Solution of the Einstein-Strauss problem with a Λ term

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The conditions for a continuous matching between the Schwarzschild–de Sitter line element and the one corresponding to a cosmological solution with a Λ term are given under the hypothesis that the cosmological fluid has zero pressure. In the limiting case of $\Lambda=0$ Schücking's results are recovered.

I. INTRODUCTION

The problem of embedding a Schwarzschild mass into cosmology has been extensively studied. Classical references are the work of Einstein and Strauss,[1] Mc Vittie,[2] and Dirac.[3]

The physical model proposed by Einstein and Strauss, which we shall adopt, assumes that inside a cosmological fluid with zero pressure a spherical vacuum region is cut out and a Schwarzschild mass is placed in it. Then one works out the relationships for the vacuum Schwarzschild metric to join smoothly to the cosmological metric.

When the cosmological constant Λ is zero, the explicit solution of this problem was obtained by Schücking[4] working in curvature coordinates. In this paper the matching condition will be given for a case in which the cosmological constant is not vanishing.

It is worthwhile to mention that recently Gautreau[5] has developed a quite different approach to the problem of embedding a Schwarzschild mass into a given cosmology. In his physical model no vacuum region of space-time exists: the cosmological fluid is now in contact with the central mass.

II. JUNCTION CONDITION

To properly match two different metrics along a boundary surface characterized by jumps in the energy-momentum tensor various equivalent approaches can be used.

Following Israel[6] one can consider a timelike three-space Σ which describes the motion of the discontinuity surface, which separates the Riemannian space-time into two distinct four-dimensional manifolds V^- and V^+ of class C^4, each containing Σ as its boundary. Let

$$ds^2=g_{ij}d\xi_i d\xi_j \quad (i=1,2,3) \tag{2.1}$$

be the intrinsic metric on Σ and

$$ds^2=g_{\alpha\beta}dx^\alpha dx^\beta \quad (\alpha=0,1,2,3) \tag{2.2}$$

be the relevant metrics in $V^\pm$. Now approaching Σ in V^- or V^+ we will have to demand

$$(ds_-^2)_\Sigma=(ds_+^2)_\Sigma=ds_\Sigma^2 \tag{2.3}$$

where $(\ )_\Sigma$ signifies the limits of the relevant functions as Σ is approached. If one introduces $K_{ij}^\pm$, the extrinsic curvature of Σ in terms of the unit spacelike normal vector to Σ, $\eta_\alpha^\pm$, as

$$K_{ij}^\pm=-\eta_\alpha^\pm\frac{\partial^2 x_\pm^\alpha}{\partial\xi^i\partial\xi^j}-\eta_\alpha^\pm\Gamma_{\mu\nu}^\alpha\frac{\partial x_\pm^\mu}{\partial\xi^i}\frac{\partial x_\pm^\nu}{\partial\xi^j}\ , \tag{2.4}$$

one can obtain from the second continuity condition imposed on Σ the relation

$$[K_{ij}]=K_{ij}^+-K_{ij}^-=0\ . \tag{2.5}$$

One can further show that the continuity conditions (2.3) and (2.5) are equivalent to the Lichnerowicz and O'Brien-Synge function conditions. These latter require the discontinuity in the Einstein tensor of $V^\pm$ to satisfy[7]

$$(C_\beta^\alpha\eta_\alpha)=0\ . \tag{2.6}$$

In the case of spherical symmetry, that will concern us, the relevant metrics in $V^\pm$ will be of the form

$$ds^2=D\,d\tau^2-A\,dr^2-r^2(d\theta^2+\sin^2\theta\,d\varphi^2) \tag{2.7}$$

with obvious significance of the coordinates used. $A(\tau,r)$ and $D(\tau,r)$ are C^0 functions on Σ. Let

$$r=r_0(\tau) \tag{2.8}$$

be the equation of Σ. With

$$\eta_\beta=\left[\frac{dr_0}{d\tau},-1,0,0\right] \tag{2.9}$$

we obtain from Eqs. (2.6) the two equations

$$\frac{A}{D}\left[\frac{dr_0}{d\tau}\right]^2=-\frac{[T_1^1]}{[T_0^0]}\ , \tag{2.10a}$$

$$\frac{D}{A}[T_1^0]^2=-[T_1^1][T_0^0]\ , \tag{2.10b}$$

where T^{α}_{β} is the stress-energy tensor and use of the Einstein field equations has been made. In this paper we shall extend the matching proposed by Schücking to the case when a nonvanishing cosmological constant is allowed in the field equations and show that the O'Brien-Synge conditions, Eqs. (2.10), are automatically satisfied.

