Dynamical chiral-symmetry breaking and determination of the quark masses

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Using an effective-potential approach for composite operators, we study dynamical symmetry breaking in QCD-like theories with massless quarks. The analysis is extended to massive quarks in QCD with three flavors and the masses of the pseudoscalar-octet mesons and their decay constants are calculated. Renormalization-group corrections are taken into account. The effective potential depends on the standard parameters of QCD: $\Lambda_{\rm QCD}$, m_u, m_d, m_s and on a mass scale μ which discriminates between the infrared and the ultraviolet regimes. A good fit for the meson masses (agreement within 3%) and for the decay constants is obtained for the following values of the quark masses at 1 GeV: $m_u = 5.8$ MeV, $m_d = 8.4$ MeV, and $m_s = 118$ MeV. These values essentially agree with the values obtained by quite different methods.

I. INTRODUCTION

In this work we will study the problem of chiral symmetry in QCD and the implications of its spontaneous and explicit breaking.

From the vast amount of experimental information available some highly fruitful ideas were developed long before QCD was invented. We are referring to what is known as "current algebra." In this approach the quark mass matrix appears only as a phenomenological quantity to be related to observables through the commutators involving the currents and the energy-momentum tensor.

The approximate SU(3) symmetry of the strong interactions of u,d,s implies that the constituent masses of the light quarks are not much different. The breaking of this global flavor symmetry was identified with an octet term.[1] Isospin conservation is a much better symmetry than the whole flavor SU(3). The approximate equality of the u and d constituent masses holds to a higher degree of accuracy than for d and s. The masses of c,b,t are all much larger than those of u,d,s and we do not see evidence for flavor-SU(4) or higher symmetries in the hadronic spectrum.

The smallness of the pion mass ($M_\pi/M_p = 0.14$) makes the pion very special among the hadrons. To understand this point Nambu[2] suggested that there is a limit in which the pion is a massless Goldstone boson associated with spontaneous symmetry breaking. To apply this idea one first considers the chiral limit $m_u = m_d = 0$ and ignores strangeness. This system is classically invariant under the global-symmetry group $SU(2)_L \otimes SU(2)_R \otimes U(1)_V \otimes U(1)_A$. The $U(1)_V$ is directly manifest as baryon-number conservation, whereas $U(1)_A$ is broken by the Adler, Bell, and Jackiw axial anomaly. The $SU(2)_V = SU(2)_{L+R}$ invariance leads to isospin conservation and in fact the hadrons fall into easily recognizable isospin multiplets. On the other hand, there is no evidence for a chiral hadronic spectrum.

For a generator L^j of a symmetry transformation of the dynamical system we have two possibilities:

$$L^j \mid 0\rangle = 0 \ , \tag{1.1}$$

the Wigner-Weyl realization, or

$$L^j \mid 0\rangle \neq 0 \ , \tag{1.2}$$

the Nambu-Goldstone realization. Two theorems are specially relevant.

The first, due to Coleman,[3] asserts that "the invariance of the vacuum is the invariance of the world." The physical states (including bound states) are then invariant under the Wigner-Weyl symmetry transformation. It is then strongly suggested that the $SU(2)_V$ is an approximate Wigner-Weyl symmetry and that the chiral $SU(2)_L \otimes SU(2)_R$ contains Nambu-Goldstone-type generators.

The second relevant theorem is due to Goldstone.[4] It states that for each global generator that fails to annihilate the vacuum there must exist a massless spinless boson with the quantum numbers of that generator. We may thus explain the smallness of the π masses if in the limit $m_u, m_d \to 0$ the π becomes a Goldstone boson. The mass of the physical pions then originates from the explicit chiral-symmetry-breaking parameters m_u and m_d. Since m_u and m_d are small, the pions are *almost* massless and the axial-vector currents to which they couple are *almost* conserved.

Setting

$$\langle 0 | J^k_{\mu 5}(x) | \pi_j(p) \rangle = i p_\mu \delta_{kj} f_\pi e^{-ipx} , \tag{1.3}$$

where i,j,k are isospin indices and $J^k_{\mu 5}$ is the axial-vector current, one has

$$\langle 0 | \partial^\mu J^k_{\mu 5}(x) | \pi_j(p) \rangle = \delta_{kj} f_\pi M_\pi^2 e^{-ipx} , \tag{1.4}$$

which expresses the property of the pions being the Goldstone bosons in the chiral limit. One defines

$$\phi_k(x) \equiv \frac{1}{M_\pi^2 f_\pi} \partial^\mu J^k_{\mu 5}(x) \tag{1.5}$$

so that

$$\langle 0 | \phi_k(x) | \pi_j(p) \rangle = \delta_{kj} e^{-ipx} . \tag{1.6}$$

The content of PCAC (partial conservation of axial-vector current) is the identification of $\phi_k(x)$ as the pion field in the chiral limit.

The six generators of the chiral $G = SU(2)_L \otimes SU(2)_R$ group, $Q^i = Q^i_L + Q^i_R$ and $Q^i_5 = Q^i_L - Q^i_R$ $(i = 1,2,3)$, are such that

$$Q^i | 0 \rangle = 0 , \tag{1.7}$$

$$Q^i_5 | 0 \rangle \neq 0 . \tag{1.8}$$

The isospin generators annihilate the vacuum and generate a subgroup H of G, the stability group of the vacuum [here $H = SU(2)_V$], while the generators Q^i_5 lie in the quotient space

$$Q^i_5 \in \mathrm{Lie}G / \mathrm{Lie}H . \tag{1.9}$$

The symmetry G breaks down spontaneously to the symmetry H (Ref. 5).

The approximate SU(3) suggests an extended chiral-symmetric limit corresponding to $m_u = m_d = m_s = 0$. This extended chiral symmetry will be spontaneously broken and manifested by Goldstone bosons π, K, η. In this limit the global group is $SU(3)_L \otimes SU(3)_R \otimes U(1)_V$ with $SU(3)_A = SU(3)_{L-R}$ realized in the Goldstone mode and $SU(3)_V = SU(3)_{L+R}$ directly manifest in the "eightfold way." The explicit chiral-symmetry violation raises the masses of the pseudoscalar mesons to finite values. The violation of $SU(3)_V$ leads to mass splittings in the flavor multiplets.[6] The term which explicitly breaks $SU(3)_L \otimes SU(3)_R \otimes U(1)_V$ is

$$-(m_u \bar{u} u + m_d \bar{d} d + m_s \bar{s} s) . \tag{1.10}$$

Its effects are usually treated by "chiral perturbation theory"[7] which contains both "current algebra" and "extended PCAC."

The main problem at this point is to understand the dynamical reasons why $SU(3)_A$ manifests itself in the Goldstone mode. In the nonrenormalizable pre-QCD model of Nambu and Jona-Lasinio[8] the cause of spontaneous symmetry breakdown was a direct strong nucleon-nucleon attraction. The scheme was motivated by the observation of an interesting analogy between the properties of Dirac particles and the quasiparticle excitations that appear in the theory of superconductivity of Bardeen, Cooper, and Schrieffer[9] (BCS). The characteristic feature of the BCS theory is that it leads to the energy gap between the ground state and the excited states of a superconductor. This gap is due to the fact that the attractive phonon-mediated interaction between electrons produces correlated pairs of electrons, with opposite momenta and spin, near the Fermi surface. In the same way as the energy gap in a superconductor is created by an effective electron-electron attraction, one may envisage that the Dirac mass is also due to some interaction between massless bare fermions. In Ref. 8 a simplified nonrenormalizable model of a chirally invariant four-fermion interaction is considered. The implications are that the nucleon mass is generated by some primary interaction between originally massless fermions and that the same interaction is also responsible for the formation of pseudoscalar zero-mass bound states of fermion-antifermion pairs which may be regarded as idealized pions. The presence in the physical spectrum of massless particles is a manifestation of spontaneous symmetry breaking. The Goldstone bosons are here composite particles. The playing of the Goldstone mechanism is visible in the effective Lagrangian more explicitly than in the fundamental Lagrangian. The symmetry is spontaneously broken by the dynamics, and one speaks of dynamical symmetry breaking (DSB).

Just as the effective electron-electron attraction in superconductivity arises from the more fundamental electron-phonon interaction, the Nambu–Jona-Lasinio model can be taken as an effective low-energy description of the strong quark-gluon gauge interaction. So the further problem is to explain how the color forces lead to the dynamical breakdown of chiral symmetry. In such a way the problem is reduced to that of the dynamical realization of a linear σ model.[10] In the unstable (symmetric) phase of the σ model there are n^2 scalar and n^2 pseudoscalar tachyons transforming as the $(n,n^*) \oplus (n^*,n)$ representation of the chiral $SU(n)_L \otimes SU(n)_R$ group. In addition there are n left and n right massless fermions assigned to the $(n,0)$ and $(0,n)$ representations, respectively. The occurrence of tachyons indicates that the vacuum of the normal phase with massless fermions is unstable. Under the vacuum rearrangement (phase transition), the symmetry is lowered to $SU(n)_V \otimes U(1)_V$, and $n^2 - 1$ pseudoscalar tachyons go into massless Goldstone bosons, whereas n^2 scalars and one pseudoscalar [associated with the $U(1)_A$ broken by the axial anomaly] are transformed into massive bosons. Fermions acquire mass due to the Yukawa-type interaction with scalar bosons. In the framework of QCD, hadrons are represented by bound states of quarks and antiquarks. So one must determine the forces which can lead to such tightly bound states as tachyons and Goldstone bosons in the unstable and stable phases, respectively. One expects that the binding of the fermions, coming from the strong action of the color forces at distances of the size of the bound bosons, results in the appearance of condensates breaking chiral symmetry spontaneously.

The crude but basic idea is the following. Consider a bound state of a pair of massless quark and antiquark. Because of the uncertainty principle, the energy of the ground state in a fully relativistic formulation will be given by

$$E^2 \simeq p^2 - g^2/r^2 \simeq p^2(1-g^2) , \tag{1.11}$$

where p and r denote the relative momentum and coordinate, respectively, and g is the gauge coupling constant. When g exceeds something of order of one, there will be a tachyon bound state, indicating the instability of the vacuum. In order to cure this instability, the vacuum rearranges itself and gives mass to the quarks. The existence of a critical value for the coupling is essential for the mechanism of dynamical mass generation.

