)$ takes its values in a subalgebra $\mathcal{J}$ of Sp(2N) and ξ^2 is defined as $\xi^m\xi^n\eta_{mn}$. Using (15) we derive the line element

$$\begin{aligned} dS^2&=dX^m dX^N G_{NM}(x)\\ &=g_{\mu\nu}dx^\mu dx^\nu+\rho^2\eta_{ab}(\xi A+d\xi)^b(\xi A+d\xi)^a . \end{aligned}$$

It is easily shown that dS^2 is invariant under the restricted transformations:

$$\begin{aligned} x^\mu\to x'^\mu=f^\mu(x),\quad \xi^m\to\xi'^m=\xi^n U_n{}^m(\Lambda(x)) ,\\ U\in G,\ \text{the group with algebra } \mathcal{J} , \end{aligned} \tag{16}$$

provided that A_μ transforms according to

$$A_\mu\to[U^{-1}A_\nu U+(\partial_\nu U^{-1})U]\frac{\partial x^\nu}{\partial x'^\mu} . \tag{17}$$

In (17) we recognize the gauge covariance property of a Yang-Mills connection: indeed A_μ plays the role of a gauge field as we shall presently see.

It is important to point out that the superbein one-form does not retain its form under the coordinate changes (16). However, the combination of (16) with the frame rotation

$$L_A{}^B(X)=\begin{bmatrix} l_\alpha{}^\beta(x) & 0 \\ 0 & U_a{}^b(\Lambda(x)) \end{bmatrix}$$

does leave its form intact. This is unsurprising, because choosing the ansatz (15) is equivalent to partly fixing the gauge of $E_M{}^A$. The coordinate change (16) takes us out of the gauge, but a compensating frame rotation brings it back. In fact this situation is reminiscent of supergravity models in Wess-Zumino gauges.

Hereafter we shall refer to the combination of transformations

$$\begin{aligned} x^\mu&\to x^\mu ,\\ \xi^m&\to\xi^n U_n{}^m(\Lambda(x)) ,\\ L_A{}^B(X)&=\begin{bmatrix} \delta_\alpha{}^\beta & 0 \\ 0 & U_a{}^b(\Lambda(x)) \end{bmatrix} \end{aligned} \tag{18}$$

as a gauge transformation. We can clean up the structure by redefining the basis of one-forms

$$\omega^M=(dx^\mu,(\xi A+d\xi)^m) . \tag{19a}$$

Then the tangent-space basis, orthogonal to ω^M is given by

$$D_M=(\partial_\mu-(\xi A_\mu)^n\partial_n,\partial_m) \tag{19b}$$

and the commutator of two derivatives yields

$$[D_\mu,D_\nu]=-(\xi F_{\mu\nu})^m\partial_m$$

showing that D acts like a covariant derivative. In these bases the superbein and the metric simplify enormously:

$$\begin{aligned} \hat{E}_M{}^A&=\begin{bmatrix} e_\mu{}^\alpha & 0 \\ 0 & \rho\delta_m{}^a \end{bmatrix} ,\\ \hat{E}_A{}^M&=\begin{bmatrix} e_\alpha{}^\mu & 0 \\ 0 & \rho^{-1}\delta_m{}^a \end{bmatrix} , \end{aligned} \tag{20}$$

and

$$\hat{G}_{MN}=\begin{bmatrix} \eta_{\alpha\beta}e_\mu{}^\alpha e_\nu{}^\beta & 0 \\ 0 & \rho^2\eta_{mn} \end{bmatrix} . \tag{21}$$

It remains to find the connection and the curvature in terms of the superbein. In order to determine $\Phi_A{}^B$, we first look at its behavior under the gauge transformation (18):

$$\begin{aligned} \Phi_A{}^B(X)\to&\begin{bmatrix} \Phi'_\alpha{}^\beta(X') & \Phi'_\alpha{}^c(X')U_c{}^b \\ U^{-1}{}_a{}^c\Phi'_c{}^\beta(X') & U_a^{-1c}\Phi'_c{}^d(X')U_d{}^b \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ 0 & dU^{-1}{}_a{}^c\cdot U_c{}^b \end{bmatrix}\\ =&\begin{bmatrix} \Phi_\alpha{}^\beta(X) & \Phi_\alpha{}^c(X)U_c{}^b \\ U^{-1}{}_a{}^c\Phi_c{}^\beta(X) & U_a{}^{-1c}\Phi_c{}^d(X)U_d{}^b+dU^{-1}{}_a{}^c\cdot U_c{}^b \end{bmatrix} . \end{aligned} \tag{22}$$