III. THE COSMOLOGICAL LINE ELEMENT

Our first task, in order to apply the O'Brien-Synge conditions method, is to write the cosmological metric in a form in which the rotational symmetry is more evident. Indeed the cosmological line element for a homogeneous and isotropic model of the Universe can be written as ($G=c=1$ units are hereafter used)

$$ds^2=dt^2-R^2(t)\left[\frac{dx^2+dy^2+dz^2}{(1+\epsilon l^2/4)^2}\right], \qquad (3.1)$$

where ϵ is the curvature constant $(0,\pm 1)$, x, y, and z are Cartesian coordinates, $l^2=x^2+y^2+z^2$, $R(t)$ is the scale factor, and the velocity of light c has been set equal to one. It is easy, introducing spherical polar coordinates, to get from Eq. (2.1) the line element in the form

$$ds^2=dt^2-R^2(t)\left[\frac{d\rho^2}{1-\epsilon\rho^2}+\rho^2 d\omega^2\right], \qquad (3.2)$$

where the radial distance ρ is given in function of l by

$$\rho=\frac{l}{1+\epsilon l^2/4} \qquad (3.3)$$

and

$$d\omega^2=d\theta^2+\sin^2\theta\, d\varphi^2 , \qquad (3.4)$$

θ and φ are the usual angular coordinates of a spherically symmetric polar coordinate system. As is known the cosmological metric Eq. (3.1) is hypersurface-homogeneous and admits an isometry group transitive on spacelike orbit. Following the classification of Kramer, Stephani, Herlt, and MacCallum[8] the group of motion allowed by this metric is a G_6 on a $S_{(3)}$. Now a subgroup of G_6 is the group of space rotation H_3. The rotational invariance of the cosmological line element is evident in Eq. (3.2). The spherically symmetric line element corresponding to the empty field solution of Einstein's equations with a cosmological constant Λ describing the space-time outside a central body of mass m is given by the Schwarzschild–de Sitter metric

$$ds^2=\psi^2(\tau)\left[1-\frac{2m}{r}-\lambda r^2\right]d\tau^2 - \left[1-\frac{2m}{r}-\lambda r^2\right]^{-1} dr^2-r^2 d\omega^2 , \qquad (3.5)$$

where $\psi(\tau)$ is an arbitrary function of the "Schwarzschild time" τ and $\lambda=\Lambda/3$.

This line element allows a G_3 group of motion on a $S_{(2)}$ which again admits as a subgroup the group of space rotation H_3. It is evident that the radial coordinate r of the Schwarzschild line element is connected to the scale factor $R(t)$ and the radial coordinate ρ entering the cosmological line element by

$$r=R(\tau)\rho . \qquad (3.6)$$

One can therefore, in virtue in Eq. (3.6), write the cosmological line element in terms of the set (t,r,θ,φ) of coordinates as

$$ds^2=dt^2-\frac{(R\,dr-rh\,dt)^2}{R^2-\epsilon r^2}-r^2 d\omega^2 . \qquad (3.7)$$

Here $h(t)$ is defined by $h(t)\equiv dR(t)/dt$ and is given by Einstein field equations. In the case that the stress tensor for the cosmological fluid corresponds to a "pure dust" we have

$$\frac{dR(t)}{dt}\equiv h(t)=\left[\frac{\chi A-\epsilon R+\lambda R^3}{R}\right]^{1/2} , \qquad (3.8)$$

$\chi=8\pi$ and $A\equiv\frac{1}{3}\bar{\mu}R^3$ are two constants and $\bar{\mu}$ is the mean mass density of the Universe.

Now in Eq. (3.7) the radical coordinate is the same radial curvature coordinate as defined in the Schwarzschild line element. However the cosmological metric in this form is no longer diagonal. To get rid of the nondiagonal term in Eq. (3.7) we introduce a "time curvature coordinate" τ as follows:[4] first we invert the dependence of R from t, i.e., $t=t(R)$, then we transform R according to the equation

$$R=\phi(r,\tau) . \qquad (3.9)$$

Now the scale factor is a function of both curvature coordinates $\{\tau,r\}$; the same will hold for the cosmological time t:

$$t=T(r,\tau) . \qquad (3.10)$$

Of course now the curvature time coordinate τ is not more simply defined in terms of comoving observers as it was for t.