The gauge coupling in quantum chromodynamics is asymptotically free and becomes strong at large distances. So there will be a scale at which the ground state of the theory has an indefinite number of massless fermion pairs which can be created by the strong coupling. Since we still expect the bound state to be invariant under Lorentz and color-SU(3)$_c$ transformations, it will only contain pairs with vanishing total momentum, angular momentum, and color charge but with a net chiral charge. More generally the situation is that the vacuum $|\Omega\rangle$ will have the property that an operator which destroys a fermion pair has a nonzero vacuum expectation value

$$\langle \Omega | \bar{\Psi}_{La} \Psi_{Rb} | \Omega \rangle = \langle \Omega | \bar{\Psi}_{Ra} \Psi_{Lb} | \Omega \rangle = v \delta_{ab} . \tag{1.12}$$

To carry out the analysis one has first to decide how to perform a quantitative computation of chiral-symmetry breaking. Basically we need to test whether the energy of the vacuum is lowered when fermion bilinears acquire a nonzero vacuum expectation value. The standard method, when the quantity acquiring a vacuum expectation is a scalar field ϕ, is to evaluate the effective potential.[11] The main property of the effective potential is that it turns out to be equal to the energy of the vacuum under the constraint that the vacuum expectation value of ϕ has some definite value ϕ_c. So one only needs to minimize this functional with respect to ϕ_c in order to determine the vacuum value of ϕ and the various phases of the theory.

A series expansion for the effective potential was derived by Jackiw.[12] Each order of the series corresponds to an infinite set of Feynman diagrams with a fixed numbers of loops. This functional evaluation of the effective potential is very useful, since it is important to be able to study the higher-order multiloop graphs, if not explicitly, at least in general terms. In fact there exist phenomena which cannot be easily seen in perturbative series. A clear example is the formation of bound states, which cannot be observed at a finite order in a loop expansion. Necessarily it requires at least an infinite subset of all orders (in a chirally symmetric theory the invariance of the Lagrangian guarantees that the mass term in the fermion propagator will never appear in any finite order of perturbation theory). One needs an approximation scheme that preserves those nonlinear features of field theory which lead to the relevant cooperative effects. With the effective potential series, as introduced by Jackiw, it is possible to sum large classes of ordinary perturbation-series diagrams so one has a formalism specially appropriate for the study of DSB.

In the present case one expects that the breaking of the theory is due to the formation of bound states (condensates) playing the role of the elementary scalar fields ϕ. So one needs an appropriate generalization of the effective potential for composite operators. This was introduced by Cornwall, Jackiw, and Tomboulis[13] (CJT). The idea is to introduce, inside the generating functional of the Green's functions of the theory, sources $J(x,y)$ coupled to the composite operators one is interested in, and then to Legendre transform to a generalized effective action. The functional Γ one obtains for a scalar theory depends not only on the expectation value of the scalar field $\phi_c(x)$ but also on $G(x,y)$, the expectation value of $T[\phi(x)\phi(y)]$, and it represents the generating functional in ϕ_c of the two-particle-irreducible Green's functions expressed in terms of the propagator G. [The conventional effective action is merely $\Gamma(\phi_c, G)$ at $J(x,y)=0$.] Physical solutions must satisfy

$$\frac{\delta\Gamma(\phi_c, G)}{\delta\phi_c} = 0 , \tag{1.13}$$

$$\frac{\delta\Gamma(\phi_c, G)}{\delta G} = 0 . \tag{1.14}$$

Equation (1.13) reproduces the equations of motion, while (1.14) is nothing but the Schwinger-Dyson equation for the full propagator G. This formalism is thus especially appropriate to the study of dynamical-symmetry violation, characterized by the fact that one may have for (1.13) and (1.14) the symmetric solution for ϕ_c, $\phi_c = 0$, and a symmetry-breaking solution for G.

In Ref. 13 a formal series expansion was derived for the generalized effective action consisting in a systematic resummation of graphs with a fixed number of loops. So one has to evaluate $\Gamma_{\rm CJT}$ to a certain loop order and derive the stationary conditions given by Eqs. (1.13) and (1.14) for vanishing sources. A nonsymmetric solution of Eqs. (1.13) and (1.14) for the composite operator G is a signal for dynamical symmetry breaking.

For example, in the case of spontaneous chiral-symmetry breaking (χSB), all this procedure is equivalent to turning on some external field (analogous to a magnetic field orienting a potentially ferromagnetic system) coupled to the bilinear $(\bar{\psi}\psi)$, construct the ordered vacuum in the presence of this field, and then see if the vacuum remains ordered when one turns off the field. The vacuum expectation value of the composite operator $\langle \bar{\psi}\psi \rangle$ is the order parameter characterizing the phase transition.

However, as it has been pointed out in Refs. 14 and 15, the CJT functional has an intrinsic defect: it is not bounded from below. In particular, in the case of a SU(N) fermion gauge theory, a recent detailed numerical study[16] shows that all chiral-symmetry-breaking stationary points are saddle points. This is a very unpleasant property if one wants to perturb the vacuum of the theory to find its excitations.

To cure this instability problem of the CJT formulation a modification of the CJT effective action has been introduced, which corresponds to a different choice of the source-dependent term.[17,18] The modified functional has the same stationary points as that of CJT, but does not suffer from the problem of unboundness from below, and has also the property that the symmetry-breaking solutions of the Schwinger-Dyson equation for the propagator correspond to minima. The main difference with respect to CJT resides in the choice of the dynamical variables. In the case of fermionic gauge theories, the CJT effective action is a functional of the full fermion propagator S, while the modified effective action is completely expressed in terms of the fermion proper self-energy Σ.

We have applied the modified effective action formalism to the study of dynamical breaking of chiral symmetry in QCD-like gauge theories. Our main hypothesis is that the relevant contribution to the χSB phenomenon comes from relatively short-distance effects (this kind of problem has been considered also in the lattice calculation framework[19]). This assumption will be mainly justified by the numerical results obtained in this approach, and it will allow us to evaluate the effective action at the two-loop order. Our strategy consists in introducing a parameter μ as an infrared cutoff. Practically we have taken the self-energy of the fermions constant below μ, and above we have used the behavior suggested by the operator-product expansion (OPE). The fermion self-energy is expressed in terms of the renormalized fermionic condensates. The condensates are indeed our variational parameters, to be determined by looking at the minimum of the generalized effective potential.

For a first approximate understanding[17,20,21] we have discussed the so-called "rigid case," in which the logarithmic corrections coming from the renormalization-group analysis were neglected. In this approximation it is possible to derive analytically the complete expression for the effective potential at two fermion loops. The result is that in the case of massless fermions, the theory has two phases: the chirally symmetric phase and the phase broken into the diagonal flavor subgroup whenever the gauge coupling constant exceeds a critical value. For massive quarks it is necessary to specify the renormalization of the composite operator wave function. This leads to a condition equivalent to the Adler-Dashen requirement in the limit of vanishing quark masses, and also ensures the absence of spontaneous breaking of parity. In this framework the lowest vacuum corresponds to a local minimum of our effective potential, and from this it is possible to calculate the masses of the pseudoscalar mesons. They are simply related to the second derivatives of the effective potential which, in our model, are indeed positive definite (our stationary points are minima). This first approach, in QCD with three flavors, already gave[21] a good fit to the pseudoscalar-meson masses (singlet sector excluded) giving rise to the quark mass ratios

$$m_d/m_u=1.94,\quad m_s/m_d=21.7\ ,$$

$$(m_s-\hat{m})/(m_d-m_u)=43.14\quad [\hat{m}=(m_u+m_d)/2]$$

to be compared, for example, to the respective values 1.76 ± 0.13, 19.6 ± 1.6, 43.5 ± 2.2, given in Ref. 22.

For a complete calculation one has to use the gauge coupling constant and the fermion self-energy corrected by the renormalization-group analysis in the leading-logarithmic approximation. For massless quarks we find, in particular, that, when the leading-logarithmic expressions are used for both g and Σ, χSB does occur in QCD with three flavors for

$$\alpha_s=\frac{g^2(\mu)}{4\pi}>0.73\pi \tag{1.15}$$

with a $U(3)_V$ residual symmetry.[18]

For the massive case a preliminary presentation has been given in Ref. 23. We must evaluate the effective potential, a functional of the proper fermion self-energy Σ, in QCD with three flavors, in the general case in which both spontaneous and explicit breakdown of the chiral symmetry are present. The calculations are in the two-loop approximation, use is made of the Landau gauge, and the renormalization-group-improved expression for the gauge coupling is used. From the analysis of the asymptotical equations satisfied by the Green's functions we deduce the form of the test function to adopt for Σ. We assume a constant behavior in the infrared region of momentum and a decrease as $1/p^2$(logs) for $p>\mu$, consistent with the OPE analysis. By substituting in the effective action we find an expression which is ultraviolet finite. This fact is connected with the use of renormalization-group-improved expressions for Σ and g, which completely regularize the theory in the two-loop approximation considered. However a finite part of the effective potential remains again to be fixed through a suitable normalization condition. The natural choice comes from the expression of the effective potential for small masses and, in this limit, it is equivalent to the Adler-Dashen requirement. Our method then consists in making a convenient *Ansatz* for Σ in terms of a set of parameters related to the fermionic condensates and in minimizing the effective potential with respect to these parameters. We find that, similarly as for the massless case, the effective potential evaluated at the minimum decomposes into the sum of separate contributions, one for each flavor. We determine in this way the values of the condensates for the quarks u, d, and s at the minimum. They depend on the parameters of our model: the renormalization invariant mass $\Lambda_{\rm QCD}$, the three quark masses m_u, m_d, and m_s, and the further scale μ we have introduced in order to separate the infrared from the ultraviolet region of momenta. Our task is to determine these parameters from the experimental data. We can calculate the masses of the pseudoscalar mesons (pseudo-Goldstone bosons) and we can derive their decay coupling constants by following the treatment of Ref. 24.