It is noteworthy that $\Phi_a{}^b$ behaves exactly like the gauge potential $A_a{}^b$ under changes of gauge: hence we are emboldened to make the identification

$$\Phi_a{}^b = dx^\mu (A_\mu)_a{}^b \ . \tag{23}$$

To discover the remaining connection components, we note that even when $A_\mu = 0$, $e_\mu{}^\alpha = \delta_\mu{}^\alpha$, $\Phi = 0$, the torsion tensor does not vanish; hence it is incorrect to set the torsion to zero as in ordinary Riemannian geometry. Instead we must take

$$T^A = (0, \omega^m \delta_m{}^a d\rho) \ . \tag{24}$$

Recalling the definition (5) of T^A and Eq. (23), we arrive at the following equations by using (24):

$$\begin{aligned} &de^\alpha + e^\beta \Phi_\beta{}^\alpha + \rho \delta_m{}^b \omega^m \Phi_b{}^a = 0 \ , \\ &e^\beta \Phi_\beta{}^a + \rho (\xi F)^a = 0 \ , \end{aligned} \tag{25}$$

where $F = dA + A \wedge A$. It is straightforward to solve (25) by using the symmetry property (3) of Φ and one finds

$$\Phi_A{}^B = \begin{bmatrix} \Phi_\alpha^{(g)\beta} - \rho^2 \omega^c (\xi F_\alpha{}^\beta)_c & -\rho (\xi F_{\mu\alpha})^b dx^\mu \\ -\rho (F_\mu{}^\beta \xi)_a dx^\mu & A_a{}^b \end{bmatrix} . \tag{26}$$

with $\Phi^{(g)}{}_\alpha{}^\beta$ denoting the ordinary gravitational connection, calculated from $e_\mu{}^\alpha$.

We are now in a position to evaluate the scalar curvature. Some elementary manipulations lead to

$$R = R^{(T)} + R' \ , \tag{27}$$

with

$$R^{(T)} = [B] E_B{}^N E_A{}^M [AN] T_{MN}{}^C (\Phi_C)_D{}^B I^{AD} \ , \tag{28}$$

$$\begin{aligned} R' = [B][&2\partial_B (\Phi_D)_A{}^B - (\Phi_B)_D{}^F (\Phi_F)_A{}^B \\ &+ [BF] (\Phi_D)_A{}^F (\Phi_B)_F{}^B] I^{AD} \ , \end{aligned} \tag{29}$$

and

$$(\Phi_A)_B{}^C = E_A{}^M (\Phi_M)_B{}^C \ .$$

$R^{(T)}$ is relatively trivial to evaluate and we get

$$R^{(T)} = -\kappa^{-2} \rho^2 \xi A_\mu \xi \cdot \xi A_\nu F^{\mu\nu} \xi \ .$$

But because A_{ab} takes its values in a subalgebra of Sp($2N$), $A_{ab} = A_{ba}$, so $\xi A_\mu \xi = 0$. This in turn makes $R^{(T)}$ vanish. The evaluation of R' is much lengthier and messier; we will content ourselves by giving the principal steps:

$$[B] \partial_B (\Phi_D)_A{}^B I^{AD} = e_\beta{}^\mu \partial_\mu (\Phi_\alpha^{(g)})^{\alpha\beta} \ ,$$

$$[B] (\Phi_B)_D{}^F (\Phi_F)_A{}^B I^{AD} = (\Phi_\rho^{(g)})^{\alpha\gamma} (\Phi_\gamma^{(g)})_\alpha{}^\beta + \tfrac{1}{4} \rho^2 \xi F_{\mu\nu} F^{\mu\nu} \xi \ ,$$

$$[B][BF] (\Phi_D)_A{}^F (\Phi_B)_F{}^B I^{AD} = (\Phi_\alpha^{(g)})^{\alpha\gamma} (\Phi_\beta^{(g)})_\gamma{}^\beta \ .$$

The final result is

$$R' = R^{(g)} + \tfrac{1}{4} \rho^2 \xi F_{\mu\nu} F^{\mu\nu} \xi \ , \tag{30}$$

where $R^{(g)}$ is the scalar curvature calculated from the gravitational connection alone. As expected (30) just involves quantities which are both gauge and general coordinate invariant and qualifies as a Lagrangian density.

