PHYSICAL REVIEW D VOLUME 38, NUMBER 8 15 OCTOBER 1988

Harmonic synchronizations of spacetime

C. Bona and J. Massó

Departament de Física, Universitat Illes Balears, 07071 Palma de Mallorca, Spain

(Received 13 April 1988)

The spacetime slicings associated with a harmonic time coordinate (harmonic synchronizations) are considered and their usefulness in the field of numerical relativity is studied. Harmonic synchronizations are shown to avoid singularities in the same way that the widely used maximal slicings do. Both kinds of slicing are compared in stationary axisymmetric spacetimes, homogeneous cosmological models, and Kerr-Newman black holes.

I. INTRODUCTION

The study of the initial-value problem in general relativity requires slicing spacetime by a family of spacelike hypersurfaces. This can be performed by choosing a time coordinate t such that every slice is a $t=\text{const}$ hypersurface. In a generic local coordinate system, this amounts to choosing a single function ϕ:

$$\phi(x^A)=t \quad (A=0,1,2,3) \tag{1}$$

so that $d\phi$ is temporal. The parametrized slicing defined by (1) will be called synchronization for short.

The choice of a synchronization is crucial when one attempts to construct a spacetime by evolving a given set of initial data in the framework of numerical relativity.[1] In that context, the "maximal slicing" condition has been widely used,[2-6] its success being due in part to its "singularity avoidance" properties.[3,4] Other authors[7,8] have proposed a different condition to ensure the hyperbolicity of the (appropriately written) system of Einstein field equations: we shall call it "harmonic synchronization" (the term will be justified later).

The purpose of this work is to study the singularity avoidance properties of the harmonic synchronization and to compare it with maximal time slicing in many important cases: stationary and axisymmetric spacetimes, homogeneous cosmological models and black holes. This is a very preliminary step in the way of constructing a general-relativistic numerical code with harmonic synchronization.

II. SPACETIME SYNCHRONIZATIONS

Let us consider the field n^A of unit normals to every slice of a synchronization (1). One can interpret n^A as the field of velocities of a system of observers (Eulerian observers) whose local three-spaces are tangent to every slice. One can choose the space coordinates x^I in one slice and propagate this coordinate system to the other slices along the integral curves of n^A. In the resulting Eulerian coordinate system, the spacetime line element can be written as

$$-\alpha^2(t,x)dt^2+\gamma_{IJ}(t,x)dx^I dx^J , \tag{2}$$

where γ_{IJ} is the three-dimensional metric induced on every slice and α is the lapse function.

The lapse function α measures the metric interval between two slices corresponding to two infinitesimally close values of the time parameter t. It can be defined in a covariant way from ϕ: note that, in our coordinate system,

$$\phi=t, \quad d_A\phi=\delta^t{}_A \tag{3}$$

so that

$$\alpha=|g^{AB}d_A\phi\, d_B\phi|^{-1/2} . \tag{4}$$

Let us consider the effect of a reparametrization of the slicing: that is,

$$t=f(t') . \tag{5}$$

The lapse function will now be

$$\alpha'=\alpha\,|\dot{f}\,|$$

and the field of unit normals n^A will remain unchanged. The Eulerian observers by themselves determine the slicing, but a synchronization (a time coordinate) requires the complete specification of the lapse function also.

The maximal time slicing[3] is obtained when one demands the Eulerian observers to be expansion-free: that is,

$$\nabla_A n^A=0 , \tag{6}$$

where ∇ stands for the four-dimensional covariant derivative. In the local coordinate system (2), it can be written

$$\partial_t\gamma=0 , \tag{7}$$

where γ stands for the determinant of the space metric γ_{IJ}.