Necessary ingredients are the normalization of our pseudoscalar dynamical variables and the expression for the vertex functions of the pseudoscalar composite fields. The decay constants are then evaluated from the couplings of the mesons $|\pi_j\rangle$ to the axial-vector currents $J^k_{\mu 5}(x)$ $(j,k=1,\ldots,8)$. (The mixing in the 3-8 sector has been explicitly taken into account.) We have in this way a system of coupled equations which allow to determine the parameters of the model by an iterative procedure. We obtain a very good fit for the meson masses (agreement within 3%) and for the decay coupling constants for the following values of the quark masses at 1 GeV:

$$m_u(1)=5.8\text{ MeV},\quad m_d(1)=8.4\text{ MeV}\,,\qquad m_s(1)=118\text{ MeV}\,. \tag{1.16}$$

These values agree with the values obtained by quite different methods, except for certain sum-rule estimates of $m_s(1)$ which give larger values.

We have tried to make this paper self-contained as much as possible, also at the expense of repeating some material which can be found in the references. In Sec. II we review some fundamental aspects of alternative forms of the effective potential for composite operators and we look at the nature of their stationary points. Then we derive our modified version for the effective action and discuss its properties. In Sec. III we evaluate the effective action for a QCD-like gauge theory of massive fermions. In Sec. IV we derive the ultraviolet behavior of the fermion self-energy in the general case in which both spontaneous and explicit breakdown of the chiral symmetry are present. We also make some comments in favor of the use of the so-called "regular solution" for the self-energy. In Sec. V we discuss the variational *Ansatz* for Σ, and we introduce a suitable renormalization condition for our functional. In Sec. VI, in order to better understand the pattern of the dynamical breakdown of the chiral symmetry, we analyze the properties of the massless effective potential. Section VII is devoted to the comparison with other studies of the dynamical-symmetry-breaking phenomenon. In Sec. VIII we discuss the massive case for QCD with three flavors. Then we calculate the masses and the decay coupling constants of the octet-pseudoscalar mesons in terms of the parameters of our model in Secs. IX and X, respectively. The numerical results are given in Sec. XI while in Sec. XII our discussion focuses on the comparison of the values we get for the quark masses with the values obtained by different methods. Conclusions are in Sec. XIII. In Appendix A we show explicitly the cancellation of the ultraviolet divergences in Γ, whereas Appendix B is devoted to the analysis of some properties of the extrema of the effective potential.

II. EFFECTIVE ACTION FOR COMPOSITE OPERATORS, REVIEW OF ALTERNATIVE FORMULATIONS

The main tool in the following analysis is a modified version of the effective action for composite operators introduced by Cornwall, Jackiw, and Tomboulis[13] (CJT). The physical interest in the study of the effective action and, more particularly in the study of the effective potential, is the fact that the minima of this functional determine the possible vacua of the theory. This is particularly relevant when one expects a nontrivial vacuum as in the case of spontaneously broken symmetries. When the breaking is due to the formation of condensates the technique of the effective action for composite operators turns out to be very useful. In fact it consists in a systematic resummation of graphs which is capable of describing nonperturbative phenomena in a sequence of approximations.

The more direct way to construct an effective action describing the interactions between the elementary degrees of freedom of the theory and the collective modes (composite fields) is to introduce an auxiliary field in the generating functional and to develop a loop expansion. This approach leads to the so-called "collective variables" or "auxiliary field" (AF) method (see, for example, Ref. 25). Unfortunately the AF technique suffers from severe limitations because it can be usefully applied only in the case of quartic interactions. A more general formalism to study dynamical symmetry breaking was introduced by Domokos and Suranyi,[26] and by Cornwall, Jackiw, and Tomboulis.[13] In this method, one introduces a "classical" bilocal field, which turns out to be the one-particle propagator of the theory, and defines a generalized effective action such that its variations with respect to the usual "classical" fields and to the bilocal fields reproduce the equations of motion of the theory and generate the Schwinger-Dyson (SD) equations for the propagators (gap equation). This seems to be the most efficient way at our disposal to discuss DSB.

We will review the AF and the CJT methods for a fermion gauge theory showing that, in the lowest approximation, these formulations are equivalent, in the sense that the stationary points in the two cases give the same dynamics. We will then introduce a modified version of the CJT functional having the same local extrema as the CJT one, but a different asymptotic behavior, and which is bounded from below.[17,18] The main advantage of the new form of the action is that the symmetry-breaking solutions of the gap equation correspond to minima. This can be seen from the explicit expression and from the general analysis performed by various authors.[27,28] This stability property is essential if we want to do something more than just finding the extrema of the effective potential and in particular if we want to perturb the vacuum to find its excitations.

Let us recapitulate here the salient points of the CJT formalism in the case of a fermion gauge theory in its Euclidean formulation. The technique consists in introducing a bilocal source $J(x,y)$ coupled to the operator $\bar\psi(x)\psi(y)$ in the generating functional $Z_{\rm CJT}[J]$:

$$Z_{\rm CJT}[J]=e^{-W_{\rm CJT}[J]}$$
$$=\frac{1}{\mathcal{N}}\int \mathcal{D}\psi\mathcal{D}\bar\psi\mathcal{D}A_\mu e^{-[I(\psi,\bar\psi,A_\mu)+\bar\psi J\psi]}\ , \qquad (2.1)$$

where $\mathcal{N}$ is a normalization constant (which will be omitted from now on), and $I(\psi,\bar\psi,A_\mu)$ is the classical Euclidean action for the gauge theory evaluated in the Landau gauge (see Sec. III). Clearly I also contains ghost terms, but, for the sake of simplicity, we have written down only the dependence on the fermion and gauge fields. Use is made of the shorthand notation

$$\bar\psi J\psi=\int d^4x\, d^4y\ \bar\psi_\alpha(x)J_{\alpha\beta}(x,y)\psi_\beta(y)\ , \qquad (2.2)$$

where α and β are collective indices for spinor, flavor, and color variables. Also we have not explicitly introduced in $Z_{\rm CJT}[J]$ the usual linear sources coupled to ψ and $\bar\psi$ because we are not interested in their effects. Let us define the "classical" bilocal field S

$$\frac{\delta W_{\rm CJT}}{\delta J}=-S \qquad (2.3)$$

and introduce the effective action $\Gamma_{\rm CJT}$ as the Legendre transform of the generating functional of the connected Green's functions $W_{CJT}=-\ln Z_{\rm CJT}$:

$$\Gamma_{\rm CJT}[S]=W_{\rm CJT}-\frac{\delta W_{\rm CJT}}{\delta J}J\ . \qquad (2.4)$$

It follows that

$$\frac{\delta\Gamma_{\rm CJT}}{\delta S}=J\ . \qquad (2.5)$$

The Legendre variable S conjugate to J will be the CJT dynamical variable. For physical processes ($J=0$), S has to satisfy the stationary condition for the effective action, $\delta\Gamma_{\rm CJT}/\delta S=0$. We will show that this is nothing but the Schwinger-Dyson equation for the fermion propagator, and so S coincides with the exact fermion propagator when the source J is turned off.

If one is interested only in translationally invariant (TI) solutions of the SD equation, one can take the composite field S to be a function of the space-time difference $(x-y)$. In this way an overall space-time volume term factorizes out and the effective potential for composite operators $V_{\rm CJT}$ may be defined as

$$V_{\rm CJT}[S]\Omega=\Gamma_{\rm CJT}[S]\,|_{\rm TI},\quad \Omega=\int d^4x\ , \qquad (2.6)$$

which, in the CJT formulation, is equivalent to consider the generating functional for zero-momentum two-particle-irreducible Green's functions, expressed in terms of the propagator S.

$\Gamma_{\rm CJT}$ (and equivalently $V_{\rm CJT}$) can be expressed in the Euclidean space as the following formal series:

$$\Gamma_{\rm CJT}(S)=-{\rm Tr}\ln S^{-1}-{\rm Tr}(S_0^{-1}S)-\Gamma_2(S)\ , \qquad (2.7)$$

where S_0 is the free fermion propagator and $\Gamma_2(S)$ is the sum of all the two-particle-irreducible vacuum diagrams of the theory evaluated with fermionic propagator equal to S. By inserting (2.7) in (2.5) one gets the following expression for J:

$$J=\frac{\delta\Gamma_{\rm CJT}}{\delta S}=S^{-1}-S_0^{-1}-\frac{\delta\Gamma_2}{\delta S}\ . \qquad (2.8)$$

From Eq. (2.8) the SD equation follows by setting $J=0$. It is clear that

$$\Sigma=-\frac{\delta\Gamma_2}{\delta S} \qquad (2.9)$$

represents the fermion self-energy when the source is turned off. We will evaluate Γ_2 at the lowest order, that is, at the two-loop level, corresponding to a single-gluon exchange (Fig. 1). This approximation will be improved by taking into account renormalization-group effects, namely, by using the running coupling constant at the vertices (see Sec. III). The important fact is that Γ_2 in this approximation is a quadratic expression in S. In fact $(\delta^2\Gamma_2/\delta S^2)$ is nothing but the gluon propagator plus possible corrections not explicitly involving fermions. For this reason it follows (the trace operation is understood) that

$$\Gamma_2=\frac{1}{2}S\frac{\delta^2\Gamma_2}{\delta S^2}S=\frac{1}{2}S\frac{\delta\Gamma_2}{\delta S} \qquad (2.10)$$

or

$$\frac{\delta^2\Gamma_2}{\delta S^2}S=\frac{\delta\Gamma_2}{\delta S}\ . \qquad (2.11)$$

It is easy to see that the approximation we are using for Γ_2 corresponds to formally integrating over the gluon fields in the generating functional Z and then to expand up to the fourth order in the fermionic fields. In this way one obtains an effective four-fermion interaction and, instead of using $\Gamma_{\rm CJT}$, one can use the effective action expressed in terms of a collective variable, the auxiliary field Φ related to the bilocal fermion-antifermion composite field. The strength of the effective four-fermion interaction, within the specified approximation, is given by $(\delta^2\Gamma_2/\delta S^2)$.

