Probability of Bianchi type-I inflation

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The Henneaux-Gibbons-Hawking-Stewart canonical measure is calculated for spatially flat but anisotropically expanding universes (diagonal Bianchi type I) minimally coupled to a homogenous massive scalar field. Both inflationary and noninflationary solutions have infinite measure, giving an ambiguous classical probability for inflation in these models. A one-parameter family of methods for taking the ratio is described, and it is shown how it can give a probability ranging from zero to unity.

I. INTRODUCTION

Our present Universe is highly isotropic on the largest observed distance scales. One goal of quantum cosmology is to explain this fact,[1–3] but it is often hoped that a purely classical explanation can be found, thus avoiding our present uncertainties concerning the correct theory of quantum gravity, needed for the very early Universe.

Probably the most popular explanation today for the isotropy of the Universe is the inflationary cosmological scenario. Since inflation is generally assumed to occur when the curvature is at least a few orders of magnitude less extreme than the Planck values, one might ask whether inflation can provide a purely classical explanation of isotropy. This would be the case if almost all of the solutions of the classical dynamical equations of cosmology had enough inflation to make the Universe highly isotropic today.

In this paper we consider a simple cosmological model allowing both anisotropy and inflation, namely, a diagonal Bianchi type-I homogenous spacetime geometry minimally coupled to a homogenous massive scalar field, which can drive Linde's so-called "chaotic" inflation.[4,5] We find approximate solutions to the resulting dynamical equations, some of which have a period of inflation and others of which do not. In order to determine whether or not almost all solutions have sufficient inflation to lead isotropy we calculate the Henneaux-Gibbons-Hawking-Stewart measure[6,7] for the solutions.

Just as in the Friedmann-Robertson-Walker (FRW) models with a massive scalar field,[8] or with an $R+\epsilon R^2$ Lagrangian[9] we find that both the inflationary and the noninflationary solutions have infinite measure, so normalizing the total measure to unity gives an undetermined probability of inflation. If one restricts considerations to a finite range of the parameters governing the size and anisotropy of the Universe at an initial data surface of fixed logarithmic expansion rate for the volume, then both kinds of solutions have finite measure, so one can try to define the probability of inflation for this restricted set of solutions. However, the result depends upon the initial data surface at which the restriction is applied. Applying it at very early times, when the expansion rate is much greater than the natural frequency of the scalar field (i.e., the mass), gives the probability of inflation near unity. The same restriction, applied at late times, when the expansion rate is much less than the scalar field frequency, gives a very small probability for the occurrence of inflation at any time along a randomly chosen solution from the restricted set. Thus one can explicitly calculate the probability of inflation to be either large or small, depending upon the procedure used to define an inherently ambiguous ratio of infinite quantities.

II. FIELD EQUATIONS AND THEIR APPROXIMATE SOLUTIONS

In this paper we consider the diagonal Bianchi type-I metric

$$ds^2=-N^2dt^2+(e^{\beta}dx)^2+(e^{\gamma}dy)^2+(e^{\delta}dz)^2 , \tag{2.1}$$

where N, β, γ, and δ are functions of time t only. We assume that this metric is minimally coupled to a real scalar field ϕ of mass m.

We use units such that $4\pi G/3=1$. One can rescale the classical action for that system, $S=\int L\,dt$, by the comoving volume of the three-dimensional spacelike sections. Then the Lagrangian is

$$L=\frac{1}{2N}e^{\beta+\gamma+\delta}\left[\frac{\dot\beta\dot\gamma+\dot\gamma\dot\delta+\dot\beta\dot\delta}{3}+\dot\phi^2-N^2m^2\phi^2\right] . \tag{2.2}$$

Following Misner,[10] instead of using variables β, γ, and δ we use auxiliary functions α, β_+, and β_- such that

$$\beta=\alpha+\beta_++\sqrt{3}\beta_- , \tag{2.3}$$

$$\gamma=\alpha+\beta_+-\sqrt{3}\beta_- , \tag{2.4}$$

$$\delta=\alpha-2\beta_+ . \tag{2.5}$$

Now the Lagrangian becomes

$$L=\tfrac{1}{2}N^{-1}e^{3\alpha}(-\dot\alpha^2+\dot\beta_+^2+\dot\beta_-^2+\dot\phi^2-N^2m^2\phi^2) . \tag{2.6}$$

Varying the action with respect to the lapse function N gives the constraint

$$\dot\alpha^2=\dot\beta_+^2+\dot\beta_-^2+\dot\phi^2+N^2m^2\phi^2 . \tag{2.7}$$

Once this constraint is imposed we may choose the lapse function arbitrarily before solving the other equations of motion. If one sets

$$Nm^2\phi^2e^{3\alpha}=1\ , \tag{2.8}$$

one gets a timelike geodesic, with proper time t, in the following conformally flat pseudo-Euclidean metric:

$$ds^2=m^2\phi^2e^{-6\alpha}(-d\alpha^2+d\beta_+^2+d\beta_-^2+d\phi^2)\ . \tag{2.9}$$

