# Response of accelerated detectors in coherent states and the semiclassical limit

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It is known that a uniformly accelerating detector gets excited in the inertial vacuum of a field, as if it were immersed in a thermal bath. We study the effect that the choice of the quantum state has on the response of the detector. The "correlation function" $\langle \Psi | \phi(x)\phi(x') | \Psi \rangle$ is computed in a Fock state, in the coherent state, and in the thermal state. It is then shown that the response function is a sum of two terms—the first being the same as the response in the vacuum state and a second term which is state dependent. The limit of the response function as $\hbar \to 0$ is discussed and the response of the inertial and accelerated detectors compared in this limit. The response in the large-$n_{\mathbf{k}}$ limit of the Fock state is compared with the response in the coherent state. We conclude that the thermal response of an accelerated detector in the inertial vacuum has interesting similarities to its response in other states.

## I. INTRODUCTION

The response of idealized detectors which couple to fields is determined by the nature of the coupling, the state of the field, and the state of motion of the detector. One example is that of the idealized Unruh-Dewitt point detector which is described by the interaction Hamiltonian

$$H_I = \alpha \int d\tau\, \mu(\tau) \phi[x(\tau)] \ , \tag{1}$$

where $\alpha$ is a small coupling constant, $\mu(\tau)$ is a detector variable, and the detector moves along the trajectory $x(\tau)$, $\tau$ being the proper time of the detector. The quantum field $\phi(x)$ is usually chosen to be in the inertial vacuum $|0\rangle_I$. If the detector is initially in the ground state $|E_0\rangle$, it does not get excited to a higher state $|E_1\rangle$ while moving along an inertial trajectory. For a detector moving with a uniform acceleration g, the rate of excitation is

$$R(\omega) \propto \frac{\omega}{e^{2\pi\omega g^{-1}} - 1} \ , \tag{2}$$

where $\omega = (E_1 - E_0)/\hbar$ (Refs. 1 and 2). The detector responds as if it were unaccelerated, but immersed in a thermal bath of temperature $T = g\hbar/2\pi$.

It is appropriate to choose the field in the inertial vacuum, if one simply wants to demonstrate that the response of the detector depends on its world line. However, it is of some interest to study response in other quantum states of the field as well. For instance, one may ask the following questions. (a) How does the detector respond if the quantum state $|\Psi\rangle$ of the field is chosen in such a way that the expectation value $\langle \Psi | \phi | \Psi \rangle$ resembles a classical field configuration? ($|\Psi\rangle$ may be a coherent state or a Fock-basis state $|n_{\mathbf{k}}\rangle$ with large $n_{\mathbf{k}}$.) Clearly one expects that the response function of this "quantum detector" should approach that of a classical detector, as $\hbar \to 0$. (By a classical detector we mean a classical system interacting with a classical field—e.g., a charged particle in a Coulomb field.) (b) How does the detector respond if $|\Psi\rangle$ is chosen to be a thermal state, the classical analog of which is a field at finite temperature? Is there a simple relation between the temperature of the field and the acceleration of the detector (in suitable units)? (c) Is the Planckian response of the accelerated detector peculiar to the *vacuum* state, or a feature present in other quantum states as well?

In this paper we investigate the response of the Unruh detector in the various states mentioned above. The purpose of such a study is twofold—(i) to understand the limit of the response function as $\hbar \to 0$ (semiclassical limit) and (ii) to find out how the response depends on the motion of the detector, while letting $|\Psi\rangle$ be as general as possible.

We find that in a state $|\Psi\rangle$ the response function typically splits into two parts. The first part is the Fourier transform of the Wightman function $\langle 0 | \phi(x)\phi(x') | 0 \rangle$, while the second part is essentially proportional to the square of the Fourier transform of $\langle \Psi | \phi(x) | \Psi \rangle$. In the semiclassical limit, the second term is expected to dominate over the first; and as we shall demonstrate, in this limit the response approaches that of a classical detector.

The contents of the paper are divided as follows. In Sec. II we briefly review the construction of the Unruh detector and the derivation of the response function. In Sec. III we summarize the relevant properties of coherent states. We then compute the correlation function $\langle \Psi | \phi(x)\phi(x') | \Psi \rangle$ for the various cases. In Sec. IV the above correlation functions are used to compute the response of detectors in various quantum states. The results for the different cases are compared in Sec. V.

## II. DETECTOR MODEL AND RESPONSE FUNCTION

Suppose the Unruh detector, described by Eq. (1), was in its ground state $|E_0\rangle$ in the asymptotic past and the field is in some quantum state $|\Psi\rangle$. The probability amplitude that the detector makes a transition to an excited state $|E_1\rangle$ $(E_1>E_0)$ and the field makes a transition to a state $|\Psi_f\rangle$ is given by[2]

$$A(\omega)=\alpha\langle E_1|\mu(0)|E_0\rangle\int_{-\infty}^{\infty}d\tau\, e^{i\omega\tau}\langle\Psi_f|\phi(x)|\Psi\rangle \quad (3)$$

to first order in perturbation theory. Here, $\omega=(E_1-E_0)/\hbar$ and we have assumed a Hamiltonian evolution

$$\mu(\tau)=e^{iH_d\tau}\mu(0)e^{-iH_d\tau} \quad (4)$$

for the detector variable, $H_d$ being the Hamiltonian operator for the detector. Since we are only interested in the probability $P(\omega)$ that the detector is excited to $|E_1\rangle$, we sum $|A(\omega)|^2$ over all final states to get

$$P(\omega)=E_{10}\int_{-\infty}^{\infty}d\tau\int_{-\infty}^{\infty}d\tau' e^{i\omega(\tau-\tau')} \times G_\Psi[\phi(x(\tau)),\phi(x(\tau'))] \; , \quad (5)$$