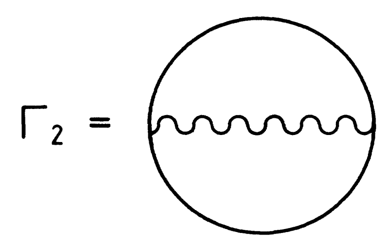

FIG. 1. Γ_2 in the lowest-order approximation for a vector gauge theory. The solid line represents a fermion propagator equal to S.

One makes use of the functional identity

$$\int \mathcal{D}\Phi \exp[-\tfrac{1}{2}(\Phi-\psi\bar{\psi})D(\Phi-\psi\bar{\psi})]=\text{const}\ , \tag{2.12}$$

where D is an arbitrary operator. In order to eliminate the quadrifermionic term, we will choose $D=\delta^2\Gamma_2/\delta S^2$. One then defines a functional $Z_{\rm AF}$ depending on a bilocal source $J(x,y)$ which is now coupled to the auxiliary field Φ:

$$Z_{\rm AF}[J]=e^{-W_{\rm AF}[J]}=\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\Phi \exp\left[-\left[I_{\rm eff}(\psi,\bar{\psi})+\tfrac{1}{2}(\Phi-\psi\bar{\psi})\frac{\delta^2\Gamma_2}{\delta S^2}(\Phi-\psi\bar{\psi})-J\Phi\right]\right]\ , \tag{2.13}$$

where $I_{\rm eff}(\psi,\bar{\psi})$ results from the integration over the gauge fields and from the expansion up to the fourth order in the fermionic fields. From the equations of motions for the auxiliary field Φ one gets

$$\Phi_{\alpha\beta}(x,y)=\psi_\alpha(x)\bar{\psi}_\beta(y)\ . \tag{2.14}$$

In this sense Φ can be identified with the operator $\psi\bar{\psi}$.

In order to build up an effective action relative to the auxiliary composite field, let us integrate on the fermion fields

$$Z_{\rm AF}[J]=\int \mathcal{D}\Phi \exp\left\{-\left[-{\rm Tr}\ln\left[S_0^{-1}+\frac{\delta^2\Gamma_2}{\delta S^2}\Phi\right]+\frac{1}{2}\Phi\frac{\delta^2\Gamma_2}{\delta S^2}\Phi-J\Phi\right]\right\}\ . \tag{2.15}$$

If we make a stationary-phase approximation, i.e., a tree approximation in the Φ field, we get

$$W_{\rm AF}[J]=-{\rm Tr}\ln\left[S_0^{-1}+\frac{\delta^2\Gamma_2}{\delta S^2}\Phi_0\right]+\frac{1}{2}\Phi_0\frac{\delta^2\Gamma_2}{\delta S^2}\Phi_0-J\Phi_0\ , \tag{2.16}$$

where Φ_0 is the solution of the classical equation of motion. One can Legendre transform $W_{\rm AF}[J]$ in the usual way by defining

$$\frac{\delta W_{\rm AF}}{\delta J}=-\Phi_c \tag{2.17}$$

and

$$\Gamma_{\rm AF}[\Phi_c]=W_{\rm AF}-\frac{\delta W_{\rm AF}}{\delta J}J \tag{2.18}$$

from which it follows

$$\frac{\delta\Gamma_{\rm AF}}{\delta\Phi_c}=J\ . \tag{2.19}$$

In the lowest-order approximation we can choose $\Phi_c=\Phi_0$. By substituting Eq. (2.16) in Eq. (2.18) we obtain the auxiliary field effective action in the tree approximation for the composite field Φ_c (Ref. 27):

$$\Gamma_{\rm AF}[\Phi_c]=-{\rm Tr}\ln\left[S_0^{-1}+\frac{\delta^2\Gamma_2}{\delta S^2}\Phi_c\right]+\frac{1}{2}\Phi_c\frac{\delta^2\Gamma_2}{\delta S^2}\Phi_c\ . \tag{2.20}$$

Then from Eq. (2.19) one gets

$$J=\left[-\left[S_0^{-1}+\frac{\delta^2\Gamma_2}{\delta S^2}\Phi_c\right]^{-1}+\Phi_c\right]\frac{\delta^2\Gamma_2}{\delta S^2} \tag{2.21}$$

which gives Φ_c as a functional of J.

Let us look at Eqs. (2.8) and (2.21). They represent the stationary conditions for $\Gamma_{\rm CJT}$ and $\Gamma_{\rm AF}$, respectively. Switching off the source J and using Eq. (2.11) in (2.8) they, respectively, read as

$$S^{-1}=S_0^{-1}+\frac{\delta^2\Gamma_2}{\delta S^2}S\ , \tag{2.22}$$

$$\Phi_c^{-1}=S_0^{-1}+\frac{\delta^2\Gamma_2}{\delta S^2}\Phi_c\ . \tag{2.23}$$

This means that the CJT and AF formulations are equivalent in the lowest approximation [which is the tree approximation for the auxiliary field and the lowest nontrivial (two-loop) order in the CJT formalism], in the sense that S and Φ_c satisfy the same gap equation at the physical point and so describe the same dynamics. However, the functional forms of $\Gamma_{\rm CJT}$ and $\Gamma_{\rm AF}$ are different, and the effective actions for these two cases do differ outside the stationary points.

We can easily show that the difference between $W_{\rm CJT}$ and $W_{\rm AF}$ is a quadratic term in $J(x,y)$ due to the fact that the source in the generating functional is coupled to $\bar{\psi}(x)\psi(y)$ in the first case, and to $\Phi(x,y)$ in the second case. If we start from Eq. (2.1) and use the same functional trick (2.12), we obtain, after integrating over the fermion fields,

$$e^{-W_{\rm CJT}}=\int \mathcal{D}\Phi \exp\left\{-\left[-{\rm Tr}\ln\left[S_0^{-1}+\frac{\delta^2\Gamma_2}{\delta S^2}\Phi+J\right]+\frac{1}{2}\Phi\frac{\delta^2\Gamma_2}{\delta S^2}\Phi\right]\right\}\ . \tag{2.24}$$

If we use the invariance of the volume element under translations, we can change the integration variable

$$\Phi\to\Phi+\left[\frac{\delta^2\Gamma_2}{\delta S^2}\right]^{-1}J \tag{2.25}$$

and get

$$e^{-W_{\rm CJT}}=\int \mathcal{D}\Phi \exp\left\{-\left[-{\rm Tr}\ln\left[S_0^{-1}+\frac{\delta^2\Gamma_2}{\delta S^2}\Phi\right]+\frac{1}{2}\Phi\frac{\delta^2\Gamma_2}{\delta S^2}\Phi+\frac{1}{2}J\left[\frac{\delta^2\Gamma_2}{\delta S^2}\right]^{-1}\times J-J\Phi\right]\right\}, \tag{2.26}$$

that is,

$$W_{\rm CJT}=W_{\rm AF}+\frac{1}{2}J\left[\frac{\delta^2\Gamma_2}{\delta S^2}\right]^{-1}J\ . \tag{2.27}$$

So, the whole effect of introducing the auxiliary field Φ and coupling a source to it is to add to $W_{\rm CJT}$ a term with a quadratic J dependence.

We will see that this term is responsible for changing the stability properties of the effective potentials in the two formulations. In fact, let us introduce in the standard way the "auxiliary field" effective potential

$$V_{\rm AF}[\Phi_c]\Omega=\Gamma_{\rm AF}[\Phi_c]\,|_{\rm TI},\quad \Omega=\int d^4x\ . \tag{2.28}$$

It is clear that, in the AF formalism, the second derivative of $V_{\rm AF}[\Phi_c]$ can be interpreted as the mass of the Φ_c field. So, its positivity is a necessary condition for the validity of the composite field loop expansion. We will show that the auxiliary field effective potential does have a local minimum corresponding to the lowest vacuum of the theory. On the other hand, even in the free field case, $V_{\rm CJT}$ turns out to be unbound from below (see Refs. 14 and 15). The absence of a lower bound and the related saddle-point behavior for the solutions of the gap equation of $V_{\rm CJT}$ is an intrinsic defect of the CJT formulation. Haymaker, Matsuki, and Cooper[16,27] have shown that, in the case of a SU(N) fermion gauge theory, under certain physical conditions imposed on the solution of the gap equation, the lowest-energy stationary point is a saddle point for $V_{\rm CJT}$ while it is a local minimum for $V_{\rm AF}$.

To determine whether a solution of the gap equation is a local minimum, a saddle point, or a maximum, we need to solve an eigenvalue equation for the curvature operator, which is defined by expanding the effective potential around a solution $\bar{\Sigma}$ of the gap equation:

$$V(\bar{\Sigma}+\delta\Sigma)=V(\bar{\Sigma})+\tfrac{1}{2}\int dp\,dq\,\delta\Sigma(p^2)\times\left.\frac{\delta^2 V}{\delta\Sigma(p^2)\delta\Sigma(q^2)}\right|_{\bar{\Sigma}}\delta\Sigma(q^2)+\cdots\ . \tag{2.29}$$

It is possible to show that in the CJT case, the second term in (2.29) can be negative or positive by choosing appropriate variations $\delta\Sigma$, i.e., the solution is a saddle point, and also that the same solution of the gap equation which is a saddle point of the CJT effective potential is instead a local minimum of the AF effective potential.[27]

Taking into account these properties of the CJT and the AF effective potentials, we can introduce a further functional which is a modification of the CJT one, but which does not suffer from the problem of unboundness from below. In particular our effective action will be as general as the CJT action (not being restricted to the case of four-linear interactions), will have the same stationary points as $\Gamma_{\rm CJT}$ and $\Gamma_{\rm AF}$, and it will have the same functional form and therefore the same asymptotic behavior as the AF functional whenever applicable. In this way it will be clear that the instability due to the presence of saddle points is an artifact of the particular choice of the effective potential and that it disappears when one chooses an alternative but physically equivalent form.