However, for simplicity we shall instead set $N=m^{-1}$. Varying the action with respect to α, β_+, β_-, and ϕ and using the constraint equation (2.7), one gets the following equations of motion:

$$\ddot{\alpha}+3\dot{\alpha}^2-3\phi^2=0\ , \tag{2.10}$$

$$\ddot{\beta}_++3\dot{\alpha}\dot{\beta}_+=0\ , \tag{2.11}$$

$$\ddot{\beta}_-+3\dot{\alpha}\dot{\beta}_-=0\ , \tag{2.12}$$

$$\ddot{\phi}+3\dot{\alpha}\dot{\phi}+\phi=0\ . \tag{2.13}$$

Only three of these are independent of each other and of the constraint (2.7). Equations (2.11) and (2.12) give constants of motion which may be combined into one constant:

$$\Sigma^2=e^{6\alpha}(\dot{\beta}_+^2+\dot{\beta}_-^2)=\tfrac{1}{6}e^{6\alpha}\sigma_{\mu\nu}\sigma^{\mu\nu}\ , \tag{2.14}$$

where $\sigma_{\mu\nu}$ is the shear of the anisotropic metric (2.1).

Let us define the potential energy density ρ_ϕ of the scalar field ϕ as

$$\rho_\phi=\tfrac{1}{2}\phi^2\ , \tag{2.15}$$

the kinetic energy density $\rho_{\dot{\phi}}$ as

$$\rho_{\dot{\phi}}=\tfrac{1}{2}\dot{\phi}^2\ , \tag{2.16}$$

and the shear or anisotropy energy density as

$$\rho_\Sigma=\tfrac{1}{2}\Sigma^2e^{-6\alpha}\ . \tag{2.17}$$

Then the constraint, Eq. (2.7), can be rewritten in the form

$$\dot{\alpha}^2=\Sigma^2e^{-6\alpha}+\dot{\phi}^2+\phi^2\ , \tag{2.18}$$

or, equivalently,

$$\rho=\rho_\Sigma+\rho_\phi+\rho_{\dot{\phi}}\ , \tag{2.19}$$

where ρ, referred to as the total density, is defined to be

$$\rho=\tfrac{1}{2}\dot{\alpha}^2\ . \tag{2.20}$$

We will also use the Hubble parameter $H=\dot{\alpha}$ and the scale factor of the Universe $a=e^{\alpha}$.

Our classical analysis of the time evolution of the model is generally based on Ref. 8. We say that the Universe is matter dominated if

$$\rho_\phi+\rho_{\dot{\phi}}\gg\rho_\Sigma \tag{2.21}$$

and shear or anisotropy dominated if

$$\rho_\phi+\rho_{\dot{\phi}}\ll\rho_\Sigma\ . \tag{2.22}$$

Another way to classify the stages of the solution is to categorize them by the behavior of the scalar field ϕ, which gives the stress energy tensor of a perfect fluid at rest with respect to our coordinate frame with the following density ρ_m and pressure P:

$$\rho_m=\tfrac{1}{2}(\dot{\phi}^2+\phi^2)=\rho_{\dot{\phi}}+\rho_\phi\ , \tag{2.23}$$

$$P=\tfrac{1}{2}(\dot{\phi}^2-\phi^2)=\rho_{\dot{\phi}}-\rho_\phi\ . \tag{2.24}$$

The scalar field ϕ can be either strongly overdamped or weakly damped, depending basically on the value of the density ρ.

The first case, i.e.,

$$\rho\gg\tfrac{2}{9}\ , \tag{2.25}$$

we can divide into two subcases, depending on whether the kinetic or the potential energy dominates. (1) If the kinetic energy dominates,

$$\rho_{\dot{\phi}}\gg\rho_\phi\ , \tag{2.26}$$

then $P\approx\rho_m$ and we are in the stiff regime. (2) If the potential energy dominates,

$$\rho_{\dot{\phi}}\ll\rho_\phi\ , \tag{2.27}$$

then $P\approx-\rho_m$ and we are in a "false vacuum" regime. If the Universe is also matter dominated at this time one gets

$$-\frac{\dot{H}}{H^2}=3\frac{\rho_\Sigma+\rho_{\dot{\phi}}}{\rho}\ll 1\ , \tag{2.28}$$

which is the inflationary regime.

The second case, in which,

$$\rho\ll\tfrac{2}{9}\ , \tag{2.29}$$

is called the dustlike regime. The scalar field ϕ is weakly damped and oscillates with a period roughly equal to 2π. The pressure averaged over the period vanishes and the average stress-energy tensor has effectively the form of perfect dust.

In our analysis of the solutions of the system of Eqs. (2.10), (2.13), and (2.18), we first obtain approximate solutions valid during the inflationary regime. Those generic solutions depend on two parameters. Then we show that any general solution can be approximated in the overdamping regime by our formulas (possibly with a different range of parameters), even if the solution never undergoes inflation. Thus our approximations are valid for any general solution until it enters into the dustlike regime. Finally we investigate the dustlike region.