where

$$E_{10}=\alpha^2|\langle E_1|\mu(0)|E_0\rangle|^2 \quad (6)$$

and

$$G_\Psi[\phi(\tau),\phi(\tau')]\equiv\langle\Psi|\phi(\tau)\phi(\tau')|\Psi\rangle \quad (7)$$

is the analog of the Wightman function. Clearly, $P(\omega)$ depends on the trajectory assigned to the detector. If it so happens that $G_\Psi$ is a function only of the interval $(\tau-\tau')$ (e.g., when $|\Psi\rangle$ is the vacuum), then $P(\omega)$ is infinite, and it is more appropriate to talk of the transition rate $dP(\omega)/d(\tau+\tau)$, which is a constant. As it turns out, for a general $|\Psi\rangle$, $G_\Psi$ is not invariant under time translations.

To evaluate $P(\omega)$, we need the explicit form of $G_\Psi$ in Eq. (7). In the next section we calculate the correlation function for different states.

## III. CALCULATION OF CORRELATION FUNCTIONS

In the Introduction we outlined the reasons for studying detector response in a state other than $|0\rangle_I$. The specific quantum states we consider will be (i) a coherent state, (ii) a one-particle state $|i_{\mathbf{k}}\rangle$, (iii) an arbitrary Fock state, and (iv) a thermal state. The computation of $G_\Psi$ for these states is straightforward in the Heisenberg picture.

A real massless scalar field can be expanded in basis modes as

$$\phi(x)=\int d^3\mathbf{k}[a_{\mathbf{k}}u_{\mathbf{k}}(x)+a_{\mathbf{k}}^\dagger u_{\mathbf{k}}^*(x)] \quad (8)$$

with

$$u_{\mathbf{k}}(x)=[(2\pi)^3 2\omega_{\mathbf{k}}]^{-1/2}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)}, \quad \omega_{\mathbf{k}}=|\mathbf{k}| \; , \quad (9)$$

where $(\mathbf{x},t)$ are inertial coordinate labels. As usual the inertial vacuum is defined by $a_{\mathbf{k}}|0\rangle_I=0$. The Wightman function ${}_I\langle 0|\phi(x)\phi(x')|0\rangle_I$ can be shown to be

$$G_{\rm vac}(x,x')=\frac{-1}{4\pi^2}\frac{1}{(t-t'-i\epsilon)^2-(\mathbf{x}-\mathbf{x}')^2} \; . \quad (10)$$

Let us proceed to calculate $G_\Psi$ for other states.

### A. Coherent state

A coherent state $|\alpha\rangle$ for the harmonic oscillator described by the modes $(a_{\mathbf{k}},a_{\mathbf{k}}^\dagger)$ is an eigenstate of the annihilation operator,

$$a_{\mathbf{k}}|\alpha\rangle=\alpha_{\mathbf{k}}|\alpha\rangle \; , \quad (11)$$

or, equivalently,

$$|\alpha\rangle=\exp(\alpha_{\mathbf{k}}a_{\mathbf{k}}^\dagger-\alpha_{\mathbf{k}}^*a_{\mathbf{k}})|0\rangle \; , \quad (12)$$

$|0\rangle$ being the ground state of the harmonic oscillator. For the field $\phi(x)$, the coherent state can then be defined as

$$|\Psi_c\rangle=\exp\left[\sum_{\mathbf{k}}(\alpha_{\mathbf{k}}a_{\mathbf{k}}^\dagger-\alpha_{\mathbf{k}}^*a_{\mathbf{k}})\right]|0\rangle_I \; . \quad (13)$$

The inertial vacuum is a special coherent state, in which $\alpha_{\mathbf{k}}=0\ \forall\mathbf{k}$.

To compute the two-point function

$$G_c(x,x')=\langle\Psi_c|\phi(x)\phi(x')|\Psi_c\rangle \quad (14)$$

we write the field operator $\phi(x)$ as

$$\phi(x)=\phi_+(x)+\phi_-(x) \; , \quad (15)$$

where

$$\phi_+(x)=\int d^3\mathbf{k}\, a_{\mathbf{k}}u_{\mathbf{k}}(x), \quad \phi_-(x)=[\phi_+(x)]^\dagger \; . \quad (16)$$

It is easily seen that

$$\phi_+(x)|\Psi_c\rangle=f(x)|\Psi_c\rangle \; , \quad (17)$$

where

$$f(x)=\int d^3\mathbf{k}\,\alpha_{\mathbf{k}}u_{\mathbf{k}}(x) \; . \quad (18)$$

Using (17) and the expansion of $\phi(x)$ in (15), it can be shown in a straightforward manner that

$$G_c(x,x')=[f(x)+f^*(x)][f(x')+f^*(x')] + [\phi_+(x),\phi_-(x')] \quad (19)$$

and noting that the commutator

$$[\phi_+(x),\phi_-(x')]=\int d^3\mathbf{k}\,u_{\mathbf{k}}(x)u_{\mathbf{k}}(x') = G_{\rm vac}(x,x') \quad (20)$$

we get

$$G_c(x,x')=G_{\rm vac}(x,x')+\bar\phi(x)\bar\phi(x') \; . \quad (21)$$

$G_{\rm vac}(x,x')$ was defined in (10), and

$$\bar\phi(x)\equiv\langle\Psi_c|\phi(x)|\Psi_c\rangle=f(x)+f^*(x) \; , \quad (22)$$

as may be easily verified using Eqs. (15)–(17). Thus the correlation function in the coherent state is the sum of the Wightman function and the $\bar\phi\bar\phi$ term. It is reasonable to think of $G_{\rm vac}(x,x')$ as the quantum part, and the second term as the "classical part," which is expected to survive as $\hbar\to 0$. The form of $G_c(x,x')$ makes an interpretation of the detector response function quite simple.

