Invariant path integration and the covariant functional measure for Einstein gravitation theory

Luiz C. L. Botelho
Universidade Federal do Pará, Campus do Guamá, Belém, Pará, Brazil
(Received 1 May 1987; revised manuscript received 24 February 1988)

We propose an invariant path-integral approach for the Einstein gravitation theory suitable to the analysis of the associated covariant functional measure problem.

I. INTRODUCTION

The path integral for gravitational interactions has been discussed several times in the past and most recently the important problem of the gravitational path-integral measure has been reexamined.[1–4]

In this paper we intend to propose an approach for the quantization of Einstein's gravitational theory in the framework of path integrals suitable to the analysis of the above-mentioned problem of the path-covariant local measure.

The basic idea in our discussion[1,5] is the introduction of a Riemann structure into the functional manifold of the metric field variables compatible with the invariance group of the theory and consider the associated partition functional as an infinite-dimensional version of an invariant integral in a Riemann manifold.[5] As a result we will not need to introduce the *ad hoc* insertion of the Faddeev-Popov unity resolution into the path-integral measure in order to extract the gauge orbit volume,[6] since we will be able to implement this calculation in a purely geometric way. So, in the proposed framework, it is not necessary to use *a posteriori* a constraint Hamiltonian path integral[7] to justify the Faddeev-Popov procedure; besides our approach leads to a natural and adequate local covariant path measure.

II. INVARIANT INTEGRATION

We start our analysis by briefly reviewing the basic results of the theory of invariant integrals in Riemann manifolds.[5]

Let T be a homomorphism of a compact Lie group G in the isometry group of a given Riemann manifold M. Let us consider the integral

$$\int_M f(x)[d\mu](x) \; , \tag{1}$$

where $f(x)$ is invariant under the action of G $[f(T(g)x)=f(x),\ \forall g\in G]$ and $[d\mu]$ is the measure in M induced by its Riemann metric. The orbit of a point $x\in M$ [the submanifold of M formed by all the points $\{T(g)x\},\ g\in G$] will be denoted by $O(x)$. The orbit quotient space M/G can be realized as a submanifold of M formed by all those points of M which are not related by a group element. The measure induced by the M-Riemann metric in M/G is denoted by $[d\bar{\mu}]$ and that induced in $O(x)$ by $[d\nu]$. Now we can state the basic result of the theory.[5] We have the following relationship between the integral (1) and an integral defined only over the orbit quotient space M/G:

$$\int_M f(x)[d\mu](x)=\int_{M/G} f(x)[d\bar{\mu}](x)v(x) \tag{2}$$

with

$$v(x)=\int_{O(x)}[d\nu](x) \; . \tag{3}$$

We remark that $[d\nu](x)$ is a G-invariant measure over the group G, since $O(x)$ can be realized as a "copy" manifold of G.

This result is fundamental for our analysis.

Another result of differential geometry which we will use is the coordinate expression for the induced metric in a given submanifold of M. Let $\{g_{hj}(x)\}$ denote the matrix of the metric tensor in M with $1\le h,j\le N$ (N being the dimension of M). Here, x belongs to an M coordinate domain. Let H be a submanifold of M described by the parametric equations

$$X_j=R_j(z_l) \tag{4}$$

with $\{z_l\}$ $(1\le z_l\le k;\ k\le N)$ belonging to a domain D (coordinate domain for H). Assuming that the matrix $[A]_{jk}(z_l)=\partial R_j/\partial z_k(z_l)$ has maximal characteristic k in D, the metric $\{g_{hj}(x)\}$ induces the following metric in H:

$$g^{(\mathrm{ind})}_{pq}(z_k)=(g_{hj}A^{hp}A^{jq})(z_k) \tag{5}$$

with the volume element given by

$$[dp](z_k)=[\det g^{(\mathrm{ind})}_{pq}(z_R)]^{1/2}dz^1\cdots dz^k \; . \tag{6}$$

After having displayed the basic results of invariant integration we pass to the problem of the path-integral quantization for the Einstein theory.