We assume that $\dot{\alpha}>0$ so that the Universe is expanding. Of course, Eqs. (2.10), (2.13), and (2.18) exhibit time-reversal symmetry. If $\dot{\alpha}$ is greater than zero at any time it will always remain positive due to Eq. (2.18).

Only one of the two equations (2.10) and (2.13) is independent providing Eq. (2.18) is satisfied. If $\Sigma\neq 0$ we can rescale that quantity to have any value by a constant shift of α, which is a symmetry of the equations of motion. Given an arbitrary Σ one can always shift the time so that $t=0$ at the singularity, which will be assumed henceforth. Thus any general solution of the system of independent equations (2.13) and (2.18), being a system of coupled first- and second-order differential equations, can depend on two parameters, in addition to Σ.

Next we derive the approximate general solutions. The asymptotic form of the solutions of Eqs. (2.10) and (2.13) for small t is

$$\alpha=\tfrac{1}{3}\ln t+\alpha_1 , \tag{2.30}$$

$$\phi=2B\ln t+C , \tag{2.31}$$

where α_1, B, and C are constants. The constraint equation (2.18) implies that

$$\Sigma^2=(\tfrac{1}{9}-4B^2)\exp(6\alpha_1) . \tag{2.32}$$

This asymptotic form becomes invalid when the ϕ and ϕ^2 terms in Eqs. (2.10), (2.13), and (2.18) become comparable to the other terms. If we include the $-3\phi^2$ term in Eq. (2.10) but assume for the moment that its variation has a negligible effect, then

$$\ddot{\alpha}+3\dot{\alpha}^2=3\phi_0^2 \tag{2.33}$$

has the solution

$$\dot{\alpha}=|\phi_0|\coth(3|\phi_0|t) . \tag{2.34}$$

When this "first improved" form of $\dot{\alpha}$ is inserted into the wave equation (2.13), that equation has the exact hypergeometric solution

$$\phi(t)=A\phi_1(t)+B\phi_2(t) , \tag{2.35}$$

where

$$\phi_1(t)=(1-x)^{\mu}F(\mu,\mu;1;x) \tag{2.36}$$

and

$$\phi_2(t)=(1-x)^{\mu}\left[F(\mu,\mu;1;x)\ln x+2\sum_{k=1}^{\infty}x^k\frac{(\mu)_k^2}{(k!)^2}[\psi(\mu+k)-\psi(\mu)-\psi(k+1)+\psi(1)]\right] . \tag{2.37}$$

In these equations

$$x=\tanh^2(\tfrac{3}{2}\phi_0 t) , \tag{2.38}$$

$$\mu=\frac{1}{2}\left[1-\left[1-\frac{4}{9\phi_0^2}\right]^{1/2}\right] , \tag{2.39}$$

and ψ is the Euler digamma function

$$\psi(z)=\frac{d\ln\Gamma(z)}{dz} . \tag{2.40}$$

For real A and B the solution given by Eq. (2.35) is real even if $\phi_0^2<\frac{4}{9}$.

For $3|\phi_0|t\ll 1$ Eqs. (2.34) and (2.35) give the asymptotic forms (2.30) and (2.31), where the constant B is the same as in Eq. (2.31), if

$$C=A+2B\ln(3|\phi_0|)-2B\ln 2 . \tag{2.41}$$

Since ϕ is not constant, it is not strictly correct to assume $\phi=\phi_0$ in Eq. (2.10) as was done to get Eq. (2.34). As $t\to 0$ Eq. (2.35) gives $\phi(t)$ diverging logarithmically for the generic case of $B\neq 0$, which we shall assume unless explicitly stated otherwise. However, each of the other terms in Eq. (2.10) diverges as $(3t)^{-2}$, so the logarithmic divergence of ϕ away from ϕ_0 has a negligible effect on $\alpha(t)$ at $t\approx 0$. In order that Eq. (2.33) remain approximately valid when the first two terms have decreased in magnitude to become comparable to $3\phi_0^2$, we want ϕ_0 to be chosen to be closely equal to ϕ at that time. Hence we must examine the behavior of the hypergeometric solution (2.35).

For the time being assume $|\phi_0|\gg 1$ so

$$\mu\approx\frac{1}{9\phi_0^2} . \tag{2.42}$$

Then $\ddot{\alpha}$ and/or $3\dot{\alpha}^2$ will become and remain comparable to $3\phi_0^2$ for $3|\phi_0|t\geq 1$. For $1\ll 3|\phi_0|t\ll 3|\phi_0|(3|\phi_0|-1)$ one gets the following approximation of Eq. (2.35):

$$\phi(t)=A-\frac{A}{3|\phi_0|}t+O\left[\frac{1}{|\phi_0|}\right] . \tag{2.43}$$

We have used the fact that Eq. (2.32) implies $|B|<\frac{1}{6}$ and that $\phi_2(t)=O(|\phi_0|^{-1})$ in this range of t. Since we assumed that $\phi\approx\phi_0$, then $A\approx\phi_0$ and

$$\phi(t)=\phi_0-\tfrac{1}{3}\mathrm{sgn}(\phi_0)t+O\left[\frac{1}{|\phi_0|}\right] . \tag{2.44}$$

This is the generic behavior of the scalar field during the inflationary regime.