We have shown that $W_{\rm AF}$ can be obtained by adding a source-dependent term to $W_{\rm CJT}$. Let us change the definition of the source $J(x,y)$,

$$J=\frac{\delta^2\Gamma_2}{\delta S^2}L\ . \tag{2.30}$$

Then Eq. (2.27) can be rewritten as

$$W_{\rm AF}=W_{\rm CJT}-\Gamma_2(S+L)+\Gamma_2(S)+\frac{\delta\Gamma_2}{\delta S}L\ , \tag{2.31}$$

where we have used the property of $\Gamma_2(S)$ of being a quadratic functional of S [Eqs. (2.10) and (2.11)]. Let us consider the explicit expression for $W_{\rm CJT}$ in terms of $\Gamma_{\rm CJT}$:

$$\begin{aligned}W_{\rm CJT}&=\Gamma_{\rm CJT}-JS\\&=\Gamma_{\rm CJT}-\frac{\delta\Gamma_2}{\delta S}L\\&=-{\rm Tr}\ln(S^{-1})-{\rm Tr}(S_0^{-1}S)-\Gamma_2(S)-\frac{\delta\Gamma_2}{\delta S}L\ ,\end{aligned} \tag{2.32}$$

where we have used the formal series representation (2.7) for $\Gamma_{\rm CJT}$. Substituting Eq. (2.32) in (2.31) one gets

$$W_{\rm AF}[L]=-{\rm Tr}\ln(S^{-1})-{\rm Tr}(S_0^{-1}S)-\Gamma_2(S+L) \tag{2.33}$$

and

$$\frac{\delta W_{\rm AF}}{\delta L}=\left[S^{-1}-S_0^{-1}-\left.\frac{\delta\Gamma_2}{\delta S}\right|_{S+L}\right]\frac{\delta S}{\delta L}-\left.\frac{\delta\Gamma_2}{\delta S}\right|_{S+L}\ . \tag{2.34}$$

Let us insert (2.30) into the gap equation (2.8),

$$\frac{\delta^2\Gamma_2}{\delta S^2}L=S_0^{-1}-S^{-1}-\frac{\delta\Gamma_2}{\delta S}\ . \tag{2.35}$$

Then, recalling that Γ_2 is a quadratic functional, that is,

$$(S+L)\frac{\delta^2\Gamma_2}{\delta S^2}=\left.\frac{\delta\Gamma_2}{\delta S}\right|_{S+L}\ , \tag{2.36}$$

it follows that

$$S_0^{-1}=S^{-1}+\left.\frac{\delta\Gamma_2}{\delta S}\right|_{S+L} . \tag{2.37}$$

So, by substituting (2.37) in (2.36), we obtain

$$\frac{\delta W_{\rm AF}}{\delta L}=-\left.\frac{\delta\Gamma_2}{\delta S}\right|_{S+L}=\mathbf{\Sigma}\,|_{S+L}\equiv\tilde{\mathbf{\Sigma}} , \tag{2.38}$$

which means that the variable conjugate to the source L in the AF formalism turns out to be $\tilde{\mathbf{\Sigma}}$. Finally, let us perform the Legendre transform of $W_{\rm AF}[L]$ with respect to L in order to get the "auxiliary field" effective action as a functional of $\tilde{\mathbf{\Sigma}}=-(\delta\Gamma_2/\delta S)\,|_{S+L}$:

$$\begin{aligned}\Gamma_{\rm AF}[\tilde{\mathbf{\Sigma}}]&=W_{\rm AF}-\frac{\delta W_{\rm AF}}{\delta L}L\\ &=-\mathrm{Tr}\ln\left[S_0^{-1}+\left.\frac{\delta\Gamma_2}{\delta S}\right|_{S+L}\right]\\ &\quad+\mathrm{Tr}\left[\left.\frac{\delta\Gamma_2}{\delta S}\right|_{S+L}(S+L)\right]-\Gamma_2(S+L) ,\end{aligned} \tag{2.39}$$

where again Eq. (2.37) has been used. The different functional form of $\Gamma_{\rm CJT}$ and $\Gamma_{\rm AF}$ is due to the use of different sources, J and L, respectively. This means that the two effective actions describe the dynamics of the different composite fields S and $\tilde{\mathbf{\Sigma}}$ to which the sources are linearly attached.

A third alternative for the source term gives our result. Since we want $\mathbf{\Sigma}$ to be our dynamical variable (and not $\tilde{\mathbf{\Sigma}}$), it is quite natural to define a new action by simply redefining S as $(S+L)$ in (2.39):

$$\Gamma[\mathbf{\Sigma}]=-\mathrm{Tr}\ln\left[S_0^{-1}+\frac{\delta\Gamma_2}{\delta S}\right]+\mathrm{Tr}\left[\frac{\delta\Gamma_2}{\delta S}S\right]-\Gamma_2(S) . \tag{2.40}$$

This effective action was proposed in Ref. 17. It is clear from the derivation that $\Gamma_{\rm AF}[\tilde{\mathbf{\Sigma}}]$ and $\Gamma[\mathbf{\Sigma}]$ have the same functional form. This means that the second derivatives of the two effective potentials with respect to their respective variables, evaluated with sources turned off, are equal ($\tilde{\mathbf{\Sigma}}=\mathbf{\Sigma}$ at the physical point). This makes sure that our potential has local minima as stationary points. There is now a general proof of this property,[28] and also our analytical and numerical calculations confirm the validity of the statement.

Let us now derive the relation between our functional Γ and $\Gamma_{\rm CJT}$. By using Eq. (2.8) in (2.7) one obtains

$$\begin{aligned}\Gamma_{\rm CJT}[S]&=-\mathrm{Tr}\ln\left[S_0^{-1}+\frac{\delta\Gamma_2}{\delta S}+J\right]+\mathrm{Tr}\left[\frac{\delta\Gamma_2}{\delta S}+J\right]S-\Gamma_2(S)\\ &=-\mathrm{Tr}\ln\left[S_0^{-1}+\frac{\delta\Gamma_2}{\delta S}\right]-\mathrm{Tr}\ln\left[1+\left[S_0^{-1}+\frac{\delta\Gamma_2}{\delta S}\right]^{-1}J\right]+\mathrm{Tr}\left[\frac{\delta\Gamma_2}{\delta S}S\right]+\mathrm{Tr}(JS)-\Gamma_2(S) .\end{aligned} \tag{2.41}$$

Then, if we use again (2.8), we may write

$$\begin{aligned}\mathrm{Tr}\ln\left[1+\left[S_0^{-1}+\frac{\delta\Gamma_2}{\delta S}\right]^{-1}J\right]&=\mathrm{Tr}\ln[1+(S^{-1}-J)^{-1}J]\\ &=\mathrm{Tr}\ln(1-SJ)^{-1} .\end{aligned} \tag{2.42}$$

Let us now insert (2.42) in (2.41) and compare with (2.40):

$$\Gamma_{\rm CJT}=\Gamma+\mathrm{Tr}\ln(1-SJ)+\mathrm{Tr}(JS) . \tag{2.43}$$

From Eq. (2.43) the following relations follow:

$$\Gamma_{\rm CJT}\,|_{J=0}=\Gamma\,|_{J=0} , \tag{2.44}$$

$$\left.\frac{\delta\Gamma_{\rm CJT}}{\delta S}\right|_{J=0}=\left.\frac{\delta\Gamma}{\delta S}\right|_{J=0} . \tag{2.45}$$

However,

$$\left.\frac{\delta^2\Gamma_{\rm CJT}}{\delta S^2}\right|_{J=0}\neq\left.\frac{\delta^2\Gamma}{\delta S^2}\right|_{J=0} . \tag{2.46}$$

Furthermore, let us perform the functional derivative of Γ as given in (2.40) with respect to $\mathbf{\Sigma}$ ($\mathbf{\Sigma}=-\delta\Gamma_2/\delta S$):

$$\frac{\delta\Gamma}{\delta\mathbf{\Sigma}}=(S_0^{-1}-\mathbf{\Sigma})^{-1}-S-\mathbf{\Sigma}\frac{\delta S}{\delta\mathbf{\Sigma}}-\frac{\delta\Gamma_2}{\delta S}\frac{\delta S}{\delta\mathbf{\Sigma}} . \tag{2.47}$$

Then, the stationary condition $\delta\Gamma/\delta\mathbf{\Sigma}=0$ leads to the correct SD equation

$$S^{-1}=S_0^{-1}+\mathbf{\Sigma} . \tag{2.48}$$

Another good reason to use Γ instead of $\Gamma_{\rm CJT}$ is related to Eq. (2.38):

$$\begin{aligned}\frac{\delta W_{\rm AF}}{\delta L}&=-\left.\frac{\delta\Gamma_2}{\delta S}\right|_{S+L}=\frac{\delta W_{\rm AF}}{\delta J}\frac{\delta J}{\delta L}\\ &=-\Phi_c\frac{\delta^2\Gamma_2}{\delta S^2}=\mathbf{\Sigma}\,|_{S+L}\end{aligned} \tag{2.49}$$

which shows the simple relation between the self-energy and the vacuum expectation value of the composite field Φ ($\Phi_c=\langle\Phi\rangle$). It follows that a series expansion of the effective action in $\mathbf{\Sigma}=-\delta\Gamma_2/\delta S$ gives essentially the one-particle-irreducible (1PI) Green's functions relative to the field Φ, while a series expansion in S, as in the CJT case, does not have such a direct physical interpretation. In other words, $\mathbf{\Sigma}$ describes the physical excitations of the theory around the vacuum, whereas an analogous situation does not hold in the CJT formulation due to the fact that $(\delta^2 V_{\rm CJT}/\delta\Sigma^2)$ is not positive definite.

III. THE EFFECTIVE ACTION IN QCD-LIKE GAUGE THEORIES

Let us evaluate the effective action for an SU(N) QCD-like gauge theory within our modification of the CJT functional formalism, in the realistic situation when spontaneous and explicit breakdown of the global chiral symmetry take place. The calculations are for $\theta=0$ (θ is the parameter connected with the axial anomaly).