One should note that the coherent state defined in this section is an eigenstate of $a_{\mathbf{k}}$, which acts on the *inertial* vacuum $|0\rangle_I$. A coherent state defined with respect to, say, the Rindler vacuum, will not, of course, be the same as the above state.

### B. Fock state

A general Fock-basis state may be written as

$$|\Psi\rangle_F = |{}^{1_n}\mathbf{k}_1, {}^{2_n}\mathbf{k}_2, \ldots\rangle , \tag{23}$$

where ${}^{i_n}\mathbf{k}_i$ indicates that there are $i_n$ particles in the mode $\mathbf{k}_i$. The correlation function can be computed using the mode expansion (8) and is given by

$$\begin{aligned} G_{\mathrm{Fock}}(x,x') &\equiv {}_F\langle\Psi|\phi(x)\phi(x')|\Psi\rangle_F \\ &= G_{\mathrm{vac}}(x,x') \\ &\quad + 2\sum_i {}^{i_n}\mathbf{k}_i \mathrm{Re}[u_{\mathbf{k}}(x)u_{\mathbf{k}}^*(x')] . \end{aligned} \tag{24}$$

[Here, the summation index runs over all those modes whose particles are present in the state $|\Psi_F\rangle$. The correlation function in the one-particle state $|1_{\mathbf{k}}\rangle$ can be easily read off from (24).]

### C. Thermal state

Strictly speaking, thermal *state* is a misnomer because a quantum system in thermodynamic equilibrium is described by a density matrix and not by a pure state. If $|E_\alpha\rangle$ are the energy eigenstates of the field $\phi(x)$, we say that it is at a finite temperature $T = k^{-1}\beta$ if the probability for it to be in the state $|E\alpha\rangle$ is $\exp(-\beta E_\alpha)$. The thermal average of an operator $A$ is defined to be

$$\{A\} \equiv \frac{\sum_\alpha \exp(-\beta E_\alpha)\langle\alpha|A|\alpha\rangle}{\sum_\alpha \exp(-\beta E_\alpha)} , \tag{25}$$

which may be written using a density matrix $\rho \equiv \exp(-\beta H)$ as

$$\{A\} = \mathrm{Tr}(A\rho)/\mathrm{Tr}\rho . \tag{26}$$

How does one compute detector response if the field is at a finite temperature? Equation (5) gives the response function when the initial state is $|\Psi\rangle$. Suppose there is a probability $\exp(-\beta E_\alpha)$ that $|\Psi\rangle$ is the state $|E_\alpha\rangle$. The response function $P_{\mathrm{th}}(\omega)$ will then be equal to the weighted sum of response functions for different initial states $|E_\alpha\rangle$:

$$\begin{aligned} P_{\mathrm{th}}(\omega) &= \frac{\sum_\alpha e^{-\beta E_\alpha} P(\omega; |E_\alpha\rangle)}{\sum_\alpha e^{-\beta E_\alpha}} \\ &= \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' e^{i\omega(\tau-\tau')} G_{\mathrm{th}}(x,x') , \end{aligned} \tag{27}$$

where the "thermal" correlation function is defined by

$$\begin{aligned} G_{\mathrm{th}}(x,x') &\equiv \{\phi(x)\phi(x')\} \\ &= \frac{\sum_\alpha e^{-\beta E_\alpha}\langle E_\alpha|\phi(x)\phi(x')|E_\alpha\rangle}{\sum_\alpha e^{-\beta E_\alpha}} . \end{aligned} \tag{28}$$

(Curly brackets indicate thermal averages.) $G_{\mathrm{th}}(x,x')$ may be computed using the mode expansion in Eq. (8). The thermal averages of the various combinations of $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$ are[3]

$$\begin{aligned} &\{a_{\mathbf{k}}a_{\mathbf{k}'}\} = \{a_{\mathbf{k}}^\dagger a_{\mathbf{k}'}^\dagger\} = 0 , \\ &\{a_{\mathbf{k}}a_{\mathbf{k}'}^\dagger\} = (e^{\beta\omega_{\mathbf{k}}} - 1)^{-1}\delta_{\mathbf{k},\mathbf{k}'} , \\ &\{a_{\mathbf{k}}^\dagger a_{\mathbf{k}'}\} = \frac{e^{\beta\omega_{\mathbf{k}}}}{e^{\beta\omega_{\mathbf{k}}} - 1}\delta_{\mathbf{k},\mathbf{k}'} . \end{aligned} \tag{29}$$

Thus, $G_{\mathrm{th}}(x,x')$ can be written as

$$\begin{aligned} G_{\mathrm{th}}(x,x') &= G_{\mathrm{vac}}(x,x') \\ &\quad + \int \frac{d^3\mathbf{k}}{(2\pi)^3(2\omega_{\mathbf{k}})} \frac{2\cos k(x,x')}{e^{\beta\omega_{\mathbf{k}}} - 1} . \end{aligned} \tag{30}$$

It is convenient to work with the form in (30), without carrying out the integration in (30).