The compatibility of (6) with the Einstein field equations leads to a linear elliptic equation on the lapse function on every slice:[3]

$$\Delta\alpha=\lambda\alpha , \tag{8}$$

where Δ is the Laplace operator associated with the three-dimensional metric and λ depends both on the geometry of the slice and the energy content of the space-

 ©1988 The American Physical Society

time. When supplemented with suitable boundary conditions, (8) completely determines the lapse function. Let us note that (8) is invariant under (5) so that the boundary condition provides the parametrization of the slicing given by (7) and (8).

Harmonic synchronizations are obtained when one demands the time coordinate be given by a harmonic function,

$$\Box\phi = g^{AB}\nabla_A d_B \phi = 0 \;, \tag{9}$$

which can also be written in a form similar to (6),

$$\nabla_A (n^A/\alpha) = 0 \tag{10}$$

and in the local coordinate system (2),

$$\partial_t(\sqrt{\gamma}/\alpha) = 0 \;. \tag{11}$$

Equation (9) is a strictly kinematical condition on α in the sense that it is always compatible with Einstein's field equations. The form (11) is just the one proposed by Choquet-Bruhat and Ruggeri,[7,8] but the same choice of time coordinate is explicitly contained in well-known harmonic coordinate systems, as it follows from (9). The time lines in harmonic coordinate systems are not chosen to be normal to the slices, but this depends only on the choice of the space coordinates in every slice and does not affect the properties of the time slicing itself.

III. SINGULARITY AVOIDANCE

As we have restricted ourselves to Eulerian coordinate systems, it is evident that the singularities of the slicing must correspond to singularities in the congruence of time lines and vice versa. This is so because the three-dimensional metric γ_{ij} induced on every slice coincides with the space (quotient) metric of the normal observers.

Let us consider for instance the time line with local equation $x^i = x_0^i$ and let us use the proper time τ to label points along this line. Let us suppose now that the three-dimensional volume element $\sqrt{\gamma}$ vanishes at a given value τ_S of τ (Ref. 9),

$$(\sqrt{\gamma})(\tau_S) = 0 \;. \tag{12}$$

This is a singularity of both the time lines[10] and the slicing because the induced metric γ_{ij} on the slice passing through that point stops being invertible there. This is fatal for the numerical construction of the spacetime because the numerical algorithm stops and one cannot proceed to the next slice.

In order to be more precise, we will say that the line $x^i = x_0^i$ has a "focusing singularity"[11] at $\tau = \tau_S$ if Eq. (12) holds and the proper-time derivative of the three-dimensional volume element $\sqrt{\gamma}$ remains bounded at $\tau = \tau_S$ so that there exists a constant B such that

$$|(\partial_T \sqrt{\gamma})(\tau_S)| < B \;. \tag{13}$$

Condition (13) is adopted in order to exclude from our consideration singularities accompanied with sudden variations of the spatial volume element.

Let us now consider a harmonic synchronization starting from a regular set of initial data on the $t = t_0$ slice. Equation (11) then gives

$$\alpha = C(x^i)\sqrt{\gamma} \;, \tag{14}$$

where we have noted $C(x^i) = \alpha_0/\sqrt{\gamma_0}$. If we choose any normal line $x^i = x_0^i$ parametrized by its proper time τ (by choosing the $t = t_0$ slice as the origin), we will find along such a line

$$\alpha(\tau, x_0^i) = C(x_0^i)\sqrt{\gamma} \tag{15}$$

and then

$$\partial_\tau \alpha = C(x_0^i)\partial_\tau \sqrt{\gamma} \;. \tag{16}$$

Allowing for (15) and (16) we see that one can replace $\sqrt{\gamma}$ by α in Eqs. (12) and (13) defining a focusing singularity. It can be reformulated as

$$\lim_{\tau \to \tau_S} (\tau - \tau_S)/\alpha \neq 0 \;, \tag{17}$$

where the limit is taken along the line $x^i = x_0^i$.