The classical Euclidean Lagrangian density of the strong interaction of the fermions Ψ, mediated by a set of vector gluons A_μ which are the gauge bosons of the symmetry group SU(N), is (here and in the following we will use boldface characters for matrices in flavor space which are not proportional to the identity matrix)

$$\mathcal{L}=\bar{\Psi}\mathbf{S}_0^{-1}\Psi-ig\bar{\Psi}\hat{A}\Psi+\text{gauge terms}$$
$$+\text{ghost terms}+\text{gauge fixing}\ , \qquad (3.1)$$

where Ψ are n multiplets of SU(N), each of them assigned to the fundamental representation of the gauge group, and $\mathbf{S}_0$ is the free fermion propagator which, in a theory renormalizated at the point $p^2=\mu^2$, has the expression

$$\mathbf{S}_0(p)=\{Z_\Psi(\mu,\Lambda)[i\hat{p}-\mathbf{m}_0(\Lambda)]\}^{-1}\ . \qquad (3.2)$$

Here Λ is an ultraviolet cutoff, $Z_\Psi(\mu,\Lambda)$ is the renormalization constant for the fermion propagator, and

$$\mathbf{m}_0(\Lambda)=\mathbf{m}(\mu)-\delta\mathbf{m}(\mu,\Lambda)\ , \qquad (3.3)$$

where $\mathbf{m}(\mu)$ is the $n\times n$ renormalized mass matrix which is responsible for the explicit breakdown of the chiral symmetry and $\delta\mathbf{m}(\mu,\Lambda)$ is the mass counterterm. In fact we can write the Lagrangian in Eq. (3.1) as a sum of two contributions:

$$\mathcal{L}=\mathcal{L}_0-Z_\Psi(\mu,\Lambda)\bar{\Psi}\mathbf{m}_0(\Lambda)\Psi\ , \qquad (3.4)$$

where $\mathcal{L}_0$ is invariant under the transformations of the flavor group $\text{U}(n)_L\otimes\text{U}(n)_R$ [more precisely it is invariant under the global chiral $\text{SU}(n)_L\otimes\text{SU}(n)_R$ and the $\text{U}(1)_{L+R}$ groups, since the divergence of the singlet axial-vector current connected with the $\text{U}(1)_{L-R}$ group is nonzero even in the chiral limit, due to the axial anomaly].

The expression to be evaluated is [see Eq. (2.40)]

$$\Gamma[\mathbf{\Sigma}]=-\text{Tr}\ln\left[\mathbf{S}_0^{-1}+\frac{\delta\Gamma_2}{\delta\mathbf{S}}\right]+\text{Tr}\left[\frac{\delta\Gamma_2}{\delta\mathbf{S}}\mathbf{S}\right]-\Gamma_2(\mathbf{S}) \qquad (3.5)$$

with

$$\mathbf{\Sigma}=-\frac{\delta\Gamma_2}{\delta\mathbf{S}}\ . \qquad (3.6)$$

In Eq. (3.5) $\mathbf{S}$ is the full fermion propagator which, at the physical point, satisfies the gap equation

$$\mathbf{S}^{-1}(p)=\mathbf{S}_0^{-1}(p)-\mathbf{\Sigma}(p) \qquad (3.7)$$

with $\mathbf{\Sigma}$ equal to the fermion self-energy function.

We will show that, in the chiral limit, the theory possesses two phases (the chiral phase and the phase broken into the diagonal subgroup) and, in particular, that spontaneous symmetry breaking occurs when the coupling constant g exceeds some critical value. This spontaneous symmetry breaking is accompanied by n^2-1 composite Goldstone bosons which are each associated with an unbroken generator of the coset space $\text{SU}(n)_L\otimes\text{SU}(n)_R/\text{SU}(n)_{L+R}$.

Actually the Lagrangian in Eq. (3.1) is not chirally invariant because of the quark mass term. However in the dynamically broken phase one has dynamical generation of fermionic masses due to the formation of quark-antiquark condensates. For the sake of simplicity, we will keep on calling this phenomenon spontaneous χSB even if, clearly, this term is no longer entirely appropriate. In this case the particle spectrum contains pseudo-Goldstone bosons which have acquired a mass induced by the explicit chiral-symmetry breaking.

We assume that the main contribution to the effective action for the spontaneous chiral-symmetry-breaking phenomenon comes from short-distance effects. For this reason we will introduce an infrared cutoff for the confinement region and we will mainly focus on the short-distance dynamics. In this range it is sensible to perform a loop expansion of the effective action. In fact for large momenta, in virtue of the asymptotic freedom of the gauge theory, one can neglect the multiloop contributions and evaluate Γ_2 to the lowest order, given by the graph in Fig. 1. Here one has to decide on the form of the vertex and of the gauge field propagator. The renormalization-group analysis and the asymptotic freedom suggest us to use the free expression for the vertex and gauge field propagator improved by the running coupling constant (see the next section). But, as far as the vertex is concerned, the situation is more subtle because, in principle, one can run in some difficulties in order to satisfy the Ward identities. Let us examine this point.

We can express the inverse of the full fermion propagator in the following general form:

$$\mathbf{S}^{-1}(p)=i\mathbf{Z}(p^2)\hat{p}-\mathbf{\Sigma}'(p^2)\ . \qquad (3.8)$$

Then the Ward identity for the vertex function reads

$$(q_1-q_2)^\mu\mathbf{\Gamma}_\mu=\mathbf{S}^{-1}(q_1)-\mathbf{S}^{-1}(q_2)$$
$$=i\mathbf{Z}(q_1^2)\hat{q}_1-i\mathbf{Z}(q_2^2)\hat{q}_2-\mathbf{\Sigma}'(q_1^2)+\mathbf{\Sigma}'(q_2^2)\ . \qquad (3.9)$$

This equation can be satisfied by taking

$$\mathbf{\Gamma}_\mu=i\gamma_\mu+\frac{(q_1-q_2)_\mu}{(q_1-q_2)^2}\{i[\mathbf{Z}(q_1^2)-1]\hat{q}_1-i[\mathbf{Z}(q_2^2)-1]\hat{q}_2$$
$$-\mathbf{\Sigma}'(q_1^2)+\mathbf{\Sigma}'(q_2^2)\}\ . \qquad (3.10)$$

However, in the evaluation of Γ_2, $\mathbf{\Gamma}_\mu$ is always saturated with the gauge field propagator. Therefore, if we adopt the Landau gauge, the gauge field propagator is transverse and we can safely use the free expression for the vertex. In other gauges the corrections to the free vertex are needed in order to satisfy the Ward identity (3.9). For this reason we shall prefer the use of the Landau gauge. Also, as we shall see, there will be some other simplifications in this gauge. For example, the wave-function renormalization constant $Z_\Psi(\mu,\Lambda)$ in the Landau gauge is equal to one in the approximation we are considering. We remark that by itself, the phenomenon of spontaneous chiral-symmetry breaking is gauge invariant since the chiral group currents are singlets with respect to the gauge group. As stated before we will construct the effective action as a functional of $\mathbf{\Sigma}$ and we will describe χSB with the help of this function which, at the physical point, represents the dynamical fermion mass. In the next section we will derive a relation between the scalar part of the fermion self-energy and the condensate $\langle \bar{\Psi}\Psi \rangle_\mu$ renormalized at the point μ. So if one chooses the vacuum expectation value $\langle \bar{\Psi}\Psi \rangle_\mu$, which is a gauge-invariant quantity, as the order parameter, it is reasonable that all the results one finds in such a picture are gauge invariant. There is also a recent demonstration that chiral-symmetry breaking occurs in vectorlike gauge theories in such a way that the critical coupling constant and dynamical mass function are gauge independent, at least at the leading order.[29] This fact ensures the gauge independence of our results which are derived in the Landau gauge.

The expression for Γ_2 is then

$$\Gamma_2=-\frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}[ig(p,q)]^2 \times \mathrm{Tr}[\mathbf{S}(p)T^a\gamma^\mu \mathbf{S}(q)T^a\gamma^\nu]D_{\mu\nu}(p-q) \times \int d^4x \ , \tag{3.11}$$

where T^a, $a=1,\ldots,N^2-1$ are the Hermitian generators of the gauge group in the fundamental representation, $g(p,q)$ is the running coupling constant, and

$$D_{\mu\nu}(k)=\frac{1}{k^2}\left[g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right] . \tag{3.12}$$

In the region of momenta larger than the renormalization-group-invariant scale of the theory M_0 (in QCD $M_0=\Lambda_{\mathrm{QCD}}$) we will assume, in the leading-logarithmic approximation, the following form for the function $g^2(p,q)$ (Ref. 30):

$$g^2(p,q)=\Theta(p^2-q^2)g^2(p)+\Theta(q^2-p^2)g^2(q) . \tag{3.13}$$

However, we know that the running coupling constant $g^2(p)$ becomes singular for $p^2=M_0^2$, a singularity due to the use of perturbation theory in a region where the coupling becomes strong. Unfortunately in Eq. (3.11) one has to integrate upon all the range of momenta and consequently one has to make an *Ansatz* for the coupling constant in the infrared region. Since the attitude we take here is that the spontaneous chiral-symmetry breaking is dominated by short-distance effects, we will substitute the infrared behavior of $g^2(p)$ with a constant value by introducing a mass scale μ characterizing the separation between the large- and the small-distance regions. On the other hand, for values of $p^2>\mu^2$ we will assume the standard renormalization-group expression for $g^2(p)$ which provides an effective cutoff of the interaction at small distances. So, the expression we will use for the running coupling constant in the leading-logarithmic approximation is

$$g^2(p)=2b\left[\Theta(\mu^2-p^2)\frac{1}{\ln(\mu^2/M_0^2)}+\Theta(p^2-\mu^2)\frac{1}{\ln(p^2/M_0^2)}\right] \tag{3.14}$$

with $b=24\pi^2/(11N-2n)$. In this way the expression (3.11) for Γ_2 is not merely the ladder approximation consisting of a single gauge field exchange but, with the insertion of the running coupling constant, it takes automatically into account the vertex perturbative corrections at least in the leading-logarithmic approximation.