Equations (21), (24), and (30) provide us with the required correlation functions. In the next section we use these results to compute the response functions. However, before we proceed to do so, let us recall why these quantum states are of interest. The expectation value of the field in the coherent state obeys a classical evolution (mode by mode), while the dispersion (for every mode) is a constant. So we expect that the response in the coherent state will have a "quantum part" which is constant, and a part which resembles the response in a classical field. This indeed turns out to be true. Second, it is known that a state $|n_{\mathbf{k}}\rangle$ with large $n_{\mathbf{k}}$ mimics a classical oscillator, though in a sense different from a coherent state (see, e.g., Ref. 4, Chap. 13). So it is of interest to compare the response in a large-$n_{\mathbf{k}}$ state with the response in a coherent state. As regards a thermal state at temperature $T$, one would like to know if the accelerated detector "sees" a Planckian of temperature $T + g\hbar/2\pi$ (nothing so simple happens).

## IV. THE RESPONSE FUNCTIONS AND THEIR INTERPRETATION

The probability that the detector makes a transition to an excited state can now be found by taking the Fourier transform of the appropriate correlation function, as in Eq. (5). We will consider each case separately.

### A. Coherent state, inertial trajectory

Consider first a detector moving along the trajectory

$$\mathbf{x}(t) = \mathbf{x}_0(t) + \mathbf{v}t \tag{31}$$

interacting with the field, initially in a coherent state. As is evident from (21), $P(\omega)$ is determined by a sum of two terms:

$$P(\omega)=P_1(\omega)+P_2(\omega)\ . \tag{32}$$

The first $P_1(\omega)$ is the Fourier transform of the Wightman function $G_{\rm vac}(x,x')$. However, it can be easily shown that $P_1(\omega)$ vanishes along the inertial trajectory; inertial detectors do not respond in the inertial vacuum state. The second part $P_2(\omega)$ is the Fourier transform of the $\bar{\phi}\,\bar{\phi}$ term [we shall adopt the notation of (32) for the remainder of this paper]. Thus for inertial detectors we have

$$P(\omega)=P_2(\omega)\equiv E_{10}\,|\,\bar{\phi}_\omega\,|^2\ , \tag{33}$$

where

$$\bar{\phi}_\omega=\int d\tau\, e^{i\omega\tau}\bar{\phi}(x)\ . \tag{34}$$

Equation (33) implies that the transition probability is proportional to the power spectrum of the mean field, evaluated at the frequency $\omega$. This is analogous to what we expect in the classical limit. For example, the dipole oscillator immersed in an electromagnetic field will absorb energy at a rate proportional to the power spectrum of the field (see, for instance, Ref. 5, Sec. 13.2)

This can be seen more clearly by writing $\bar{\phi}(x)$ using the mode expansion (8), so that

$$\bar{\phi}=\int d^3\mathbf{k}[\alpha_{\mathbf{k}}u_{\mathbf{k}}(x)+\alpha_{\mathbf{k}}^*u_{\mathbf{k}}^*(x)]\ . \tag{35}$$

(Recall that $\alpha_{\mathbf{k}}$ is the expectation value of the operator $a_{\mathbf{k}}$ in the coherent state.) For the inertial trajectory (35) gives

$$\begin{gathered}\bar{\phi}_\omega=\int d^3\mathbf{k}\,\alpha_{\mathbf{k}}N_{\mathbf{k}}\delta[\omega-\gamma(\,|\,\mathbf{k}\,|\,-\mathbf{k}\cdot\mathbf{v})]\ ,\\ \gamma=(1-\mathbf{v}^2)^{-1/2},\quad N_{\mathbf{k}}=[2\omega_{\mathbf{k}}(2\pi)^3]^{-1/2}\ .\end{gathered} \tag{36}$$

That is, only those modes contribute to $\bar{\phi}_\omega$ which are "Doppler shifted" to the frequency $\omega$, and $\alpha_{\mathbf{k}}N_{\mathbf{k}}$ is the amplitude of the corresponding mode [for simplicity we have set $\mathbf{x}_0=0$ in (31)]. Thus the response of inertial detectors in a coherent state is similar to the response in a classical field.

### B. Coherent state, accelerated detector; longitudinal mode

We now evaluate $P(\omega)$ for the more interesting case of a uniformly accelerating detector. For a detector moving along the $z$ axis of Minkowski coordinates, the trajectory is a hyperbola

$$t=g^{-1}\sinh g\tau,\quad z=g^{-1}\cosh g\tau,\quad x=y=0\ , \tag{37}$$

where $\tau$ is the proper time of the detector and $g$ the magnitude of the acceleration four-vector. As in (32), the detector response has two parts. $P_1(\omega)$, which is determined by $G_{\rm vac}(x,x')$, is the well-known Planck spectrum[1,2]

$$P_1(\omega)=E_{10}\int_{-\infty}^{\infty}d\tau\, d\tau' e^{i\omega(\tau-\tau')}G_{\rm vac}[x(\tau),x(\tau')] \tag{38}$$

$$=E_{10}\int_{-\infty}^{\infty}d\tau''\frac{1}{2\pi}\frac{\omega}{e^{2\pi\omega g^{-1}}-1}\ . \tag{39}$$

To find the $P_2(\omega)$ of (32) in the accelerated frame, we proceed as follows. We have

$$\begin{aligned}P_2(\omega)&=E_{10}\int_{-\infty}^{\infty}d\tau\int_{-\infty}^{\infty}d\tau' e^{i\omega(\tau-\tau')}\bar{\phi}(x)\bar{\phi}(x')\\ &=E_{10}\,|\,\bar{\phi}_\omega\,|^2\end{aligned} \tag{40}$$

with $\bar{\phi}_\omega$ as in (34). We need to express $\bar{\phi}_\omega$ in terms of the detector's proper time $\tau$. Using (8), (9), and (37), $\bar{\phi}(x)$ can be expressed along the accelerated trajectory as

$$\bar{\phi}(x)=\int d^3\mathbf{k}[\alpha_{\mathbf{k}}u_{\mathbf{k}}(\tau)+\alpha_{\mathbf{k}}^*u_{\mathbf{k}}^*(\tau)]\ , \tag{41}$$

where

$$u_{\mathbf{k}}(\tau)=N_{\mathbf{k}}e^{ig^{-1}(k_z\cosh g\tau-\omega_{\mathbf{k}}\sinh g\tau)}\ . \tag{42}$$