As a consequence of the construction (9), we obtain the action in the supermanifold $\mathcal{M}$:

$$S \propto \int d^4x \, d^{2N}\xi \, e \rho^{-2N} (R^{(g)} + \tfrac{1}{4} \rho^2 \xi F^2 \xi) \ . \tag{31}$$

The Grassmann integration then leaves us with the traditional result

$$S = \int d^4x \, e \left[-\frac{R^{(g)}}{4\kappa^2} + \frac{1}{4} \mathrm{tr}(F_{\mu\nu} F^{\mu\nu}) \right] . \tag{32}$$

Observe the necessity for the factor $\exp(c\xi^2/\kappa^2)$. Without it, the fermionic integral (31) would disappear; that is the reason why we introduced it in the first place.

To add matter fields, we first consider a complex scalar superfield $\Phi(X)$ which is an OSp($4/2N$) scalar. We assume that it is of first order in $\xi, \Phi(x,\xi) = \xi^m \phi_m$, and is thus intrinsically Grassmannian.[7] Applying construction (10) we get

$$\begin{aligned} S = \int d^4x \, e \Bigg\{ &g^{\mu\nu} [(\partial_\mu + A_\mu)\phi]^\dagger (\partial_\nu + A_\nu) \phi \\ &+ \left[M^2 + \frac{1}{4\kappa^2} \left[\frac{N+1}{N} \right]^{N-1} \right] \phi^\dagger \phi \Bigg\} . \end{aligned} \tag{33}$$

Note the presence of the mass term $\propto \kappa^{-2}$ for ϕ, arising from the Grassmannian derivative component D_m in combination with E. The total scalar mass is adjustable of course.

With a spinorial field we take $\psi(X) = \xi^m \psi_m(x)$, which is overall bosonic. In our Letter[1] we proposed that Γ^a, the gamma matrices in the fermionic sector should be taken as $(M\xi + \partial/M)^a i\gamma_5$, which introduced a mass factor M. In this way we were able to reduce the Dirac action to the traditional result

$$\begin{aligned} S = \int d^4x \, e \{ &\bar\psi [i\gamma^\alpha e_a{}^\mu (\nabla_\mu + A_\mu) - iM\gamma_5] \psi \\ &+ \bar\psi \sigma^{\alpha\beta} F_{\alpha\beta} (i\gamma^5/M) \psi \} \ . \end{aligned} \tag{34}$$

However, this choice of Γ is one of many available choices. Recalling our earlier discussion that the final spinor is an Sp($2N$) scalar, we can just as well invoke the action for Ψ:

$$L(x,\xi) = \bar\Psi(x,\xi) (e_\alpha{}^\mu \gamma^\alpha \hat\nabla_\mu + m) \Psi(x,\xi) \tag{35}$$

with

$$\hat\nabla_\mu = \nabla_\mu + (\xi A_\mu)^n \partial_n \ . \tag{36}$$

This too is invariant under the coordinate transformation (16) and the gauge transformation (18). It is clearly superior over (34) because the Pauli coupling is absent (and the singularity in M of that term ceases to be a worry) and permits us to include an arbitrary mass just the same.

III. INCORPORATION OF GRAVITY INTO THE STANDARD MODEL

In this section we will apply the Sec. II scheme to SU(3) and SU(2)$\times$U(1) internal groups acting on the

Grassmannian coordinates and thus achieve a synthesis of gravity and strong/weak forces. Let us tackle chromodynamics first. In order to incorporate an SU(3) group in its fundamental representation (quarks) we will need three complex Grassmann variables: these are to be assigned to the $\bar{3}$ representation of SU(3) and together with their complex conjugates make up a sixfold ξ. If λ^i denote the eight Gell-Mann matrices, we may write

$$\Lambda^i = \begin{bmatrix} \lambda^i & 0 \\ 0 & -(\lambda^i)^* \end{bmatrix}$$

and express the present superbein as

$$E_M{}^A = \begin{bmatrix} e_\mu{}^\alpha(x) & -\frac{i}{2} g \xi^m (B_\mu{}^i \Lambda^i)_m{}^a \exp\left[\frac{g^2}{36\kappa^2}\xi^2\right] \\ 0 & \delta_m{}^a \exp\left[\frac{g^2}{36\kappa^2}\xi^2\right] \end{bmatrix}, \tag{37}$$

where g is a dimensionless coupling constant.