The solutions given by Eqs. (2.35) or (2.44) are obtained under the assumption that $\dot{\alpha}$ is given by Eq. (2.34) and therefore are only approximate ones. Now that the behavior of ϕ is better known, we use it to get corrections in $\dot{\alpha}$. Assuming $1\ll 3|\phi_0|t\ll 3|\phi_0|(3|\phi_0|-1)$ one gets the following approximate solutions of Eqs. (2.10) and (2.13):

$$\alpha(t)=\tfrac{1}{3}\ln\sinh(3|\phi_0|t)-\tfrac{1}{6}t^2-\tfrac{1}{3}\ln\left|1-\frac{t}{3|\phi_0|}\right| +\tfrac{1}{6}\ln\left[\frac{\Sigma^2}{\phi_0^2(1-36B^2)}\right] , \tag{2.45}$$

$$\alpha(t)=(\sinh 3|\phi_0|t)^{1/3}e^{-(1/6)t^2}\left|1-\frac{t}{3|\phi_0|}\right|^{-1/3} \times\left[\frac{\Sigma^2}{\phi_0^2(1-36B^2)}\right]^{1/6} , \tag{2.46}$$

$$\phi(t)=\phi_0\phi_1(t)+B\phi_2(t) . \tag{2.47}$$

This approximation is valid until $t\geq 3|\phi_0|-1$, when the solution enters the dustlike regime, which is the sole final stage.

Now we discuss the generic behavior of those solutions in the stiff regime. For early time $t \ll |\phi_0|^{-1}$ Eq. (2.45) yields

$$\alpha = \tfrac{1}{3}\ln t + \alpha_1 \ . \tag{2.48}$$

With that substitution the wave equation (2.13) becomes the Bessel equation of the zeroth order:

$$\ddot{\phi} + \frac{1}{t}\dot{\phi} + \phi = 0 \ . \tag{2.49}$$

For small arguments of the Bessel functions the right choice of solution, namely,

$$\phi(t) = [\phi_0 + 2B\ln(3|\phi_0|) - 2B\mathcal{C}]J_0(t) + \pi B N_0(t)$$
$$\equiv D J_0(t) + \pi B N_0(t) \ , \tag{2.50}$$

where $\mathcal{C}=0.577\ldots$ is the Euler constant, gives the same approximation as Eq. (2.47), up to terms of order of unity.

Any solution near the singularity has the generic form

$$\alpha(t) = \tfrac{1}{3}\ln\frac{3\Sigma t}{\sqrt{1-36B^2}} + O(t^2\ln^2 t) \ , \tag{2.51}$$

$$\phi(t) = 2B\ln\frac{t}{2} + 2B\ln(3|\phi_0|) + \phi_0 + O(t^2\ln^2 t) \ . \tag{2.52}$$

In this approximation neither α nor $\dot{\phi}$ depend on ϕ_0. At this stage the shear density ρ_Σ and the kinetic energy density $\rho_{\dot{\phi}}$ are roughly proportional to each other:

$$\rho_\Sigma \approx \frac{1-36B^2}{18t^2} \ , \tag{2.53}$$

$$\rho_{\dot{\phi}} \approx \frac{2B^2}{t^2} \ , \tag{2.54}$$

whereas the potential density ρ_ϕ is equal to

$$\rho_\phi \approx \tfrac{1}{2}\left[2B\ln\frac{t}{2} + 2B\ln(3|\phi_0|) + \phi_0\right]^2 \ , \tag{2.55}$$

which is negligible by comparison during this stiff regime. As the time goes by both shear and kinetic energy decrease simultaneously with $\dot{\alpha}$ and finally lead to a stage when the stiff regime ends. For our solutions the choice between the two possible consequent ways of evolution depends on the parameter $|\phi_0|$.

(1) If $|\phi_0|$ is small, namely,

$$\frac{1}{3|\phi_0|} \gtrsim 3|\phi_0| - 1 \ , \tag{2.56}$$

the scalar field starts oscillating with a period roughly equal to 2π before the Universe could have undergone inflation. In general, we have two subcases. If the solution starts as a matter-dominated one, it exhibits the same basic properties as the flat FRW models discussed in Ref. 8. If the opposite, i.e., if the solution starts as an anisotropy-dominated one ($B \ll \frac{1}{6}$), then one can use Eqs. (2.51) and (2.52) until $\rho_\Sigma \approx \rho_\phi$.