A further observation is in order. We have chosen the point at which we renormalize the theory to be coincident with the scale μ separating the infrared and the ultraviolet region. As will be clear in the next section, this fact leads to some simplifications, for example, in the relation between the value of the minimum of the effective potential and the corresponding value of the fermion-antifermion condensate. The constant μ is a parameter of our model.

In order to evaluate Γ_2, let us parametrize the fermion propagator in the following way:

$$[\mathbf{S}(p)]_{Aa}^{Bb}=\delta_A^B\mathbf{S}(p)_a^b =\delta_A^B[i\,\mathbf{A}(p^2)_a^b\hat{p}+\mathbf{B}(p^2)_a^b+i\gamma_5\mathbf{C}(p^2)_a^b] \tag{3.15}$$

with $A,B=1,\ldots,N$, $a,b=1,\ldots,n$. Notice that, from the assumption of a fermion propagator $\mathbf{S}$ which is a function only of the space-time differences, the translational invariance of the effective action follows and, as a consequence, the space-time volume $\Omega=\int d^4x$ factorizes out in Γ_2 [see Eq. (3.11)]. Let us substitute the parametrization (3.15) in (3.11) and evaluate the trace over the color and the spinor indices

$$\Gamma_2= 6NC_2\Omega\int\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{g^2(p,q)}{(p-q)^2}\mathrm{tr}[\mathbf{B}(p^2)\mathbf{B}(q^2)+\mathbf{C}(p^2)\mathbf{C}(q^2)] -2NC_2\Omega\int\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{g^2(p,q)}{(p-q)^2}\mathrm{tr}[\,\mathbf{A}(p^2)\,\mathbf{A}(q^2)]E(p,q) , \tag{3.16}$$

where $C_2=(N^2-1)/2N=\sum_a T^aT^a$, $a=1,\ldots,N^2-1$, is the quadratic Casimir of the fermion representation, the trace is over the flavor indices, and

$$E(p,q)=1-\frac{1}{2}\left[(p^2+q^2)+\frac{(p^2-q^2)^2}{(p-q)^2}\right]\frac{1}{(p-q)^2}\ . \tag{3.17}$$

The expression for $g^2(p,q)$ we use does not depend on the angle between p and q [see Eqs. (3.13) and (3.14)]. Therefore one can perform the angular integration in (3.16) by the help of the following formulas:

$$\int d\Omega\frac{1}{(p-q)^2}=\frac{2\pi^2}{pq}e^{-|\ln q/p|}\ , \tag{3.18}$$

$$\int d\Omega\frac{1}{(p-q)^4}=\frac{2\pi^2}{pq}\frac{e^{-|\ln q/p|}}{|p^2-q^2|}\ . \tag{3.19}$$

The result is

$$\int d\Omega\, E(p,q)=0 \tag{3.20}$$

and so there is no contribution in Γ_2 from the matrix $\mathbf{A}$ defined in (3.15). This is obviously due to the nonrenormalization of the wave function in the Landau gauge at this order. We are left with a dependence of Γ_2 only on the matrices $\mathbf{B}$ and $\mathbf{C}$.

Remember that, as pointed out in the previous section, our task is to express Γ_2 as a functional of $\mathbf{\Sigma}=-\delta\Gamma_2/\delta\mathbf{S}$ which, when the Schwinger-Dyson equation is satisfied, is nothing but the fermion self-energy. In order to do that, let us separate scalar from pseudoscalar contributions by defining

$$\mathbf{\Sigma}(p^2)=\mathbf{\Sigma}_s(p^2)+i\gamma_5\mathbf{\Sigma}_p(p^2)\ . \tag{3.21}$$

Then, performing the functional derivative of Γ_2, given in (3.11), with respect to $\mathbf{S}(q^2)$, and using the parametrization (3.15), we obtain

$$\mathbf{\Sigma}_s(q^2)=-3C_2\int\frac{d^4p}{(2\pi)^4}\mathbf{B}(p^2)\frac{g^2(p,q)}{(p-q)^2}\ , \tag{3.22}$$

$$\mathbf{\Sigma}_p(q^2)=3C_2\int\frac{d^4p}{(2\pi)^4}\mathbf{C}(p^2)\frac{g^2(p,q)}{(p-q)^2}\ . \tag{3.23}$$

Here $\mathbf{\Sigma}_s$ and $\mathbf{\Sigma}_p$ are matrices in the flavor space. After inserting (3.13), we perform the angular integration in (3.22):

$$\mathbf{\Sigma}_s(q^2)=-\frac{3C_2}{16\pi^2}\left[\frac{g^2(q)}{q^2}\int_0^{q^2}dp^2p^2\mathbf{B}(p^2)+\int_{q^2}^{\infty}dp^2\mathbf{B}(p^2)g^2(p)\right]\ . \tag{3.24}$$

We can invert the relation between $\mathbf{\Sigma}_s$ and $\mathbf{B}$ by applying an appropriate differential operator to both sides of (3.24). In particular, differentiating with respect to q^2,

$$\frac{d}{dq^2}\mathbf{\Sigma}_s(q^2)=-\frac{3C_2}{16\pi^2}\frac{d}{dq^2}\left[\frac{g^2(q)}{q^2}\right]\int_0^{q^2}dp^2p^2\mathbf{B}(p^2)\ , \tag{3.25}$$

that is,

$$-\frac{16\pi^2}{3C_2}\frac{1}{\frac{d}{dq^2}\left[\frac{g^2(q)}{q^2}\right]}\frac{d}{dq^2}\mathbf{\Sigma}_s(q^2)=\int_0^{q^2}dp^2p^2\mathbf{B}(p^2) \tag{3.26}$$

and, differentiating once again, we obtain

$$\mathbf{B}(q^2)=-\frac{16\pi^2}{3C_2}\frac{1}{q^2}\frac{d}{dq^2}\left[\frac{1}{\frac{d}{dq^2}\left[\frac{g^2(q)}{q^2}\right]}\frac{d}{dq^2}\mathbf{\Sigma}_s(q^2)\right] \tag{3.27}$$

and analogously

$$\mathbf{C}(q^2)=\frac{16\pi^2}{3C_2}\frac{1}{q^2}\frac{d}{dq^2}\left[\frac{1}{\frac{d}{dq^2}\left[\frac{g^2(q)}{q^2}\right]}\frac{d}{dq^2}\mathbf{\Sigma}_p(q^2)\right]\ . \tag{3.28}$$

Let us remark that in deriving Eqs. (3.27) and (3.28) it is crucial to assume (3.13). The expressions we find are however valid for any choice of $g^2(p)$.

An important property of Γ_2 in the two-loop approximation is to be a quadratic functional of $\mathbf{S}$. We can then reexpress Γ_2 [see Eqs. (2.10) and (2.11)] as

$$\Gamma_2=\tfrac{1}{2}\mathrm{Tr}\left[\frac{\delta\Gamma_2}{\delta\mathbf{S}}\mathbf{S}\right]=-\tfrac{1}{2}\mathrm{Tr}(\mathbf{\Sigma}\mathbf{S})\ . \tag{3.29}$$

By substituting all these results in (3.16) and performing the angular integration, we obtain

$$\Gamma_2=\frac{2N}{3C_2}\Omega\int dp^2\mathrm{tr}\left[\mathbf{\Sigma}_s(p^2)\frac{d}{dp^2}\left[\frac{1}{\frac{d}{dp^2}\left[\frac{g^2(p)}{p^2}\right]}\frac{d}{dp^2}\mathbf{\Sigma}_s(p^2)\right]+\mathbf{\Sigma}_p(p^2)\frac{d}{dp^2}\left[\frac{1}{\frac{d}{dp^2}\left[\frac{g^2(p)}{p^2}\right]}\frac{d}{dp^2}\mathbf{\Sigma}_p(p^2)\right]\right]\ . \tag{3.30}$$

This is the final form for Γ_2, depending only on $\mathbf{\Sigma}_s$ and $\mathbf{\Sigma}_p$.

Let us now evaluate the term with the logarithm in Γ [see Eq. (3.5)]:

$$\mathrm{Tr}\ln\left[\mathbf{S}_0^{-1}+\frac{\delta\Gamma_2}{\delta\mathbf{S}}\right]=\mathrm{Tr}\ln[i\hat{p}-\mathbf{m}_0(\Lambda)-\mathbf{\Sigma}_s(p^2)-i\gamma_5\mathbf{\Sigma}_p(p^2)]$$

$$=N\Omega\int\frac{d^4p}{(2\pi)^4}\ln\mathrm{Det}[i\hat{p}-\mathbf{m}_0(\Lambda)-\mathbf{\Sigma}_s(p^2)-i\gamma_5\mathbf{\Sigma}_p(p^2)]\equiv N\Omega\int\frac{d^4p}{(2\pi)^4}\ln\mathrm{Det}(i\hat{p}-\mathbf{M})\ , \tag{3.31}$$

where we have defined

$$\mathbf{M}=[\mathbf{m}_0(\Lambda)+\mathbf{\Sigma}_s(p^2)]+i\gamma_5\mathbf{\Sigma}_p(p^2)\ . \tag{3.32}$$

The following relations hold:

$$\mathbf{M}\mathbf{M}^\dagger=[\mathbf{m}_0(\Lambda)+\mathbf{\Sigma}_s(p^2)]^2+\mathbf{\Sigma}_p^2(p^2)+i\gamma_5[\mathbf{\Sigma}_p(p^2),\mathbf{m}_0(\Lambda)+\mathbf{\Sigma}_s(p^2)]\ , \tag{3.33}$$

$$\mathbf{M}\hat{p}=\hat{p}\mathbf{M}^\dagger\ , \tag{3.34}$$

$$\hat{p}\gamma_5\mathbf{M}=\mathbf{M}^\dagger\hat{p}\gamma_5\ , \tag{3.35}$$

$$\mathrm{Det}(i\hat{p}-\mathbf{M})=\mathrm{Det}\left[\hat{p}\gamma_5\frac{i\hat{p}-\mathbf{M}}{p^2}\gamma_5\hat{p}\right]=\mathrm{Det}(-i\hat{p}-\mathbf{M}^\dagger)\ , \tag{3.36}$$

from which

$$[\mathrm{Det}(i\hat{p}-\mathbf{M})]^2=\mathrm{Det}(i\hat{p}-\mathbf{M})\mathrm{Det}(-i\hat{p}-\mathbf{M}^\dagger)=\mathrm{Det}(p^2+\mathbf{M}\mathbf{M}^\dagger)=\mathrm{Det}(p^2+\mathbf{M}^\dagger\mathbf{M})\ . \tag{3.37}$$