Now consider the quark superfield $Q^{(i)}(X)$, which is an chromosinglet and Dirac spinor. Expanding it in powers of ξ and omitting higher-order terms, because such terms will lead to ghosts, we have

$$\begin{aligned} Q^{(i)}(x,\xi) &= \xi^m q_m^{(i)}(x) , \\ q_m^{(i)}(x) &= (q_{(x)}^{(i)}, q_c^{(i)}(x)) . \end{aligned} \tag{38}$$

The fields q and q_c are related by charge conjugation:

$$q_c^{(i)}(x) = c\tilde{\bar{q}}^{(i)}(x)$$

and thus reside in 3 and $\bar{3}$ representations, transforming as

$$(q^{(i)}, q_c^{(i)}) \rightarrow \left[\exp\left[\frac{i}{2}\Sigma^l(x)\lambda^l\right] q^{(i)} , \; \exp\left[-\frac{i}{2}\Sigma^l(x)\lambda^l\right] q_c^{(i)}\right] . \tag{39}$$

under the gauge rotation (18).

Substituting (37) and (38) into (35), and performing the Grassmann integrations, we obtain

$$S^{(i)} = \int d^4x \, e\bar{q}^{(i)}(x) \left[i\gamma^\alpha e_\alpha{}^\mu \left[\nabla_\mu + \frac{i}{2} B_\mu{}^l \cdot \lambda^l\right] + m^{(i)} \right] q^{(i)}(x) . \tag{40}$$

It is now a small step to write down a complete action for the system of superbein coupled to all possible quark fields:

$$S = \int d^4x \, e \left[-\frac{R^{(g)}}{4\kappa^2} - \frac{1}{4} F^l_{\mu\nu} F^{l\mu\nu} \right] + \sum_i S^{(i)} . \tag{41}$$

It is worth pointing out that the ansatz (38) for q is complete but not compulsory. A simpler approach is to forget about the conjugate quark field and take $q_m{}^{(i)} = (q^{(i)}, 0)$. Providing the action is made real (by adding Hermitian conjugate terms as needed) the same correct Lagrangian ensues. We shall make this simplification for the electroweak model.

Let us therefore turn to SU(2)×U(1) and recall the assignments of particles to that group. Since they are all either SU(2) doublets or singlets, only two complex Grassmann variables ξ are required to carry that gauge symmetry. However, the fields all carry different weak hypercharges y: i.e., they belong to distinct representations of U(1). Ostensibly we would need one complex fermionic variable for each irreducible representation. It turns out nevertheless that the situation is much better because some representations can be built out of others. We have found that four complex anticommuting coordinates θ will suffice for leptons and quarks; they carry, respectively, opposite hypercharges to the right-handed spinors and the Higgs boson. In this way our superbein ansatz reduces to

$$E_M{}^A = \begin{bmatrix} e_\mu{}^\alpha(x) & -\frac{i}{2} g' A_\mu (\theta \cdot Y)^z \rho & -\frac{i}{2} g B_\mu(x) (\xi T)^a \rho \\ 0 & \delta_x{}^z \rho & 0 \\ 0 & 0 & \delta_m{}^a \rho \end{bmatrix} \tag{42}$$

with

$$Y = \begin{bmatrix} y & 0 \\ 0 & -y \end{bmatrix}, \quad y = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & \frac{4}{3} & 0 & 0 \\ 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T^i = \begin{bmatrix} \sigma^i & 0 \\ 0 & -\sigma^{i*} \end{bmatrix},$$

and

$$\theta^x=(\theta_{-2},\theta_{+4/3},\theta_{-2/3},\theta_{+1};\theta^*_{-2},\theta^*_{+4/3},\theta^*_{-2/3}\theta^*_{+1}) \ .$$

ρ is the ordinary exponential factor

$$\rho=\exp\left[\frac{1}{\kappa^2}c\Theta^2\right],\quad \Theta^2=g'^2[4\theta^2_{-2}+(\tfrac{4}{3})^2\theta^2_{+4/3}+(\tfrac{2}{3})^2\theta^2_{-2/3}+\theta^2_{+1}]+\tfrac{1}{4}g^2\xi^2,\quad c=\tfrac{1}{48}(\tfrac{5}{6})^5 \ .$$