The solution ceases to be anisotropy dominated at time $t \approx (2\pi/9)(\pi^2B^2 + D^2)^{-1}$, where D is given in Eq. (2.50). For both $|B|$ and $|\phi_0|$ small this occurs late after the scalar field begins oscillating (at $t \approx 1$). In this case one can use the WKB method to get an approximate solution valid in the dust regime and match it to the asymptotic behavior of the Bessel solutions (2.50) in the overlap region, where both approximations are valid, and get, in the dust regime,

$$\phi(t) \approx \tfrac{2}{3}[\tfrac{1}{4}t_0^2 + t^2(t+t_0)^2]^{-1/4}$$
$$\times\cos\left[t - \theta_0 - \frac{\pi}{4} - \frac{1}{8t} - \frac{1}{8(t+t_0)} + \frac{1}{4t_0}\ln\frac{t+t_0}{tt_0}\right] \ , \tag{2.57}$$

where

$$t_0 = \frac{2\pi}{9(D^2+\pi^2B^2)} \tag{2.58}$$

and

$$\tan\theta_0 = \frac{\pi B}{D} \ . \tag{2.59}$$

This approximation can be used for $t \gg t_0$.

(2) For the solutions that undergo inflation a similar reasoning is not applicable since the assumption $\rho_{\dot{\phi}} \ll \rho_\phi$ that leads to Eq. (2.35) and therefore to Eqs. (2.45) and (2.47) is no longer satisfied near the end of inflation. There are many different reasons[4,5,11,12] to desire a large amount of inflation, corresponding to large $|\phi_0|$. Therefore we consider the case

$$\frac{1}{3|\phi_0|} \ll 3|\phi_0| - 1 \ , \tag{2.60}$$

or, equivalently $|\phi_0| \gg \frac{1}{6}(1+\sqrt{5})$. Assume for the moment that the solution is matter dominated. Then the Universe begins to inflate at a time t_1 that can be very roughly estimated by

$$t_1 = \frac{1}{3|\phi_0|} \tag{2.61}$$

and undergoes inflation until approximately

$$t_2 = 3|\phi_0| - 1 \ . \tag{2.62}$$

Here the inflationary regime is defined as a stage at which the size factor increases roughly exponentially, that is, for our case, equivalent to the condition $-\dot{H} \ll H^2$. This condition requires that the solution be matter dominated during the inflationary regime.

Even if the solution starts as an anisotropy-dominated one ($B \ll \frac{1}{6}$), one can still use Eqs. (2.45) and (2.47) in those regimes where the variation of ϕ is small compared to ϕ. Then the solution is valid during the stiff regime, which ends at a time approximately equal to $t \approx 2|B\phi_0^{-1}| \ll (3|\phi_0|)^{-1}$. At that time the anisotropy density still dominates, $\rho_\Sigma \approx B^{-2}\rho_m$, so we enter the "false vacuum" regime but not quite yet the inflationary regime. The solutions given by Eqs. (2.51) and (2.52) are still valid until the matter dominates. This happens at a time roughly equal to $t \approx (3|\phi_0|)^{-1}$, when the solutions become invalid. During the short transition period before the onset of inflation the solution must become matter dominated, since from Eqs. (2.45) and (2.47), we have in the inflationary regime (in which both equations are again highly accurate)

$$-\frac{\dot{H}}{H^2}=-3\frac{\rho_\Sigma+\rho_{\dot\phi}}{\rho}\ll 1 \ . \tag{2.63}$$

During inflation the ratio ρ_Σ/ρ_ϕ decreases exponentially, and any solution must remain matter dominated at that stage. Therefore after inflation ends the model should exhibit the same properties as the flat FRW model presented in Ref. 8. If $|\phi_0|\gg 1$, then Eq. (2.47) gives the increase of the size factor from a_1 at the beginning of inflation, at time t_1 defined by Eq. (2.61), to a_2 at the end of inflation, at time t_2 given by Eq. (2.62), as

$$\frac{a_2}{a_1}\approx |\phi_0|^{1/3}\exp(\tfrac{3}{2}\phi_0^2) \ , \tag{2.64}$$

to within factors of order unity. This result is the same as for the FRW models.[8]

Another way to describe the behavior of the solutions is to use new dimensionless variables x, y, and z defined as

$$x=\frac{\phi}{\dot\alpha} \ , \tag{2.65}$$

$$y=\frac{\dot\phi}{\dot\alpha} \ , \tag{2.66}$$

$$z=\frac{\Sigma e^{-3\alpha}}{\dot\alpha} \ . \tag{2.67}$$

The system of Eqs. (2.10), (2.13), and (2.18) now becomes

$$\dot x=y-3\dot\alpha x(x^2-1) \ , \tag{2.68}$$

$$\dot y=-x(1+3\dot\alpha xy) \ , \tag{2.69}$$

$$\dot z=-3\dot\alpha x^2 z \ , \tag{2.70}$$

provided

$$\dot\alpha^2=\frac{\Sigma^2 e^{-6\alpha}}{z^2} \tag{2.71}$$

and

$$x^2+y^2+z^2=1 \ . \tag{2.72}$$

Some sample trajectories for that system are presented in Fig. 1 which is a projection of the unit hemisphere onto the xy plane. Except for a set of measure zero, each solution starts at $x=0$ and ends circling around and getting closer and closer to the equator. The inflationary solutions pass through a small region near the points $x=\pm 1$, $y=0$. In those regions, at which $|\dot x|\ll 1$, $|\dot y|\ll 1$, the solution spends a long time being in the inflationary regime. After the inflation ends the scalar field starts oscillating making the trajectory circle round the sphere nearly at the equator. At this stage