[It should be recalled that $\bar{\phi}(x)$ and hence $\alpha_{\mathbf{k}}$ are expectation values in the *inertial* coherent state.] Fourier transforming with respect to $\tau$ we get

$$\bar{\phi}_\omega=\int d^3\mathbf{k}\int d\tau\, e^{i\omega\tau}[\alpha_{\mathbf{k}}u_{\mathbf{k}}(\tau)+\alpha_{\mathbf{k}}^*u_{\mathbf{k}}^*(\tau)]\ . \tag{43}$$

Clearly, the value of $\bar{\phi}_\omega$ depends on the choice made for $\alpha_{\mathbf{k}}$, the only free parameter in a coherent state. Consider the simple case in which only one mode has a nonzero amplitude, and it moves along the positive $z$ axis; i.e.,

$$\alpha_{\mathbf{k}}\equiv\alpha_k\delta(k_z-k)\delta(k_x)\delta(k_y) \tag{44}$$

with a real $\alpha_k$ (and $k>0$). We get

$$\bar{\phi}_\omega=\int d\tau\, e^{i\omega\tau}\alpha_k[u_k(\tau)+u_k^*(\tau)] \tag{45}$$

with $\omega_k=k$. (We have chosen $k>0$.) The Fourier transform of $u_k(\tau)$ can now be done. We get

$$\begin{aligned}&N_k\int d\tau\, e^{i\omega\tau}e^{i\omega_k g^{-1}e^{-g\tau}}\\ &\quad=N_k\omega_k^{i\omega q^{-1}}g^{-i\omega g^{-1}-1}e^{(\pi/2)\omega g^{-1}}\Gamma(-i\omega g^{-1})\ .\end{aligned} \tag{46}$$

[The above integration may be performed by substituting $p=e^{-g\tau}$ and by rotating the contour to the imaginary axis; $p\to ip$ (see first citation in Ref. 1, Sec. 5.5).] The Fourier transform of $u_k^*(\tau)$ can be performed similarly, so that we finally get

$$\begin{aligned}\bar{\phi}_\omega=&\,e^{i\omega g^{-1}\ln(\omega_k/g)}N_kg^{-1}\alpha_k\\ &\times\Gamma(-i\omega g^{-1})(e^{(\pi/2)\omega g^{-1}}+e^{-(\pi/2)\omega g^{-1}})\end{aligned} \tag{47}$$

and hence

$$\begin{aligned}P_2(\omega)&=E_{10}\,|\,\bar{\phi}_\omega\,|^2\\ &=E_{10}\alpha_k^2N_k^22\pi\omega^{-1}g^{-1}\coth(\pi\omega/2g)\ .\end{aligned} \tag{48}$$

Equation (48) gives the response of the detector to the "semiclassical" part of the correlation function, for a single mode, i.e., when $\alpha_{\mathbf{k}}$ is given by (44).

The total response function is obtained by adding $P_1(\omega)$ and $P_2(\omega)$ from (39) and (48):

$$P(\omega)=E_{10}\left[\int_{-\infty}^{\infty}d\tau''\frac{1}{2\pi}\frac{\omega}{e^{2\pi\omega/g}-1}+\alpha_k^2N_k^22\pi\omega^{-1}g^{-1}\coth(\pi\omega/2g)\right]\ . \tag{49}$$

At this stage, it is important to clarify the role of Planck's constant, and the sense in which the $\hbar \to 0$ limit is called the semiclassical limit. It is generally assumed that the first term, the Fourier transform of the Wightman function, has no classical analog essentially because the vacuum state has no classical analog. How should one take the classical limit of $P_1(\omega)$? Clearly, if one does this by considering $P_1(\omega)$ as a function of $\omega$, and takes the limit $\hbar \to 0$ while keeping $\omega$ constant, we do not get what we expect:

$$\lim_{\hbar \to 0} P_1(\omega)\,|_{\text{constant } \omega} \equiv P_1(\omega) \ . \tag{50}$$

The "correct" limit is expected to be obtained by realizing that $\omega = \Delta E/\hbar$, and that while taking the limit, it is $\Delta E$, and not $\omega$, which should be kept constant:

$$\lim_{\hbar \to 0} \left[ \exp\left[2\pi g^{-1}\frac{\Delta E}{\hbar}\right] - 1 \right]^{-1} = 0 \ . \tag{51}$$

Why should the classical limit be taken by keeping $\Delta E$ constant? The usual justification is that the proportionality $\Delta E \propto \omega$ is true for the quantum detector, but not for the classical one (for which $\Delta E \propto \omega^2$). Since the relevant physical quantity is the energy absorbed by the detector (classical or quantum), and since the relation between $\Delta E$ and $\omega$ is different for the two, it makes sense to take the $\hbar \to 0$ limit in this way.

On the other hand, the term $P_2(\omega)$ is the square of the Fourier transform of the mean field $\bar{\phi}(x)$. Since the mean field corresponds to some classical configuration, this term has a classical analog *if* $\omega$ is interpreted as the natural frequency of oscillation of the dipole oscillator. The correct classical limit is obtained by noting in (48) that as $\hbar \to 0$, $P_2(\omega)$ behaves as

$$P_2(\omega) = 2\pi g^{-1} E_{10} N_k^2 \alpha_k^2 \hbar/\Delta E \ . \tag{52}$$

Moreover, it can be argued that if $f(k)$ is the amplitude of the classical wave mode, then it is related to the amplitude $\alpha_k$ as $\alpha_k^2 = (\text{const}/\hbar) f_k^2$ [Ref. 6, Eqs. (30) and (34)]. Then $P_2(\omega)$ is finite as $\hbar \to 0$.