III. A QUANTUM PATH MEASURE FOR EINSTEIN THEORY

Let us start our analysis writing the Einstein-Hilbert action for the theory of gravitation defined in a d-dimensional Minkowski space-time manifold E with fixed topology and without boundary (see Ref. 8 for the case of an open space-time):

$$S[\{g_{\alpha\beta}(x)\}]=\frac{1}{16\pi G}\int_E(\sqrt{-g}\,R)(x)d^Dx \; , \tag{7}$$

where the field variables are given by those metric tensors $\{g_{\alpha\beta}(x)\}$ that can be defined in E, i.e., compatible with its topological structure, $-g(x)=\det\{g_{\mu\nu}(x)\}$, $R(x)$ being the scalar of curvature induced by $g_{\mu\nu}$ in M and G the Newton gravitational constant.

The starting point of the Feynman path-integral quantization for the Einstein theory is the formal continuous sum over $\{g_{\mu\nu}(x)\}$ histories:

$$Z=\sum_{\{g_{\mu\nu}(x)\}}\exp\left[\frac{i}{\hbar}S[\{g_{\mu\nu}(x)\}]\right] . \tag{8}$$

The precise meaning for the continuous sum (8) is achieved by introducing a path measure in the functional space of all possible field configurations (denoted by M); $[d\mu][g_{\alpha\beta}(x)]$, such that (8) can be written as

$$Z=\int_M [d\mu][g_{\alpha\beta}(x)]\exp\left[\frac{i}{\hbar}S[g_{\alpha\beta}(x)]\right] . \tag{9}$$

The fundamental problem in Eq. (9) is to define appropriately the path measure since the Einstein action possesses the physical invariance under the action of the group of the coordinate transformations in M (the Einstein general-relativity principle) denoted by $G^{\text{diff}}(E)$:

$$x^\mu \to l^\mu(x^\alpha) , \tag{10}$$

$$g_{\mu\nu}(x)\to \frac{\partial l^\mu(x^\alpha)}{\partial x_\sigma} g_{\sigma\rho}(l^\mu(x^\alpha))\frac{\partial l^\nu(x^\alpha)}{\partial x_\rho} \equiv (Lg_{\sigma\rho})_{\mu\nu}(x^\alpha) \tag{11}$$

and which in its infinitesimal version $[G^{\text{diff}}(E)]$ is given by

$$\delta x^\mu = \epsilon^\mu(x^\alpha) , \tag{12}$$

$$\delta g_{\mu\nu}(x^\alpha)=(\nabla_\mu\epsilon_\nu+\nabla_\nu\epsilon_\mu)(x^\alpha) , \tag{13}$$

where ∇_α is the usual covariant derivative defined by the metric $\{g_{\alpha\beta}(x)\}$.

This invariance property leads us to treat the above path integral as an infinite-dimensional version $G^{\text{diff}}(E)$—invariant integral in M [see Eq. (1)].

So, we intend to use the fundamental relation Eqs. (2) and (3) in its functional version in order to get its expression in the physical path manifold $M/G^{\text{diff}}(E)$. As a first step to implement the invariant integration theory we have to introduce a metric structure in $\mathbb{M}$ compatible with the group $G^{\text{diff}}(E)$. By following DeWitt's analysis[1] we introduce a metric (functional) tensor $\gamma^{(\mu\nu;\alpha\beta)}[g_{\sigma\rho}](x,x')$ on the functional path space $\mathbb{M}$ for which the actions of $G^{\text{diff}}(E)$ are isometries.