The superbein is guaranteed to retain its form under the coordinate transformations

$$x\to x'=x'(x),\quad \theta\to\theta'=\theta\exp\left[-\frac{i}{2}\lambda(\kappa)y\right],\quad \xi\to\xi'=\xi\exp\left[-\frac{i}{2}\lambda^l(x)T^l\right], \tag{43}$$

accompanied by the frame rotation

$$L_A{}^B(x)=\begin{bmatrix} l(x) & 0 & 0 \\ 0 & \exp\left[-\frac{i}{2}\lambda(x)y\right] & 0 \\ 0 & 0 & \exp\left[-\frac{i}{2}\Lambda(x)T\right]\end{bmatrix}, \tag{44}$$

if the one-forms A, Y, and B^iT^i undergo the gauge changes

$$A\to A+\frac{i}{g'}d\lambda,\quad B\to\exp\left[\frac{i}{2}\Lambda T\right]B\exp\left[-\frac{i}{2}\Lambda T\right]+\frac{i}{g}d\exp\left[\frac{i}{2}\Lambda T\right]\exp\left[-\frac{i}{2}\Lambda T\right] \ . \tag{45}$$

Following the procedures outlined in the last section, we can also compute the connection and scalar curvature corresponding to the superbein. It is not very difficult to see that

$$\Phi_A{}^B=\begin{bmatrix} \Phi_\alpha^{(g)\beta}+\frac{i}{2}\rho^2\left[\left[-\frac{i}{2}g'\theta A+d\theta\right]g'F_\alpha{}^\beta\theta+\left[-\frac{i}{2}g\xi B+d\xi\right]gG_\alpha{}^\beta\xi\right] & \frac{i}{2}g'\rho\, dx^\mu(\theta F_{\mu\alpha})^z & \frac{1}{2}g\rho\, dx^\mu(\xi G_{\mu\alpha})^b \\ \frac{i}{2}g'\rho\, dx^\mu(F_\mu{}^\beta\theta)_x & -\frac{i}{2}g'(A)_x{}^z & 0 \\ \frac{i}{2}g\rho\, dx^\mu(G_\mu{}^\beta\xi)_a & 0 & -\frac{i}{2}g(B)_a{}^b \end{bmatrix} \tag{46}$$

and

$$R=R^{(g)}-\frac{1}{4}\rho^2\left[\frac{g'^2}{4}[4\theta^2_{-3}+(\tfrac{4}{3})^2\theta^2_{+4/3}+(\tfrac{2}{3})^2\theta^2_{-2/3}+\theta^2_{+1}]F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{4}g^2\xi G_{\mu\nu}G^{\mu\nu}\xi\right], \tag{47}$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu,\quad G_{\mu\nu}=\partial_\mu B_\nu-\partial_\nu B_\mu+ig[B_\mu,B_\nu] \ . \tag{48}$$

What we have done so far is quite straightforward. More care is needed though when we introduce matter fields into the game. We assume that all lepton and quark fields (of a particular generation) make up gauge-invariant SO(4) spinors $L(x,\theta,\xi)$ and $Q(x,\theta,\xi)$, respectively. The specific forms of these superfields that reproduce the standard answer are

$$\begin{aligned} L(X)&=\kappa^{-2}\theta_{+1}\theta_{-2}\xi^m l_m(x)+\theta_{-2}R_l(x) \ , \\ Q(X)&=\kappa^{-2}(\theta^*_{+1}\theta_{+4/3}+G_0\theta_{+1}\theta_{-2/3})\xi^m q_m(x)+\theta_{+4/3}R_1(x)+\theta_{-2/3}R_2(x) \ , \end{aligned} \tag{49}$$

where l_m and R_l, respectively, stand for the left- and right-handed leptonic fields, while q_m and R_1,R_2 correspond to the left-handed quark doublet and the two right-handed quark singlets. For instance, in the case of the first leptonic generation,

$$l_m=\begin{bmatrix} l \\ 0\end{bmatrix},\quad q_m=\begin{bmatrix} q_L \\ 0\end{bmatrix},\quad l=\begin{bmatrix} \nu \\ e\end{bmatrix}_L,\quad q_L=\begin{bmatrix} u \\ d\end{bmatrix},\quad R_l=e_R,\quad R_1=u_R,\quad R_2=d_R \ . \tag{50}$$

(As usual, the d field really represents the linear combination of d, s, and b fields, via Cabibbo-Kobayashi-Maskawa mixing; the color index has also been suppressed.)