In summary, we can make the term $P_1(\omega)$ vanish in the limit $\hbar \to 0$, keeping $P_2(\omega)$ finite. [In this limit $P_2(\omega)$ corresponds to the response of the classical oscillating dipole at a finite acceleration $g$.] But this requires a fairly nontrivial "rule" for taking the classical limit. Note that if we take the limit $\hbar \to 0$, $\omega = \text{const}$, neither term vanishes; if we take $\hbar \to 0$, $\Delta E = \text{const}$ with $\alpha_k$ and $\hbar$ unrelated, then both vanish. To eliminate the first term, retaining the second needs $\alpha_k^2 \hbar / f_k^2 = \text{const}$ and $\Delta E = \text{const}$, $\hbar \to 0$.

We should also clarify the term "classical detector." The internal energy levels of the Unruh detector get excited by interaction with the field, whereas the "center-of-mass" motion is not determined by the field. (The hydrogen atom in the Coulomb field could be one such system.) This allows one to assign an arbitrary motion to the detector. One can also think of a different kind of classical detector in which the center-of-mass motion itself is determined by the field—for instance, an electric charge in a Coulomb field. The classical analog of the Unruh-type detector could be a dipole which absorbs energy from the field, while its external motion does not depend on the field. Whenever we talk of the classical limit of the response function, we have in mind the response of such a detector with prescribed center-of-mass motion.

Let us next look at the $g \to 0$ limit of (48). Again, $P_1(\omega)$ goes to zero (for $\omega > 0$) as $g \to 0$. This matches with the response $P_1(\omega)$ of the inertial detector. As can be seen from (49), the second term $[P_2(\omega)]$ is divergent in the $g \to 0$ limit. This divergence has no deep significance and arises because of the choice of $\alpha_k$ in (44) which makes $P_2(\omega)$ a square of $\delta$ function. The corresponding response function $P_2(\omega)$ for the inertial detector is also divergent. This may be verified by substituting (44) in the expression (36) for $\bar{\phi}_\omega$, which shows that the inertial response function diverges as the square of a $\delta$ function. Thus the $g \to 0$ divergence corresponds to a divergence of the form $[\delta(x)]^2$.

Having noted this, we may interpret $P_2(\omega)$ as follows. Since $P_2(\omega) \neq 0$ for the inertial detector, the excess response of the accelerated detector over the inertial one may be obtained as

$$\begin{aligned} P_{\text{excess}} &= P_2(\omega) - \lim_{g \to 0} P_2(\omega) \\ &= 2\pi\omega^{-1}\alpha_k^2 N_k^2 g^{-1}(e^{\pi\omega g^{-1}} - 1)^{-1} \ . \end{aligned} \tag{53}$$

How does one compare $P_1(\omega)$ and $P_2(\omega)$ in (49)? The major difficulty is that $P_1(\omega)$ is infinite—the rate is an appropriate quantity to measure. However, what we are interested in is the dependence of the ratio $P_1(\omega)/P_2(\omega)$ on $\omega$. For this purpose we can think of the detector as being adiabatically "switched off," namely, in (39),

$$\int_{-\infty}^{\infty} d\tau'' \approx \int_{-T/2}^{T/2} d\tau'' = T \ , \tag{54}$$

where $T \to \infty$. Then from (39) and (48) we get

$$\begin{aligned} R(\omega) \equiv \frac{P_1(\omega)}{P_2(\omega)} &= 4\pi T g \frac{\omega_{\mathbf{k}}}{\alpha_k^2} \frac{\omega^2}{(e^{\pi\omega g^{-1}} + 1)^2} \\ &\equiv T A_{\mathbf{k}} \frac{\omega^2 g}{(e^{\pi\omega g^{-1}} + 1)^2} \ . \end{aligned} \tag{55}$$

As expected, $R(\omega)$ goes to zero as $g \to 0$. Moreover, $R(\omega)$ vanishes in the limits $\omega \to 0$ and $\omega \to \infty$. Thus, considered as a function of $\omega$, the ratio of the quantum response to the "classical response" is maximized when $\omega \simeq g$. From (55) we get $R_{\max} \simeq T A_{\mathbf{k}} g^3$. If we were to take for $T$ the natural time scale for the Rindler coordinate system: $T \sim c g^{-1}$ (see, e.g., Ref. 7, Chap. 6) we get $R_{\max} \simeq A_{\mathbf{k}} c g^2$.

### C. Coherent state, accelerated trajectory; arbitrary mode

In the previous section we evaluated $P(\omega)$ for a single mode, moving parallel to the detector. Let us also find $P(\omega)$ without constraining $\mathbf{k}$. Of course, $P_1(\omega)$ is still as in (39). To calculate $P_2(\omega)$ reconsider (41). With the substitution

$$\begin{aligned} &k_z = k_T \sinh\theta, \quad \omega_{\mathbf{k}} = k_T \cosh\theta \ , \\ &k_T^2 \equiv \omega_{\mathbf{k}}^2 - k_z^2 \end{aligned} \tag{56}$$

the $u_{\mathbf{k}}$ in (42) can be rewritten as

$$u_{\mathbf{k}}=e^{ik_T g^{-1}\sinh(\theta-g\tau)} \tag{57}$$

and the Fourier transform

$$I(\omega,\mathbf{k})\equiv\int d\tau\, e^{i\omega\tau}u_{\mathbf{k}}(x) \tag{58}$$

becomes

$$I(\omega,\mathbf{k})=g^{-1}I_{\mathbf{k}}\int_{-\infty}^{\infty} e^{-iax+ib\sinh x}\ , \tag{59}$$

where

$$\begin{aligned} x&=\theta-g\tau,\quad I_{\mathbf{k}}=N_{\mathbf{k}}e^{i\omega g^{-1}\theta}\ ,\\ a&=\omega g^{-1}>0,\quad b=k_T g^{-1}>0\ . \end{aligned} \tag{60}$$