The unique (ultralocal) functional metric satisfying the above condition is given by the following expression[1,9] (the well-known "DeWitt functional metric"):

$$ds^2=\int_E d^Dx\sqrt{-g(x)}\int_E d^Dx'\sqrt{-g(x')}\delta g_{\mu\nu}(x) \times\gamma^{(\mu\nu;\alpha\beta)}[g_{\sigma\rho}](x,x')\delta g_{\alpha\beta}(x') , \tag{14}$$

where the ultralocal tensor density $\gamma^{(\mu\nu;\alpha\beta)}[g_{\sigma\rho}](x,x')$ is explicitly given by ($c\neq -2/D$)

$$\gamma^{(\mu\nu;\alpha\beta)}[g_{\sigma\rho}](x,x')=\frac{1}{\sqrt{2}}\frac{\delta^{(D)}(x-x')}{\sqrt{-g(x')}} \times(g^{\mu\alpha}g^{\nu\beta}+cg^{\mu\nu}g^{\alpha\beta})(x) \tag{15}$$

and $(\delta g_{\alpha\beta})(x)$ denotes the functional infinitesimal displacements on $\mathbb{M}$.

After introducing a Riemann structure on the path functional manifold $\mathbb{M}$ we can use the basic relationship, Eqs. (2) and (3), to give a precise meaning for the path integral:

$$Z=\int_M [d\mu][g_{\alpha\beta}(x)]\exp\left[\frac{i}{\hbar}S[\{g_{\alpha\beta}(x)\}]\right] . \tag{16}$$

As a first step, we have to realize the abstract orbit quotient space $M/G^{\text{diff}}(E)$ in $\mathbb{M}$. For this task we consider a set of D functionals $f^\mu(g_{\sigma\rho}(x))$ defined in $\mathbb{M}$ and in such a way that the equations [see Eq. (11)] in $G^{\text{diff}}(E)$,

$$f^\mu((Lg_{\alpha\beta})(x))=0,\quad \mu=1,\ldots,D , \tag{17}$$

have only the identity solution for a given $\{g_{\alpha\beta}(x)\}$; i.e., we have fixed our gauge. In order to simplify the discussion below we restrict our analysis to the class of the linear functionals $f^\mu(g_{\sigma\rho}(x))$ satisfying the following condition:

$$\delta f^\mu(g_{\alpha\beta}(x))/\delta g_{\mu\nu}(x') \text{ is a functional independent of the field variables } \{g_{\sigma\xi}(x)\} . \tag{18}$$

For instance, the well-known harmonic gauge $\partial^\alpha g_{\mu\alpha}(x)=f^\mu(g_{\alpha\beta}(x))$ belongs to the above-cited class.

Thus, we can realize the orbit quotient space $M/G^{\text{diff}}(E)$ in M as the path inequivalent manifold solution of Eq. (17) in M:

$$\bar{g}_{\alpha\beta}(x)\in M/G^{\text{diff}}(E)\Longrightarrow f^\mu(\bar{g}_{\alpha\beta}(x))=0 . \tag{19}$$

With this implicit $M/G^{\text{diff}}(E)$ parametrization the induced path measure is, thus, given by the well-known DeWitt result [see Ref. 9, Eq. (14.52)]

$$[d\bar{\mu}][\bar{g}_{\alpha\beta}(x)]=\prod_{(x\in E)}[dg_{\alpha\beta}(x)]\det\{\gamma^{(\mu\nu;\alpha\beta)}(x,x')\} \times\delta_F(f^\mu(g_{\sigma\rho}(x))) , \tag{20}$$

where

$$\det\{\gamma^{(\mu\nu;\alpha\beta)}(x,x')\}=(-1)^{D-1}\left[1+\frac{cD}{2}\right] \times(\sqrt{-g}\,)^{(D-4)(D+1)/4} \tag{21}$$

and the functional delta $\delta_F(f^\mu(g_{\sigma\rho}(x)))$ in the functional measure (20) restricts its support to the manifold of inequivalent metrics [Eq. (19)].