$$z(\rho)\approx z(\rho_0)\left[\frac{\rho}{\rho_0}\right]^{1/2} \ , \tag{2.73}$$

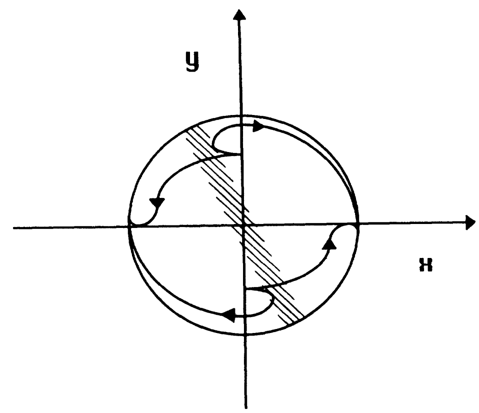

FIG. 1. The trajectories of the solutions in the variables x and y.

where $\rho=\frac{1}{2}\dot\alpha^2$ is the total energy density and ρ_0 is its value at the end of inflation. The value of ρ_0 is of order of $\frac{2}{9}$ and depends on the precise criterion for the inflation. From our definitions of the inflationary and dustlike regimes, inflation ends at, approximately, $t=3|\phi_0|-1$ and ρ_0 is $\frac{2}{9}$. Thus for $|\phi_0|\gg 1$ one has

$$z(\rho_0)\approx 9\sqrt{1-36B^2}\exp(-\tfrac{9}{2}\phi_0^2) \ . \tag{2.74}$$

Let us assume that the solution undergoes a large amount of inflation, i.e., $|\phi_0|\gg 1$. From the preceding considerations the Universe is very nearly isotropic at the end of inflation. Then it is well motivated to use the analysis of the dustlike region for the flat FRW metric given in Ref. 8. If we define θ by

$$\tan\theta=\frac{y}{x} \ , \tag{2.75}$$

the result of that analysis is that

$$\theta=t_3-t+O\left[\frac{1}{t-t_3}\right] \ , \tag{2.76}$$

$$\rho\approx\tfrac{2}{9}(\theta_3-\theta+\tfrac{1}{2}\sin 2\theta)^{-2} \ , \tag{2.77}$$

and

$$\alpha\approx\left[\frac{\lambda}{4}\right]^{1/3} a_2(t-t_3)^{2/3} \ . \tag{2.78}$$

The parameters t_3, θ_3, and λ are discussed in Ref. 8. It is argued that although t_3 diverges as $|\phi_0|$ is taken to infinity, the remaining parameters, λ and θ_3, tend to finite limits. Our parameter λ is the same as in Ref. 8 to within a factor which slightly depends on the initial shear density but is always very close to unity for large $|\phi_0|$.

Within the same accuracy the parameter $\theta_3(\phi_0)$ differs from its limiting value $\theta_3(\infty)$ by

$$\theta_3(\phi_0)-\theta_3(\infty)\approx\nu\phi_0^{-1}e^{-(9/2)\phi_0^2}\approx\nu\left[\frac{a_2}{a_1}\right]^{-3} \ , \tag{2.79}$$

where ν is a constant to be obtained numerically. Similar considerations for the ϕ_0 of the opposite sign give the opposite sign for ν.

As in FRW models one can determine how much inflation occurred by finding the difference between actual θ_3 and its asymptotic value $\theta_3(\pm\infty)$. Moreover, once ϕ_0 is known, one can determine the initial ratio of the kinetic energy to the anisotropy energy by measuring the anisotropy z and using Eqs. (2.73) and (2.74). Unfortunately, this method does not seem to be of great use because of the extremely small actual value of z.

III. THE PROBABILITY OF INFLATION

It has been long suggested (see Refs. 1–3, and 12) that shear of an anisotropic metric can be an additional factor driving inflation. However, it follows from our approximate solutions (2.45), (2.46), and (2.47), that this is not exactly the case, at least for Bianchi type-I models. The condition under which inflation occurs (given by $|\phi_0| \gg 1$) does not depend on the shear on the initial data surface at early time t. In fact, the ratio (2.64) of the scale factors after and before inflation depends slightly on the initial shear density, which is roughly

$$\rho_\Sigma \approx \frac{1-36B^2}{2t^2} . \tag{3.1}$$

However, the increase of the size due to the shear is in order of unity and is therefore comparable to various uncertainties resulting from ambiguities in the definitions of the beginning and the end of inflation.