One can easily check that if the transformation (3.10) satisfies

$$\frac{\partial T}{\partial r}\equiv T_{,r}=-\frac{\phi H r}{\phi^2-\epsilon r^2-r^2H^2} , \qquad (3.11)$$

where $H(r,\tau)$ is obtained by $h(t)=h(T(r,\tau))=H(r,\tau)$, then the cosmological line element in curvature coordinates (τ,r,θ,φ) becomes diagonal:

$$ds^2=\left[1-\frac{r^2H^2}{\phi^2-\epsilon r^2}\right](T_{,\tau})^2 d\tau^2 - \frac{\phi^2}{\phi^2-r^2(\epsilon+H^2)}dr^2-r^2 d\omega^2 . \qquad (3.12)$$

One can make use of the expression of H given by the field equation and rewrite Eq. (3.12) as

$$ds^2=\left[1-\frac{\chi A r^2}{\phi^3}-\lambda r^2\right]\frac{\phi^2}{\phi^2-\epsilon r^2}(T_{,\tau})^2 d\tau^2 - \left[1-\frac{\chi A r^2}{\phi^3}-\lambda r^2\right]^{-1} dr^2-r^2 d\omega^2 , \qquad (3.13)$$

where

$$T^2_{,\tau}=\frac{1}{H^2}(\phi_{,\tau})^2=\frac{\phi}{\kappa A-\epsilon\phi+\lambda\phi^3}(\phi_{,\tau})^2 . \tag{3.14}$$

The diagonalization condition Eq. (3.11), or, after expressing H as a function of ϕ,

$$\frac{\phi\phi_{,r}}{\chi A-\epsilon\phi-\lambda\phi^3}-\frac{r\phi-r^2\phi_{,r}}{\phi^2-\epsilon r^2}=0 \tag{3.15}$$

can be integrated and the general solution has the form

$$F(\phi)+f(\phi,r^2)=G(\tau) , \tag{3.16}$$

where $F(\phi)$ is the function tabulated in the Appendix and

$$f(\phi,r^2)=\ln\left[1-\frac{\epsilon r^2}{\phi^2}\right]^{1/\epsilon} , \tag{3.17}$$

G is an arbitrary function of τ.

Partial differentiation of Eq. (3.15) gives directly $\phi_{,\tau}$ which enters the cosmological line element

$$\phi_{,\tau}=\frac{1}{2}g_{,\tau}\frac{(\chi A-\epsilon\phi+\lambda\phi^3)(\phi^2-\epsilon r^2)\phi}{\phi^3-\chi Ar^2-\lambda r^2\phi^3} , \tag{3.18}$$

where $g(\tau)\equiv dG/d\tau$.

Thus the cosmological line element can be expressed in its final form in terms of curvature coordinates as

$$ds^2=\frac{(\phi^2-\epsilon r^2)(\chi A-\epsilon\phi+\lambda\phi^3)}{4\phi^2\left[1-\frac{\chi Ar^2}{\phi^3}-\lambda r^2\right]}(g_{,\tau})^2d\tau^2 - \left[1-\kappa\frac{Ar^2}{\phi^3}-\lambda r^2\right]^{-1}dr^2-r^2d\omega^2 . \tag{3.19}$$

IV. THE MATCHING

To match the line elements Eqs. (3.19) and (3.5) we first require the continuity of the metric coefficient across the spherical surface which separate the empty space-time spherical region around the Schwarzschild mass from the cosmological fluid which pervade the remaining space-time.

Let us call $r_0(\tau)$ the radius of this discontinuity surface and ϕ_{r_0} the value of the function $\phi(r,\tau)$ evaluated along this surface. Both $r_0(\tau)$ and $\phi_{r_0}(\tau)$ are functions only of the time parameter τ.

We shall first examine the continuity of the g_{rr} coefficient. Looking at Eqs. (3.19) and (3.5) one immediately derives

$$\chi\frac{Ar_0^2}{\phi_{r_0}^3}=\frac{2m}{r_0} \tag{4.1}$$

from which it follows

$$\phi_{r_0}=\left[\frac{\chi A}{2m}\right]^{1/3}r_0 \tag{4.2}$$

or, in our original coordinate system (ρ,t),

$$\rho=\text{const}=\left[\frac{\chi A}{2m}\right]^{-1/3} \tag{4.3}$$

which follows from Eqs. (3.6) and (3.8).

Being the angular part of metrics already identical by construction, we need only, in order to complete the matching, to impose the continuity of the $g_{\tau\tau}$ coefficient.

We have at the boundary $r=r_0(\tau)$ that the continuity of $g_{\tau\tau}$ requires

$$\psi^2(\tau)=\left.\frac{(\phi^2-\epsilon r^2)(\chi A-\epsilon\phi+\lambda\phi^3)(g_{,\tau})^2}{4\phi^2\left[1-\frac{\chi Ar^2}{\phi^3}-\lambda r^2\right]^2}\right|_{r=r_0(\tau)} . \tag{4.4}$$

The only term so far undetermined is $g_{,\tau}$, but using Eq. (3.15) and the matching condition Eq. (4.2), one has that $f(r_0(\tau),\tau)$ is a constant and that

$$g_{,\tau}=\frac{dF}{d\phi_{r_0}}\frac{d\phi_{r_0}}{dr_0}\frac{dr_0}{d\tau}=\frac{2r_0'K}{\chi A-\epsilon Kr_0-\lambda K^3r_0^3} , \tag{4.5}$$

where

$$K=\left[\frac{\chi A}{2m}\right]^{1/3} \tag{4.6}$$

and $r_0'\equiv dr_0/d\tau$.