Now we have to evaluate the orbit (functional) volume defined by a given inequivalent configuration

$\{\bar{g}_{\alpha\beta}(x) \in M/G^{\text{diff}}(E)\}$. For this purpose we need an explicit parametrization of the orbit submanifold $O(\bar{g}_{\alpha\beta}(x))$. Such an expression is given explicitly by the path integral:

$$Y_{\mu\nu}[L;\bar{g}_{\alpha\beta}] = \int_M \left[\prod_{x\in E} dg_{\rho\sigma}(x)\right] g_{\mu\nu}(x) \times \delta_F(f^\mu(g_{\rho\sigma})(x) - f^\mu((L\cdot\bar{g})_{\rho\sigma}(x))) \,. \tag{22}$$

We remark that the $\{g_{\rho\sigma}(x)\}$ functional integration in Eq. (22) is defined over the whole functional manifold $\mathbb{M}$ and the group $G^{\text{diff}}(E)$ is the parameter domain for the orbit manifold $O(\bar{g}_{\alpha\beta}(x))$.

The functional integration over $\mathbb{M}$ gives straightforwardly the result

$$Y_{\mu\nu}[L;\bar{g}_{\alpha\beta}(x)] = (L\bar{g})_{\mu\nu}(x) \times \left[\prod_{\mu=1}^{D} \det{}_F \left[\frac{\delta f^\mu(g_{\alpha\beta})}{\delta g_{\rho\sigma}}\right](x)\right]^{-1} \tag{23}$$

and since the functional determinants involved in Eq. (23) are $g_{\alpha\beta}(x)$ independent by the condition (18) we find that $Y_{\mu\nu}[L;\bar{g}_{\alpha\beta}(x)]$ is an explicit parametrization of the orbit $O(\bar{g}_{\alpha\beta}(x))$; i.e., the image of $\mathbb{G}$ under $Y_{\mu\nu}[L;\bar{g}_{\alpha\beta}(x)]$ coincides with the orbit associated with the inequivalent metric $\{\bar{g}_{\alpha\beta}(x)\}$.

In order to evaluate the induced metric in $O(\bar{g}_{\alpha\beta}(x))$ by the DeWitt metric Eq. (14) we use the functional version of Eq. (5) with Eq. (22) playing the role of Eq. (4). So, the differential line element in $O(\bar{g}_{\alpha\beta}(x))$ is given by

$$ds^2_{\text{ind}} = \int d^Dx\, d^Dx' \left[\frac{\delta}{\delta\epsilon_\rho(x)} Y_{\mu\nu}[\epsilon^\gamma,\bar{g}_{\alpha\beta}]\right] \delta\epsilon_\rho(x) \times \sqrt{-\bar{g}(x)}\gamma^{(\mu\nu;\alpha\beta)}(\bar{g}_{\alpha\beta})(x,x')\sqrt{-\bar{g}(x')} \times \left[\frac{\delta}{\delta\epsilon_\sigma(x')} Y_{\alpha\beta}[\epsilon^\gamma,\bar{g}_{\alpha\beta}]\right] \delta\epsilon_\sigma(x') \,, \tag{24}$$

where we have considered the group transformation $\mathbb{L} \in G^{\text{diff}}(E)$ being infinitesimal and characterized by the infinitesimal generators $\{\epsilon^\gamma(x)\}$ [see Eqs. (12) and (13)].

Evaluating the functional derivatives in Eq. (24),

$$\begin{aligned}\frac{\delta}{\delta\epsilon_\rho} Y_{\mu\nu}[\epsilon^\gamma,\bar{g}_{\alpha\beta}] &= \int_M \left[\prod_{\substack{x\in E\\(\beta,\sigma)}} dg_{\beta\sigma}(x)\right] g_{\mu\nu}(x) \frac{\delta}{\delta\epsilon_\rho(x)}[\delta_F(f^\mu(g_{\gamma\zeta}) - f^\mu((L\cdot\bar{g})_{\gamma\zeta}))] \\ &= \sum_{(\alpha',\beta')} \left[\int_M \left[\prod_{\substack{x\in E\\(\beta,\sigma)}} dg_{\beta\sigma}(x)\right] g_{\mu\nu}(x) \left[-\frac{\delta}{\delta g_{\alpha'\beta'}(x)}[\delta_F(f^\mu(g_{\gamma\zeta}) - f^\mu((L\cdot\bar{g})_{\gamma\zeta}))]\right] \right. \\ &\qquad \left. \times \prod_{\mu=1}^{D} \det{}_F \left[\frac{\delta f^\mu((L\cdot\bar{g})_{\gamma\zeta})}{\delta\epsilon_\rho(x)}\right]\right]\end{aligned} \tag{25}$$

and using the functional version of the usual relation

$$\int_{-\infty}^{+\infty} g(x)\frac{d}{dx}\delta(f(x)) = -\sum_{\{x_0\}\in\mathbb{S}} \left.\frac{g'(x)}{f'(x)}\right|_{x=x_0} \tag{26a}$$