In order to render L and Q gauge invariant, we must demand that under gauge transformations the component fields behave as

$$l_m \to \begin{bmatrix} \exp(-\frac{1}{2}i\lambda+\frac{1}{2}i\Lambda\cdot T)l \\ 0 \end{bmatrix}, \quad q_m \to \begin{bmatrix} \exp(\frac{1}{6}i\lambda+\frac{1}{2}i\Lambda\cdot T)q_L \\ 0 \end{bmatrix},$$

$$R_l \to e^{-i\lambda}R_l, \quad R_1 \to \exp(\tfrac{2}{3}i\lambda)R_1, \quad R_2 \to \exp(-\tfrac{1}{3}i\lambda)R_2 . \tag{51}$$

The final question concerns the Higgs field. For our purposes it is a gauge-invariant superscalar $\Phi(X)$ with the decomposition

$$\Phi(x,\theta,\xi)=\kappa^{-1}[\theta_{+1}\xi^m\phi_m(x)+\theta^*_{+1}\xi^m\bar{\Phi}_m(x)] \tag{52}$$

and

$$\phi_m = \begin{bmatrix} \phi \\ 0 \end{bmatrix}, \quad \bar{\phi}_m = \begin{bmatrix} \bar{\phi} \\ 0 \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi^0 \\ \phi^- \end{bmatrix}, \quad \bar{\phi}=i\sigma_2\phi^* .$$

Its gauge transformation properties follow from those of ξ and θ: namely,

$$\phi_m \to \begin{bmatrix} \exp(\frac{1}{2}i\lambda+\frac{1}{2}i\Lambda\cdot T)\phi \\ 0 \end{bmatrix}, \quad \bar{\phi}_m \to \begin{bmatrix} \exp(-\frac{1}{2}i\lambda+\frac{1}{2}i\Lambda\cdot T)\bar{\phi} \\ 0 \end{bmatrix} . \tag{53}$$

All is in readiness for the invariant action combining gravity and the standard model. Summing over all generations we take

$$S=\int d^4k\, d^8\theta\, d^4\xi\, E \Big[-R + \sum_g \bar{L}^{(g)} i\hat{\nabla} L^{(g)} + \bar{Q}^{(g)} i\hat{\nabla} Q^{(g)} + \kappa^{-1}G_1{}^{(g)}\bar{L}^{(g)}(\Phi+\Phi^\dagger)L^{(g)} + \kappa^{-1}G_2{}^{(g)}\bar{Q}^{(g)}(\Phi+\Phi^\dagger)Q^{(g)}$$
$$+G^{NM}D_M\Phi^\dagger D_N\Phi + m^2\Phi^\dagger\Phi + \lambda\kappa^2(\eta^{xy}D_x\Phi^\dagger D_y\Phi)^2 \Big] . \tag{54}$$

The strange-looking term $(\eta^{xy}D_x\Phi^\dagger D_y\Phi)^2$ is responsible for the quartic Higgs self-coupling. After performing the integrations over the Grassmann coordinates and properly scaling the fields with appropriate numerical factors, the action (54) reduces to the familiar Glashow-Salam-Weinberg model coupled to gravity.

IV. CONCLUSIONS

It appears that we have a consistent Grassmannian framework for accommodating general covariance and gauge symmetry. Because of the anticommuting character of the additional internal coordinates it is no longer necessary to assume internal compactification of the internal space and the spectrum of a given theory is always finite. It is also rather economical in the number of Grassmann degrees of freedom that have to be appended, bearing some resemblance to preonic schemes.

We have based most of the analysis on the superbein ansatz (12) which offers a transparent explanation of gauge symmetries in terms of motions in the internal space. It may be argued that one ought to consider the full superbein field in place of (15) so as to gain a complete picture of Grassmannian Kaluza-Klein theories. This is surely true, but we expect that the essential properties revealed by our studies will recur in the fuller theories: a truncation of a finite number of modes can only cause quantitative changes to the results but no qualitative ones. This is crucially different from the case of extra bosonic coordinates, where truncation means exorcising the infinite tower of massive spin-2 states and is qualitatively drastic.