The coordinate time interval Δt elapsed along that line between the points where $\tau = 0$ ($t = t_0$) and $\tau = \tau_S$ is given by the improper integral

$$\Delta t = \int_0^{\tau_S} \alpha^{-1}(\tau, x_0^i)\, d\tau \tag{18}$$

and it follows from (17) that this integral cannot converge. This means that focusing singularities, as defined by (12) and (13), cannot be reached by harmonic synchronizations in a finite number of timesteps.

The same thing is known to be true for maximal slicing: if one compares Eqs. (7) and (12), it is clear that the Eulerian observers in a maximal time slicing never focus.[3] In fact, condition (13) is not needed in this case: maximal slicing can avoid stronger singularities than harmonic synchronizations. In what follows, we shall give an explicit comparison between both synchronization conditions in many important cases.

IV. STATIONARY AXISYMMETRIC SPACETIMES

Let us consider the usual stationary axisymmetric line element, written in the Lewis-Papapetrou form:

$$g_{tt}dt^2 + 2g_{t\varphi}dt\, d\varphi + g_{\varphi\varphi}d\varphi^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 \;, \tag{19}$$

where the metric coefficients are independent of the t and φ variables. Let $\xi^A = \delta_t^A$ and $\eta^A = \delta_\varphi^A$ be the two commuting Killing vectors of (19). It can be easily verified[12] that, in this case,

$$\partial_A \phi = \delta^t{}_A = (-g_{\varphi\varphi}\xi_A + g_{\varphi t}\eta_A)[(g_{t\varphi})^2 - g_{\varphi\varphi}g_{tt}]^{-1} \tag{20}$$

and the lapse function can be computed from (19):

$$\alpha = [(g_{t\varphi})^2/g_{\varphi\varphi} - g_{tt}]^{1/2} \;. \tag{21}$$

It is well known[3] that the Eulerian observers associated with (20) (Bardeen observers) verify the maximal condition (6). It follows from (20) and (21) that the harmonic condition (10) is also verified. The synchronization associated with the Lewis-Papapetrou form (19) of the line element in a stationary and axisymmetric spacetime is both maximal and harmonic. The only trouble is the appearance of Cauchy horizons[4] at the points where the lapse function (21) vanishes: we will discuss this point in Sec. VI.

V. HOMOGENEOUS COSMOLOGIES

In the case of homogeneous cosmological models, one has a symmetry group acting on three-dimensional spacelike orbits. These orbits provide a preferred time slicing of the spacetime so that the line element may be written

$$-d\tau^2+\gamma_{IJ}(\tau,x^K)dx^I dx^J \ , \tag{22}$$

where γ_{IJ} is a homogeneous three-dimensional metric.

It is well known[3] that the three-dimensional volume element is then of the form

$$\sqrt{\gamma}=g(\tau)\sqrt{\gamma_0} \ , \tag{23}$$

where γ_0 stands for $\gamma(\tau_0,x^K)$. This means that the slicing (22) is not maximal in the generic case. In fact, Eq. (23) corresponds to a generalization of the maximal condition: the constant mean curvature slicing.[3]

The time slicing (22) can be reparametrized however in order to obtain a harmonic synchronization: let us choose a time coordinate t such that

$$dt=d\tau/|g(\tau)| \tag{24}$$

and the harmonic condition (11) is satisfied. The preferred slicing (22) admits a harmonic parametrization but it is not maximal in the generic case.

To be more specific, let us consider the Einstein–de Sitter particular case of (22),

$$\gamma_{IJ}=\tau^{4/3}\delta_{IJ} \ , \tag{25}$$

which can be thought of as describing the collapse of a homogeneous distribution of dust, starting at $\tau=\tau_0$ (<0), up to a singularity at $\tau=0$. In this case,

$$g(\tau)=\tau^2 \tag{26}$$

so that (24) gives

$$t=-1/\tau \quad (>0) \tag{27}$$

and the $\tau=0$ singularity corresponds to $t\rightarrow\infty$.