[where $\mathbb{S}$ denotes the set of zeros of $f(x)$] to evaluate the above functional integral; we get the (formal) result

$$\begin{aligned}\frac{\delta}{\delta\epsilon_\rho} Y_{\mu\nu}[\epsilon^\gamma;\bar{g}_{\alpha\beta}] &= \sum_{(\alpha',\beta')} \left\{\frac{\delta}{\delta g_{\alpha'\beta'}}\left[g_{\mu\nu}\prod_{\sigma=1}^{D}\det{}_F\left[\frac{\delta f^\sigma((L\cdot\bar{g})_{\gamma\zeta})}{\delta\epsilon_\rho}\right]\right]\right\} \\ &= \sum_{(\alpha',\beta')} \delta_{\mu\alpha'}\delta_{\nu\beta'}\left[\prod_{\sigma=1}^{D}\det{}_F\left[\frac{\delta f^\sigma((L\cdot\bar{g})_{\gamma\zeta})}{\delta\epsilon_\rho}\right]\right] ,\end{aligned} \tag{26b}$$

where we have used that $\delta f^\mu(g_{\alpha\beta})/\delta g_{\rho\sigma}(x)$ is a functional independent of the metric $\{g_{\gamma\zeta}(x)\}$ and $\delta/\delta g_{\alpha\beta} f^\mu(L\cdot\bar{g}) \equiv 0$ since $\{\bar{g}_{\alpha\beta}(x)\}$ is a fixed metric.

By substituting Eq. (26) into Eq. (24) we thus obtain

$$ds^2_{\text{ind}} = \int d^Dx\, d^Dx' \sqrt{-\bar{g}(x)} \det\left[\frac{\delta f^\mu(L\cdot\bar{g})}{\delta\epsilon_\rho(x)}\right][\delta\epsilon_\rho(x)] \times \text{Tr}[\bar{\gamma}^{(\mu\nu;\alpha\beta)}(\bar{g})]\sqrt{-\bar{g}(x')}\delta^{(D)}(x-x')\det\left[\frac{\delta f^\mu(L\cdot\bar{g})}{\delta\epsilon_{\rho'}(x')}\right][\delta\epsilon_{\rho'}(x')] \,, \tag{27}$$

where

$$\mathrm{Tr}[\bar{\gamma}^{(\mu\nu;\alpha\beta)}(\bar{g})]=\sum_{(\sigma_1,\sigma_2,\sigma_3,\sigma_4)}\{[\delta_\mu^{\sigma_1}\delta_\nu^{\sigma_2}(\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}+c\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta})\delta_\alpha^{\sigma_3}\delta_\beta^{\sigma_4}]\} \tag{28}$$

is the trace of the DeWitt metric defined by the fixed metric $\bar{g}_{\alpha\beta}(x)$.

The functional measure induced by Eq. (25) in $O(\bar{g}_{\mu\nu}(x))$ is then given by [see Eqs. (3)–(6)]

$$[d\nu][\bar{g}_{\alpha\beta}(x)]=\int\prod_{x\in E}(\sqrt{-\bar{g}}\,d\epsilon^\rho)(x)\{\mathrm{Tr}[\bar{\gamma}^{(\mu\nu;\alpha\beta)}(\bar{g})]\}^{1/2}\det[\delta f^\mu((L\cdot\bar{g})_{\alpha\beta})/\delta\epsilon_\rho]\ . \tag{29}$$