Finally we want to emphasize that the idea of appending extra Grassmann coordinates to space-time in order to accommodate mysterious internal symmetries is very generally applicable. Apart from the well known and beautiful utilization of the idea to the superspace realization of supersymmetry,[8] it has also been used to great effect for visualizing spin[9] and realizing the Becchi-Rouet-Stora-Tyutin (BRST) algebra in gauge models.[10]

In this paper we have taken it a step further and investigated the simplest consequences of "general relativity" in such an enlarged space-time. Optimistically, it is possible that the concept may shed light on the generation problem too.

ACKNOWLEDGMENT

This research was supported by the Australian Research Grants Scheme.

[1]R. Delbourgo and R. B. Zhang, Phys. Lett. B **202**, 296 (1988).

[2]T. Appelquist and A. Chodos, Phys. Rev. D **28**, 772 (1983); A. Chodos and E. Myers, Ann. Phys. (N.Y.) **156**, 412 (1984); M. A. Rubin and B. D. Roth, Nucl. Phys. **B226**, 444 (1983).

[3]J. Strathdee, Int. J. Mod. Phys. **A1**, 1 (1986).

[4]S. Randjbar-Daemi, A. Salam, and J. Strathdee, Nucl. Phys. **B214**, 491 (1983).

[5]V. Kac, in Representations of Classical Lie Superalgebras, edited by K. Bleuler, A. Reetz, and H. R. Petry (Lecture Notes in Mathematics, Vol. 676) (Springer, Berlin, 1978), p. 597.

[6]The reason why the exponent c in the exponential looks so odd is because we have chosen ρ to be an exponential function. Any nonpolynomial ρ will actually do providing the coefficients of ξ^{2N} and ξ^{2N-2} are appropriately adjusted to reproduce the correct relative weighting between $R^{(g)}$ and $F^2/4$ in the action. A more natural choice of ρ will surely be found in time.

[7]In principle we could have included higher terms in ξ^2 in the expansion of Φ:

$$\Phi(X)=\xi^{M_1}\phi^{(1)}_{M_1}(x)+\xi^{M_1}\xi^{M_2}\xi^{M_3}\phi^{(3)}_{M_3M_2M_1}(x)+\cdots+\xi^{M_1}\xi^{M_2}\cdots\xi^{M_{2M+1}}\phi^{(2M+1)}_{M_{2M+1}M_{2M}\cdots M_1},$$

even though wrong spin-statistics fields are discarded. The series expansion order M is limited to $N-1$ and is also constrained by the condition $2M+1<N$. The reason for the latter is simple: if it is not satisfied, $(\xi^2)^{2M}\xi^2=0$; hence, $g^{\mu\nu}(\nabla_\mu\phi^{(2M+1)})^\dagger\nabla_\nu\phi^{(2M+1)}$ cannot appear in the action: however, terms such as $g^{\mu\nu}(\nabla_\mu\phi^{(1)})^\dagger\nabla_\nu\phi^{(2M+1)}$ do arise. When rediagonalizing the fields we inevitably end up with an unacceptable ghost term $-g^{\mu\nu}(\nabla_\mu\phi^{(2M+1)})^\dagger\nabla_\nu\phi^{(2M+1)}$. The same problem strikes for spinors. That is why we truncate the expansion of Φ and Ψ to the lowest-order pieces.

[8]A. Salam and J. Strathdee, Nucl. Phys. **B76**, 477 (1974); S. Ferrara, J. Wess, and B. Zumino, Phys. Lett. **51B**, 239 (1974).

[9]R. Delbourgo, Int. J. Mod. Phys. **A3**, 591 (1988).

[10]L. Bonora, P. Pasti, and M. Tonin, Nuovo Cimento **64A**, 307 (1981); R. Delbourgo and P. D. Jarvis, J. Phys. A **15**, 611 (1982); L. Baulieu and J. Thierry-Mieg, Nucl. Phys. **B197**, 477 (1982); F. Ore and P. van Nieuwenhuizen, *ibid.* **B204**, 317 (1982); R. Delbourgo, P. Jarvis, and G. Thompson, Phys. Rev. D **26**, 775 (1982).