Another interesting problem, stated in Ref. 8, is the dependence of the probability of inflation on the chosen model. It has been argued that additional degrees of freedom (such as nonvanishing shear) would not change the fact that the measure of the inflationary solutions and the measure of the noninflationary solutions are both infinite. For a fixed range of additional variables it was suggested that the qualitative nature of FRW models would not be destroyed. We examine this in great detail below.

References 6 and 7 give a canonical measure on the space of solutions of a Hamiltonian system with an odd number of constraints. Here we have one Hamiltonian constraint on a symplectic space Γ_n of dimension $2n$. The restriction to an initial data surface, in order to count each solution only once, further reduces the dimension to $2(n-1)$, giving a subspace Γ_{n-1}. The measure is then the $(n-1)$th power of the symplectic form pulled back to Γ_{n-1},

$$\Omega_{n-1} = (-1)^{(n-1)(n-2)/2}(\omega_{n-1})^{n-1} . \tag{3.2}$$

In the case of a diagonal Bianchi type-I metric coupled to a scalar field, n is 4,and the symplectic form is equal to

$$\omega = dp_\alpha \wedge d\alpha + dp_\phi \wedge d\phi + dp_{\beta_+} \wedge d\beta_+ + dp_{\beta_-} \wedge d\beta_- . \tag{3.3}$$

Since all solutions cross a surface of constant density ρ once, that surface can be chosen as the initial data surface Γ_6, which is assumed henceforth. Using the constant of motion Σ defined as

$$\Sigma^2 = p_{\beta_+}^2 + p_{\beta_-}^2 , \tag{3.4}$$

we change the variables p_{β_+} and p_{β_-} to Σ and χ such that

$$p_{\beta_+} = \Sigma \sin\chi , \tag{3.5}$$

$$p_{\beta_-} = \Sigma \cos\chi . \tag{3.6}$$

Then the measure on the surface ρ=const, integrated over χ, becomes

$$\mu = -12\pi\Sigma\, d\Sigma \wedge dp_\phi \wedge d\beta_+ \wedge d\beta_- \wedge d\phi . \tag{3.7}$$

In order to compare this result to those of Ref. 8 we fix the range of parameters β_+ and β_-. The measure (3.7) is then proportional to

$$\mu' = \Sigma\, d\Sigma \wedge d(\dot{\phi} e^{3\alpha}) \wedge d\phi . \tag{3.8}$$

Using the parameters of our approximate solutions (2.45) and (2.47) we get, on a surface of fixed and very high density, that

$$\mu' \approx \frac{6\Sigma^2}{\sqrt{1-36B^2}} \left| \frac{2B}{\phi_0} + 1 \right| d\Sigma \wedge dB \wedge d\phi_0 . \tag{3.9}$$

This expression is finite when integrated only over the parameter B, which has the range from $-\frac{1}{6}$ to $\frac{1}{6}$. However, it diverges when integrated over ϕ_0. Thus, even for a fixed range of Σ the measure is infinite. This is what was expected for this model—additional degrees of freedom give rise to additional divergencies. To make a direct comparison of our result to that of Ref. 8 we set

$$a_* = [\Sigma^{-2}\phi_0^2(1-36B^2)]^{-1/6} , \tag{3.10}$$

$$\phi_* = \phi_0 . \tag{3.11}$$

Then, for a fixed range of Σ the measure (3.9) is proportional to

$$3a_*^2[\tfrac{1}{3}\mathrm{sgn}\phi_* + \phi_*(1-\Sigma^2 a_*^{-6}\phi_*^{-2})^{-1/2}]da_* \wedge d\phi_* . \tag{3.12}$$

For $\Sigma=0$ this is exactly the same as the measure on a surface of fixed density for the FRW models.

Since the total measure is infinite the ratio of the measure of the inflationary solutions to the measure of the noninflationary ones is ambiguous and depends on how it is evaluated. To make this statement evident we first analyze the measure on a surface of (fixed) high density ρ and then compare the result to that which we find for very low density.

For a fixed range of α, the measure (3.8) on the initial data surface of fixed ρ can be expressed in terms of "rescaled" variables x and y defined by Eqs. (2.65) and (2.66). It is proportional to

$$\mu'' = 4(1-x^2)dx \wedge dy . \tag{3.13}$$

Let ϕ_m be the least value of $|\phi_0|$ necessary to get significant inflation. If, e.g., we require the increase of the scale factor of the Universe to be greater than some predetermined

$$\frac{a_2}{a_1} = (1+Z) \approx |\phi_0|^{1/3}\exp(\tfrac{3}{2}\phi_0^2) , \tag{3.14}$$