Since we are considering the infinitesimal group transformations in Eq. (29) we can use the Taylor expansion for the functional $\delta f^\mu(L\cdot\bar{g})/\delta\epsilon_\rho$, i.e.,

$$\frac{\delta f^\mu((L\cdot\bar{g})_{\alpha\beta})}{\delta\epsilon_\rho}=\left.\frac{\delta f^\mu((L\cdot\bar{g})_{\alpha\beta})}{\delta\epsilon_\rho}\right|_{\epsilon_\rho\equiv 0}(x)+O(|\epsilon|^2(x)) \tag{30}$$

and, as a consequence of Eq. (30), we get the result where the invariant group volume is covariantly factorized from the path integral:

$$[d\nu][\bar{g}_{\alpha\beta}(x)]=\{\mathrm{Tr}[\bar{\gamma}^{(\mu\nu;\alpha\beta)}(\bar{g})]\}^{1/2}\det\left[\frac{\delta f^\mu(L\cdot\bar{g})}{\delta\epsilon_\rho(x)}\right]\left[\int_{G^{\mathrm{diff}}}\prod_{x\in E}\sqrt{-\bar{g}(x)}(d\epsilon^\rho)(x)\right]\ . \tag{31}$$

Finally by grouping together the obtained results Eqs. (20) and (31) [see Eqs. (2) and (16)] we obtain our proposed path measure for Einstein gravitation theory:

$$[d\mu](g_{\alpha\beta})=\prod_{x\in E}[dg_{\alpha\beta}\det\gamma^{(\mu\nu;\alpha\beta)}\delta_F(f^\mu(g))\mathrm{Tr}(\gamma^{(\mu\nu;\alpha\beta)})^{1/2}](x)\det\left[\left.\frac{\delta f^\mu(L\cdot\bar{g})}{\delta\epsilon_\rho}\right|_{\epsilon_\rho\equiv 0}\right]\ . \tag{32}$$

At this point of our study it is instructive to point out that the above written measure differs from the original DeWitt measure by the factor $\mathrm{Tr}(\gamma^{(\mu\nu;\alpha\beta)})$ [see Eq. (28)] which in our framework takes into account the contribution from the geometric intersection between the orbit submanifold $O(\bar{g}_{\alpha\beta}(x))$ [see Eq. (23)] with the quotient space $M/G^{\mathrm{diff}}(E)$ in M [see Eqs. (24)–(29)]. However, we can see that this factor is irrelevant in the physical space-time $D=4$, since the functional measure

$$\prod_{x\in E}(dg_{\alpha\beta}\det\gamma^{(\mu\nu;\alpha\beta)})(x)$$

becomes "flat" [see Eq. (21)]. So, we can now safely use the dimensional regularization scheme to vanish the "tadpole" contribution $\mathrm{Tr}(\bar{\gamma}^{(\mu\nu;\alpha\beta)})$. This result in turn coincides with that proposed in Ref. 9 by DeWitt in $D=4$.

Another point worth remarking is that our functional measure naturally differs from those proposed in Refs. 2 and 3 since their results do not coincide with the DeWitt results.[9]

[1]B. DeWitt, J. Math. Phys. **3**, 625 (1962).
[2]K. Fujikawa, Nucl. Phys. **B226**, 437 (1983).
[3]Ron K. Unz, Phys. Rev. D **32**, 2539 (1985).
[4]D. J. Toms, Phys. Rev. D **35**, 3796 (1987).
[5]W. J. Hsiang, in *Proceedings of the Conference on Compact Transformation Groups* (Springer, Berlin, 1967); O. Babelon and C. M. Viallet, Commun. Math. Phys. **81**, 515 (1981).
[6]L. D. Faddeev and V. N. Popov, Phys. Lett. **25B**, 29 (1967).
[7]L. D. Faddeev, Teor. Mat. Fiz. **1**, 3 (1969).
[8]S. W. Hawking, in *General Relativity: An Einstein Centenary Survey,* edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).
[9]B. S. DeWitt, Phys. Rev. **160**, 1113 (1967).