$I(\omega,\mathbf{k})$ in (59) happens to be an integral representation of the modified Bessel function (Ref. 8, p. 182), so that

$$I(\omega,\mathbf{k})=2g^{-1}I_{\mathbf{k}}e^{\pi a/2}K_{ia}(b)\ . \tag{61}$$

The Fourier transform of $u_{\mathbf{k}}^*(\tau)$ can be found similarly, so that $\bar{\phi}_\omega$ of (41) is

$$\begin{aligned}\bar{\phi}_\omega=2g^{-1}\int d^3\mathbf{k}\, I_{\mathbf{k}}K_{i\omega g^{-1}}(k_T g^{-1})\\ \times(\alpha_{\mathbf{k}}e^{\pi\omega/2g}+\alpha_{\mathbf{k}}^* e^{-\pi\omega/2g})\ ,\end{aligned} \tag{62}$$

which has a form similar to the $\bar{\phi}_\omega$ of (47) with the Bessel function replacing the $\gamma$ function.

In the case when only one mode is excited, i.e., if

$$\alpha_{\mathbf{k}}=\alpha_{\mathbf{k}_0}\delta(\mathbf{k}-\mathbf{k}_0) \tag{63}$$

with $\alpha_{\mathbf{k}_0}$ real, (62) gives

$$\begin{aligned}P_2(\omega)=4g^{-2}E_{10}\alpha_{\mathbf{k}_0}^2N_{\mathbf{k}_0}^2\,|K_{i\omega g^{-1}}(k_T g^{-1})|^2\\ \times(e^{\pi\omega/2g}+e^{-\pi\omega/2g}+2)\ .\end{aligned} \tag{64}$$

Equation (64) gives the response function for an arbitrary mode $\mathbf{k}_0$, and does not appear to have a simpler form. Of course, in the limit that $k_T\to 0$, one could show that the result goes over to that of Eq. (49). It is interesting to note that (64) has a simple limit in one special case. Suppose the response function $P_2(\omega)$ is averaged over all the transverse frequencies, while keeping $\omega_{\mathbf{k}}$ and the amplitude $\alpha_{\mathbf{k}}$ constant. Then, using the relation (Ref. 9, p. 49)

$$\int_0^\infty dx\, x\,|K_{i\nu}(x)|^2=\tfrac{1}{2}\,|\Gamma(1+i\nu)|^2 \tag{65}$$

we can show

$$[P_2(\omega)]_{\text{average}}=8\pi^2\omega g^{-3}E_{10}\alpha_{\mathbf{k}}^2N_{\mathbf{k}}^2\coth(\pi\omega/2g)\ , \tag{66}$$

which has the same $\omega$ dependence as the response function for a longitudinal mode. There does not seem to be any straightforward interpretation for this result.

This completes our analysis of response in the coherent state. In Sec. V we will compare these results with those of response in other states.

### D. Response in a Fock state

Consider first the one-particle state $|1_{\mathbf{k}}\rangle$. Using the correlation function from (24), and the by now familiar analysis of Sec. IV A, the response in an inertial frame is found to be

$$P(\omega)=E_{10}N_{\mathbf{k}}^2[\delta(\omega-\gamma(|\mathbf{k}|-\mathbf{k}\cdot\mathbf{v}))]^2\ . \tag{67}$$

This is essentially the same as the response of the detector to a plane-wave mode. It diverges as $[\delta(x)]^2$; the divergence can be expressed as an infinite time integral times a constant "rate." To do this, we note that $u_{\mathbf{k}}(\tau)u_{\mathbf{k}}^*(\tau')$ is a function $(\tau-\tau')$, so that Eq. (5) can be written for the state $|1_{\mathbf{k}}\rangle$ as

$$P(\omega)=E_{10}N_{\mathbf{k}}^2\int_{-\infty}^{\infty}d\tau''\delta(\omega-\gamma(|\mathbf{k}|-\mathbf{k}\cdot\mathbf{v}))\ . \tag{68}$$

Only $|1_{\mathbf{k}}\rangle$ states with appropriate frequencies excite the detector. Moreover, the response in an $n$-particle state $|n_{\mathbf{k}}\rangle$ is simply $n_{\mathbf{k}}$ times the response in $|1_{\mathbf{k}}\rangle$, and the response in $|\Psi\rangle_{\text{Fock}}$ of (23) is $\sum_i n_{\mathbf{k}_i}P(\omega,|1_{\mathbf{k}}\rangle)$.

In the accelerated frame, the response in the state $|1_{\mathbf{k}}\rangle$ can be found by an analysis similar to that in Sec. IV B. If $\mathbf{k}$ is along the positive $z$ axis we get

$$P(\omega)=P_1(\omega)+E_{10}N_{\mathbf{k}}^2 2\pi\omega^{-1}g^{-1}\coth(\pi\omega g^{-1}) \tag{69}$$

with $P_1(\omega)$ as in (39). By noting the identity

$$\coth(\pi\omega g^{-1})=\left[\frac{1}{e^{2\pi\omega/g}-1}+\frac{1}{1-e^{-2\pi\omega/g}}\right] \tag{70}$$

we can conclude that the second term in $P(\omega)$ is just a superposition of the detector response to plane-wave modes $u_{\mathbf{k}}^*(x)$ and $u_{\mathbf{k}}(x)$, respectively.