we choose $\phi_m \approx [\frac{2}{3}\ln(1+Z)]^{1/2}$. At constant density $\rho \gg \frac{1}{2}\phi_m^2$ all solutions that do not give sufficient increase of the size factor are in the stiff regime described by our approximate solutions (2.51) and (2.52) and lie within a narrow strip bounded by lines $x = yt \ln t \pm 3\phi_m t$, shown in Fig. 1, where $t = 1/3\sqrt{2\rho}$. The rescaled measure μ'' of this strip is $48\phi_m t$. The inflationary solutions lie outside that strip and have the measure approximately equal to $3\pi - 48\phi_m t$. Both measures are accurate to the terms of order of $t^2 \ln^2 t$ or $\phi_m t^2 |\ln t|$, whichever is greater. Thus, the ratio of the measures of the inflationary solutions and the noninflationary ones, determining the probability of inflation, is

$$\frac{\mu_I''}{\mu_{\rm NI}''} = \frac{3\pi\sqrt{2\rho}}{16\phi_m} - 1 \gg 1 \ . \tag{3.15}$$

This is essentially the same as in the FRW case,[8] up to a factor of $\frac{4}{3}$. This ratio depends on the density ρ on the initial data surface because we fixed the range of variables α, β_+, and β_- to get a finite total measure. If we assume a uniform distribution of the initial data on the hypersurface $\rho = \text{const} \gg \frac{1}{2}\phi_m^2$ within the fixed range of α, β_+, and β_-, it is highly probable to get an inflationary solution.

Now, let us consider the measure on a hypersurface of fixed, but very low, density, $\rho \ll \frac{2}{9}$. At this density all the inflationary solutions are in the dustlike regime well after the end of inflation, which happens approximately at $\rho = \frac{2}{9}$. Therefore the solutions are given by Eqs. (2.73), (2.76), and (2.77). The inflationary solutions (with $|\phi_0| \gg \phi_m$) lie near the equator of the hemisphere on Fig. 1 and have

$$z < 27 \left[\frac{\rho}{2}\right]^{1/2} \exp(-\tfrac{9}{2}\phi_m^2) \ . \tag{3.16}$$

The angular coordinate θ, defined by Eq. (2.75), is a function of the density ρ, and of the parameters ϕ_0 and B. It is given by Eq. (2.77), in which θ_3 depends on ϕ_0 and B. For a large amount of inflation (e.g., for large $|\phi_0|$), the shear becomes extremely small by the time of the dustlike region, so the dependence of θ and θ_3 on B is very weak. We will not consider this dependence. In the narrow annular region near the equator the measure (3.13) on the sphere has the leading term equal to

$$\mu'' \approx 4\sin^2\theta \, z \, dz \wedge d\theta \ . \tag{3.17}$$

The inflationary solutions are confined to two sectors of the ring defined by Eq. (3.16), the first one bounded by lines $\theta = \theta(\infty, \rho) \pm \theta(\phi_m, \rho)$ and the second one obtained by $\phi \to -\phi$. The measure of those regions is equal to

$$\mu_I'' = 1458 \nu \rho \phi_m^{-1} \exp(-\tfrac{27}{2}\phi_m^2) \ll 1 \ , \tag{3.18}$$

and tends to zero as ϕ_m is taken to infinity or the density ρ tends to zero. This behavior is different from the behavior of the analogous measure on the FRW models, where the measure tends to a nonzero limit as ρ tends to zero at fixed ϕ_m. In our case the width of the annular region decreases with ρ decreasing and so does the measure of the inflationary solutions. However, the angles in which the inflationary solutions lie do not depend on the density ρ. Those angles cut out a fraction of the circumference which is the same as that for the FRW models.[8]

To estimate the probability of getting a solution which had undergone a satisfactory inflation one can find the ratio of the measure of the inflationary solutions to the measure of the noninflationary ones, which is equal to

$$\begin{aligned} \frac{\mu_I''}{\mu_{\rm NI}''} &\approx \frac{486}{\pi} \nu \rho \phi_m^{-1} \exp(-\tfrac{27}{2}\phi_m^2) \\ &\approx \frac{486}{\pi} \nu \rho \left[\frac{a_2}{a_1}\right]^{-9} \end{aligned} \tag{3.19}$$

and is negligibly small for large ϕ_m and small density ρ. This is exactly the opposite to the result obtained at high densities. This result is similar to that of the Friedmann-Robertson-Walker case[8] in that both give small probabilities of inflation at low density ρ and for a large inflationary growth factor a_2/a_1, but here the growth factor occurs in the -9 power rather than to the -3 power, and there is also a factor of ρ which is absent in the FRW case. This latter factor means that at fixed a_2/a_1, the probability of inflation in the Bianchi type-I case can be calculated to be arbitrarily low, by choosing ρ small enough, whereas in the FRW case the probability approaches a (very small) constant as ρ is taken toward zero. Thus by varying ρ from infinity to zero, we can vary the probability of inflation, calculated by our procedure, from unity to zero. This behavior clearly exhibits the ambiguity in the probability of inflation in this classical Bianchi type-I cosmology.

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