Now substitution of Eq. (4.5) in Eq. (4.4) gives the explicit expression of the arbitrary function $\psi(\tau)$ entering the Schwarzschild–de Sitter metric. In order that the matching be continuous across the surface $r=r_0(\tau)$ one must have

$$\psi(\tau)=\frac{K^3r_0(r_0')^2(1-\epsilon K^2)}{(\chi A-\epsilon Kr_0-\lambda K^3r_0^3)\left[1-\frac{2m}{r_0}-\lambda r_0^2\right]} . \tag{4.7}$$

In the limiting case $\lambda=0$ one obviously recovers Schücking's results.

It is now straightforward to verify that with the above conditions on the metric coefficients the O'Brien-Synge conditions are automatically satisfied. In fact, stress tensor being identically null in the interior region, Eqs. (2.10) become simple identities once the cosmological stress tensor for the pure dust is expressed in curvature coordinates $\{r,\tau\}$.

As a last comment one should note that the matching takes place along the world line of a typical comoving observer, Eq. (4.3). Such a trajectory is free falling in both cosmological and Schwarzschild coordinate systems as can be seen from Eq. (3.8) which, taking into account the condition (4.3), becomes the equation for radial geodesics in the Schwarzschild–de Sitter metric, since t coincides with the proper time along the boundary.

APPENDIX

The explicit form for $F(\phi)$ in Eq. (3.16) is

$$F(\phi)=\frac{1}{2(N+M^2)}\ln\frac{(\phi-M)^2}{\phi^2+M\phi+N} - \frac{3M}{2(N+2M)}-\frac{3M\mathfrak{g}(\phi)}{2(N+2M)} , \tag{A1}$$

where

$$M=S+T,\quad N=S^2+T^2-ST\ , \tag{A2}$$

and

$$\tilde{g}(\phi)=\begin{cases}\dfrac{-2}{\sqrt{-\Delta}}\operatorname{arctanh}\dfrac{M+2\phi}{-\sqrt{\Delta}}, & \Delta<0\ ,\\ -\dfrac{2}{M+2\phi}, & \Delta=0\ ,\\ \dfrac{2}{\sqrt{\Delta}}\arctan\dfrac{M+2\phi}{\sqrt{\Delta}}, & \Delta>0\end{cases} \tag{A3}$$

and

$$\Delta=4N-M^2=3(S-T)^2\ .$$

From the definition of S and T it follows that for $\epsilon=0$, $\Delta>0$. When $\epsilon=-1$ we have

$$\Delta<0 \text{ if } -\frac{4}{9\chi^2A^2}<\Lambda<0\ ,$$

$$\Delta=0 \text{ if } \Lambda=-\frac{4}{9\chi^2A^2}\ ,$$

$$\Delta>0 \text{ if } \Lambda>0 \text{ or } \Lambda<-\frac{4}{9\chi^2A^2}\ .$$

Finally when $\epsilon=1$ we have

$$\Delta<0 \text{ if } 0<\Lambda<\frac{4}{9\chi^2A^2}\ ,$$

$$\Delta=0 \text{ if } \Lambda=\frac{4}{9\chi^2A^2}\ ,$$

$$\Delta>0 \text{ if } \Lambda<0 \text{ or } \Lambda>\frac{4}{9\chi^2A^2}\ .$$

[1]A. Einstein and E. G. Strauss, Rev. Mod. Phys. **17**, 120 (1945); **18**, 44 (1946).
[2]C. C. Mc Vittie, Mon. Not. R. Astron. Soc. **93**, 325 (1937).
[3]P. A. M. Dirac, Proc. R. Soc. London **A345**, 19 (1978).
[4]E. Schücking, Z. Phys. **137**, 595 (1954).
[5]R. Gautreau, Phys. Rev. D **27**, 764 (1983); **29**, 186 (1984); **29**, 198 (1984).
[6]W. Israel, Nuovo Cimento **44B**, 1 (1966); **48B**, 463E (1967).
[7]W. Israel, Proc. R. Soc. London **A248**, 404 (1958).
[8]D. Kramer, H. Stephani, E. Herlt, and M. Mac Callum, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 1980).