An expression similar to (69) may be found for the state $|1_{\mathbf{k}}\rangle$ with an arbitrary $\mathbf{k}$.

### E. Response in a thermal state

Starting from $G_{\text{th}}(x,x')$ in (30), the response along the inertial trajectory can be calculated to be

$$P(\omega)=E_{10}\int d^3\mathbf{k}\,N_{\mathbf{k}}^2\frac{1}{e^{\beta\omega_{\mathbf{k}}}-1}[\delta(\omega-(|\mathbf{k}|-\mathbf{k}\cdot\mathbf{v}))]^2\ . \tag{71}$$

Once again, only the right frequencies excite the detector, and there is a corresponding thermodynamic weightage $[\exp(\beta\omega_{\mathbf{k}})-1]^{-1}$. Along the accelerated trajectory, the response function is given by

$$\begin{aligned}P(\omega)=P_1(\omega)\\ +4g^{-2}\int d^3\mathbf{k}\,N_{\mathbf{k}}^2\frac{1}{e^{\beta\omega_{\mathbf{k}}}-1}\,|K_{i\omega g^{-1}}(k_T g^{-1})|^2\\ \times(e^{\pi\omega g^{-1}}+e^{-\pi\omega g^{-1}})\ .\end{aligned} \tag{72}$$

Clearly, both in (71) and (72) the response function has a basic difference from that in the coherent state. The amplitude $\alpha_{\mathbf{k}}$ is not for us to choose, it is fixed to be $(e^{\beta\omega_{\mathbf{k}}}-1)^{-1}$.

We have now completed the calculation of response functions, and interpreted the results in the various cases. In the next section, we provide a comparison of these results.

## V. DISCUSSION AND CONCLUSIONS

Let us give a brief summary of the results (at the risk of repetition). In the different states which we considered, the response function is a sum of the response to the Wightman function, and the response to a state-dependent part. The inertial detector does not respond to the vacuum correlation function. So the only contribution is picked up form the state-dependent part $P_2(\omega)$. The excitation is caused by those modes which are appropriately Doppler shifted. All these results for the inertial detector are as expected and fairly trival.

With regards to the accelerated detector, one part of the response is a Planckian, while the additional part $-P_2(\omega)$, in general, has a complicated dependence on the field configuration. However, the physics of $P_2(\omega)$ is more transparent when written as a mode sum. Unlike the inertial detector, the accelerated detector is excited by all modes. This is simple to understand if we note that as the detector accelerates, every mode appears Doppler shifted to the detector frequency $\omega$ at some instant or the other.

It is interesting to note that the response of the accelerated detector to a plane-wave mode

$$P_2(\omega)=\left|\int d\tau\, e^{i\omega\tau}u_{\mathbf{k}}^*(\tau)\right|^2 \tag{73}$$

is a Planckian (for $\mathbf{k}$ along the $z$ axis). (A similar result has been effectively used by Boyer[6] to obtain a classical interpretation of acceleration radiation.) Moreover, the response in the different states considered is typically a superposition of the response $P_2(\omega)$ of (73), over various modes. Equation (73) suggests that the thermal response is not peculiar to the vacuum $|0\rangle_I$—the central role appears to be played by the (accelerated) trajectory rather than the quantum state.

Consider next, the responses in the "single-mode" coherent state, and the Fock state $|n_{\mathbf{k}}\rangle$, and for simplicity restrict $\mathbf{k}$ along the $z$ axis. From Eqs. (48) and (69) we find that the functional form of $P_2(\omega)$ is somewhat different in the two cases. The ratio of $P_2(\omega)$ for the two cases is

$$\frac{P_2^{\text{coherent}}(\omega)}{P_2^{\text{Fock}}(\omega)}=\frac{\alpha_k^2}{n_k}\frac{\coth(\pi\omega/2g)}{\coth(\pi\omega/g)} \ . \tag{74}$$

For $g\rightarrow 0$, it tends to $\alpha_k^2/n_k$, and is independent of the detector frequency $\omega$; the same limiting form is obtained as $\omega\rightarrow\infty$. If the coherent state for mode $\mathbf{k}$ corresponds to a classical state with amplitude $\alpha_{\mathbf{k}}$, and the Fock state $|n_{\mathbf{k}}\rangle$ corresponds to the *same* classical state (for large $n_{\mathbf{k}}$) then one can show that $n_{\mathbf{k}}\sim\alpha_{\mathbf{k}}^2$ (see, e.g., Ref. 4, Chap. 13). [Note from (48) that $\alpha_{\mathbf{k}}N_{\mathbf{k}}$ corresponds to the classical amplitude.] This means the response of inertial detectors matches in the coherent state and large-$n$ Fock state, but the response of accelerated detectors does not (except as $\omega\rightarrow\infty$).

It does not seem possible to simplify the response function in a thermal state [Eq. (72)], essentially because the amplitudes $\alpha_{\mathbf{k}}$ cannot be chosen arbitrarily. This makes it difficult to compare it with response in other states.

Our analysis suggests that for any state $|\Psi\rangle$, the response function splits into two parts—the response to the vacuum and a part exclusively determined by the expectation value of the field in that state. Because of this, it becomes particularly simple to study the classical limit of the response. We conclude that there are interesting similarities between the response of the accelerated detector to (i) the vacuum and (ii) the "classical" field. This may provide a deeper understanding of the "thermal" nature of the excitation in the inertial vacuum.


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