Note that the asymptotic past singularity ($\tau\rightarrow-\infty$) in (25) appears at a finite value of the time coordinate ($t=0$). This effect is a predictable consequence[1] of the harmonic condition (11). In fact $t=0$ is a spacetime boundary and the stopping of the (backwards) numerical algorithm there would mean that the whole past of the initial slice has been constructed. One does not expect this kind of singularity to occur in the future time development of regular initial data for collapsing systems.

VI. BLACK HOLES

Let us consider the Kerr-Newman line element.[13] It can be written in the form (19) with the metric coefficients given by

$$\begin{aligned} &g_{tt}=-[1+(e^2-2mr)/\Sigma] \ , \\ &g_{t\varphi}=a\sin^2\theta(e^2-2mr)/\Sigma \ , \\ &g_{\varphi\varphi}=\sin^2\theta[r^2+a^2-a^2\sin^2\theta(e^2-2mr)/\Sigma] \ , \\ &g_{rr}=\Sigma/(r^2+a^2+e^2-2mr), \quad g_{\theta\theta}=\Sigma \ , \end{aligned} \tag{28}$$

where m, a, and e are constants and we have taken for short $\Sigma=r^2+a^2\cos^2\theta$. The particular case when $e=0$ (vacuum spacetime) corresponds to the Kerr metric. The spherically symmetric cases ($a=0$) are the Reissner-Nordström metrics, the Schwarzschild case being recovered when both a and e vanish.

Let us compute the lapse function from Eq. (21):

$$\alpha=\left|\frac{r^2+a^2+e^2-2mr}{r^2+a^2-a^2\sin^2\theta(e^2-2mr)/\Sigma}\right|^{1/2} \tag{29}$$

so that, in the case of $m^2>a^2+e^2$, Cauchy horizons[4] appear at the surfaces given by

$$r=r_\pm=m\pm[m^2-(a^2+e^2)]^{1/2} \ . \tag{30}$$

The slices are spacelike where $r>r_+$ or $r<r_-$, but become timelike in the region where $r_-<r<r_+$. Note that no physical singularity appears there: it is only the coordinate system (19) which becomes ill defined in that region.

To study the zone between r_- and r_+, alternative maximal slicings have been proposed. All of them present a boundary at $r=r_L$ ($r_-<r_L<r_+$). In the Schwarzschild case[2,14] one has

$$r_L=3m/2 \ . \tag{31}$$

The same thing occurs with the Reissner-Nordström metrics at a value[4]

$$r_L=3m/4\pm(9m^2/16-e^2/2)^{1/2} \ . \tag{32}$$

In the Kerr case, a lower bound for the value of r_L is given in Ref. 4.

Let us look for alternative harmonic synchronizations by solving the wave equation (9), that is

$$\partial_A(\sqrt{-g}\,g^{AB}d_B\phi)=0 \ , \tag{33}$$

where g stands for the determinant of the metric g_{AB}. Note that the Cauchy horizons (30) are independent of the angular variables θ, φ. We shall then limit ourselves to spherically symmetric solutions $\phi(t,r)$ of (33) satisfying the following asymptotic conditions:

$$\lim_{r\rightarrow\infty}\phi=t \ , \tag{34}$$

$$\lim_{r\rightarrow\infty} g^{AB}d_A\phi d_B\phi<0 \ , \tag{35}$$

where uniform convergence is demanded in (35) to ensure that it holds even in the limit when $t\rightarrow\infty$, independently of the order in which the two limiting processes are performed.

These conditions lead to the following form of the harmonic function ϕ:

$$\phi=t+f(r) \tag{36}$$

with $f(r)$ being a solution of the wave equation (33), so that

$$\sqrt{-g}\, g^{rr} f'(r) = C \ , \tag{37}$$

where C is a constant. This can be easily integrated to give

$$\phi = t + \frac{C}{r_+ - r_-} \ln \left| \frac{r - r_+}{r - r_-} \right| \ , \tag{38}$$

where (34) has been taken into account. Note that the original synchronization (20) corresponds to $C=0$.

The lapse function corresponding to the new time coordinate ϕ can be computed from its invariant definition (4). Allowing for (38), one gets

$$\alpha = \left| \frac{(r-r_+)(r-r_-)\Sigma}{(r^2+a^2)^2 - C^2 - a^2(r-r_+)(r-r_-)\sin^2\theta} \right|^{1/2} \tag{39}$$

and any of the two choices

$$C = \pm(r_+^2 + a^2) \tag{40}$$

ensures that no Cauchy horizon appears for $r > r_-$.

The harmonic synchronizations of a generic Kerr-Newman metric (28) given by

$$\phi = t \pm \frac{r_+^2 + a^2}{r_+ - r_-} \ln \left| \frac{r - r_+}{r - r_-} \right| \tag{41}$$

are then well defined in the whole region $r > r_-$. The surface $r = r_-$ is a Cauchy horizon. Note that the expression (41) is similar to the one appearing in the Eddington-Finkelstein transformation[15] for the Schwarzschild metric; the double sign in (41) can then be interpreted as in that case.

VII. CONCLUSIONS

The singularity avoidance behavior of harmonic synchronizations has been studied in Sec. III, where focusing singularities have been precisely defined. The numerical development of initial data will avoid focusing singularities in the sense that they are not reached in a finite number of time steps. It is also clear that maximal slicing can avoid stronger singularities than harmonic synchronizations. This may be connected with the fact that maximal slicing usually stops by the vanishing of α (collapse of the lapse) before getting very close to the singularities,[4] whereas it follows from (14) that harmonic synchronizations can get arbitrarily close to focusing singularities without actually reaching them.

In Secs. IV–VI we give, for many important cases, explicit choices of harmonic synchronizations providing a more complete (or the same) time development than the one arising from the optimal choice of maximal slicing in every case.[4]

ACKNOWLEDGMENTS

We are indebted to Dr. B. Coll for fruitful suggestions and discussions. This work was supported by the CAICYT of Spain under Project No. 1005/84.

[1] T. Piran, in *Gravitational Radiation,* 1982 Les Houches Lectures, edited by N. Derruelle and T. Piran (North-Holland, Amsterdam, 1983).
[2] F. Estabrook *et al.*, Phys. Rev. D **7**, 2814 (1973).
[3] L. Smarr and J. W. York, Phys. Rev. D **17**, 1945 (1978); **17**, 2529 (1978).
[4] D. M. Eardley and L. Smarr, Phys. Rev. D **19**, 2239 (1979).
[5] L. I. Petrich, S. L. Shapiro, and S. A. Teukolsky, Phys. Rev. D **31**, 2459 (1985).
[6] J. Thornburg, Class. Quantum Gravit. **4**, 1119 (1987).
[7] Y. Choquet-Bruhat, in *Gravitational Radiation* (Ref. 1).
[8] Y. Choquet-Bruhat and T. Ruggeri, Commun. Math. Phys. **89**, 269 (1983).
[9] We are supposing that $\sqrt{\gamma}$ does not vanish identically along the line $x^i = x_0^i$, as it would happen if the value x_0^i corresponds to a singularity (center, polar axis) of the spatial coordinate system on every slice.
[10] The normal lines are focusing to a caustic. See, for instance, Ref. 3.
[11] Note that our definition does not coincide with that of "crushing singularities" as given in Ref. 4.
[12] J. M. Bardeen, Astrophys. J. **162**, 71 (1970).
[13] D. Kramer *et al.*, *Exact Solutions of Einstein's Field Equations* (VEB, Berlin, 1980).
[14] B. L. Reinhart, J. Math. Phys. **14**, 719 (1973).
[15] We refer here to the version presented in the book by M. Carmeli, *Classical Fields* (Wiley, New York, 1982).