Our result is then

$$\mathrm{Tr}\ln\left[\mathbf{S}_0^{-1}+\frac{\delta\Gamma_2}{\delta\mathbf{S}}\right]=\frac{N\Omega}{32\pi^2}\int dp^2p^2\ln\mathrm{Det}(p^2+\mathbf{M}\mathbf{M}^\dagger)\ . \tag{3.38}$$

We can now write down the final form of the effective action Γ as a function of $\mathbf{\Sigma}$. Observing that in the two-loop approximation

$$\mathrm{Tr}\left[\frac{\delta\Gamma_2}{\delta\mathbf{S}}\mathbf{S}\right]-\Gamma_2=\Gamma_2\ , \tag{3.39}$$

we get (det is the determinant in flavor space)

$$\Gamma[\mathbf{\Sigma}]=\Omega\left\{-\frac{N}{8\pi^2}\int dp^2p^2\ln\det\{p^2+[\mathbf{m}_0(\Lambda)+\mathbf{\Sigma}_s(p^2)]^2+\mathbf{\Sigma}_p^2(p^2)+i[\mathbf{\Sigma}_p(p^2),\mathbf{m}_0(\Lambda)+\mathbf{\Sigma}_s(p^2)]\}\right.$$
$$\left.+\frac{2N}{3C_2}\int dp^2\mathrm{tr}\left[\mathbf{\Sigma}_s(p^2)\frac{d}{dp^2}\left[\frac{1}{\frac{d}{dp^2}\left[\frac{g^2(p)}{p^2}\right]}\frac{d}{dp^2}\mathbf{\Sigma}_s(p^2)\right]+\mathbf{\Sigma}_p(p^2)\frac{d}{dp^2}\left[\frac{1}{\frac{d}{dp^2}\left[\frac{g^2(p)}{p^2}\right]}\frac{d}{dp^2}\mathbf{\Sigma}_p(p^2)\right]\right]\right\}\ . \tag{3.40}$$

As expected, the volume element Ω factorizes out, and we can define the effective potential

$$\Gamma=\Omega V\ .$$

Our method will now consist in making a convenient *Ansatz* for $\mathbf{\Sigma}(p^2)$ in terms of a set of parameters related to the fermionic condensates and then in evaluating these parameters by minimizing the effective potential with respect to them.

IV. THE ULTRAVIOLET BEHAVIOR OF THE FERMION SELF-ENERGY FUNCTION

In order to get the necessary information about the asymptotic behavior of the self-energy, we will adopt the following strategy. First of all, we will derive the renormalized Schwinger-Dyson (SD) equation in the UV regime by using renormalization-group considerations. This circumstance will allow us to show that the choice we have made in the previous section for the vertex function and for the gluon propagator to evaluate Γ_2 is indeed the correct one. Next we convert the SD equation in the UV regime into an ordinary linear differential equation. From the discussion of this equation we will be able to find the asymptotic behavior of the self-energy. We will show the consistency of this result by proving that the SD equation is UV finite, assuming the standard relation between the bare mass and the renormalized one, in the leading-logarithmic approximation. Then, in the next section, we will discuss, through the SD equation, the infrared behavior of the self-energy, and we will introduce a variational *Ansatz* for the self-energy based on these results. Of course, this procedure leaves us the freedom to add a finite counterterm to the mass matrix. The resulting ambiguity in the effective action can be resolved by requiring an appropriate renormalization condition.

We will obtain some restrictions on the mechanism of spontaneous chiral-symmetry breaking directly from the equations of the theory.[31] In particular we will consider the Ward identities relating the unrenormalized proper axial-vector vertex function to the fermion bare propagator in the massive case. These quantities depend on an ultraviolet cutoff Λ. Only after having introduced the renormalized functions and going over to the deep Euclidean region of momenta we will perform the limit $\Lambda \to \infty$.

For the sake of simplicity, we will consider the case of an SU(N) gauge theory of n fermions having the same bare mass; that is, we will restrict to a bare mass matrix proportional to the identity in the flavor space $[\mathbf{m}_0(\Lambda)=m_0(\Lambda)\mathbf{1}]$. The Ward identities read

$$ip^\mu \Gamma^{(0)i}_{5\mu}(q_2,q_1,\Lambda)=-2im_0(\Lambda)\Gamma^{(0)i}_5(q_2,q_1,\Lambda) +\gamma_5\frac{\lambda_i}{2}S^{(0)-1}(q_1,\Lambda) +S^{(0)-1}(q_2,\Lambda)\frac{\lambda_i}{2}\gamma_5 , \tag{4.1}$$

where $p=q_2-q_1$, $\Gamma^{(0)i}_{5\mu}$ are the bare vertices of the colorless axial-vector currents $J^i_{\mu 5}=\bar{\Psi}\gamma_\mu\gamma_5(\lambda_i/2)\Psi$, $\Gamma^{(0)i}_5$ are the bare vertices of the colorless pseudoscalar densities $J^i_5=\bar{\Psi}\gamma_5(\lambda_i/2)\Psi$, λ_i are the matrices of the fundamental representation of the SU(n) algebra normalized to $\mathrm{tr}(\lambda_i\lambda_j)=2\delta_{ij}$, $i,j=1,\ldots,n^2-1$, and $S^{(0)-1}$ is the inverse bare fermion propagator.

The vertices $\Gamma^{(0)i}_{5\mu}$ and $\Gamma^{(0)i}_5$ satisfy equations of the Bethe-Salpeter type:

$$\Gamma^{(0)i}_{5\mu}(q_2,q_1,\Lambda)_{\alpha\beta}=\frac{\lambda_i}{2}(\gamma_\mu\gamma_5)_{\alpha\beta}+\int^\Lambda \frac{d^4k}{(2\pi)^4}K^{(0)}(q_2,q_1,k,\Lambda)_{\alpha\beta\alpha'\beta'}[S^{(0)}(k+p,\Lambda)\Gamma^{(0)i}_{5\mu}(k+p,k,\Lambda)S^{(0)}(k,\Lambda)]_{\alpha'\beta'} , \tag{4.2}$$

$$\Gamma^{(0)i}_{5}(q_2,q_1,\Lambda)_{\alpha\beta}=\frac{\lambda_i}{2}(i\gamma_5)_{\alpha\beta}+\int^\Lambda \frac{d^4k}{(2\pi)^4}K^{(0)}(q_2,q_1,k,\Lambda)_{\alpha\beta\alpha'\beta'}[S^{(0)}(k+p,\Lambda)\Gamma^{(0)i}_{5}(k+p,k,\Lambda)S^{(0)}(k,\Lambda)]_{\alpha'\beta'} , \tag{4.3}$$

where $K^{(0)}$ is the bare fermion-antifermion scattering kernel.

By substituting (4.2) and (4.3) in (4.1) we get

$$i\hat{p}\gamma_5+\int^\Lambda \frac{d^4k}{(2\pi)^4}K^{(0)}(q_2,q_1,k,\Lambda)[S^{(0)}(k+p,\Lambda)\gamma_5+\gamma_5 S^{(0)}(k,\Lambda)]=2m_0(\Lambda)\gamma_5+\gamma_5 S^{(0)-1}(q_1,\Lambda) +S^{(0)-1}(q_2,\Lambda)\gamma_5 . \tag{4.4}$$

By taking the limit $p\to 0$ and defining $q=(q_1+q_2)/2$, we get

$$\gamma_5 S^{(0)-1}(q,\Lambda)+S^{(0)-1}(q,\Lambda)\gamma_5=-2m_0(\Lambda)\gamma_5+\int^\Lambda \frac{d^4k}{(2\pi)^4}K^{(0)}(q,k,\Lambda)[S^{(0)}(k,\Lambda)\gamma_5+\gamma_5 S^{(0)}(k,\Lambda)] . \tag{4.5}$$

Let us consider the renormalized functions

$$S(k)=Z_\Psi^{-1}(\mu,\Lambda)S^{(0)}(k,\Lambda),\quad K(q,k)=Z_\Psi^2(\mu,\Lambda)K^{(0)}(q,k,\Lambda) , \tag{4.6}$$

where $Z_\Psi(\mu,\Lambda)$ is the usual renormalization constant for the fermion propagator and μ is the renormalization point, which, as before, is chosen to be coincident with our mass scale μ. One then obtains the equation

$$\gamma_5 S^{-1}(q)+S^{-1}(q)\gamma_5=-2Z_\Psi(\mu,\Lambda)m_0(\Lambda)\gamma_5+\int^\Lambda \frac{d^4k}{(2\pi)^4}K(q,k)[S(k)\gamma_5+\gamma_5 S(k)] . \tag{4.7}$$

Let us parametrize

$$S^{-1}(k)=iZ(k^2)\hat{k}-\Sigma'(k^2) . \tag{4.8}$$

Substituting in (4.7) we get

$$\gamma_5\Sigma'(q^2)=Z_\Psi(\mu,\Lambda)m_0(\Lambda)\gamma_5 +\int^\Lambda \frac{d^4k}{(2\pi)^4}K(q,k)[S(k)\gamma_5\Sigma'(k^2)S(k)] . \tag{4.9}$$

As usual in considering the ultraviolet asymptotics, let us go to the deep Euclidean region of momenta in Eq. (4.9). Since the ultraviolet asymptotics of $Z(k^2)$ and $K(q,k)$ are insensitive to the mass term, they should not be modified from spontaneous chiral-symmetry breaking. Therefore, in the leading-logarithmic approximation, one can take for them the expressions following from the renormalization-group analysis.[32] We will assume the validity of the usual argument that the main contribution to the integral on the right-hand side of Eq. (4.9) in the limit

$q^2 \to \infty$ comes from the region $k^2,(q-k)^2 \gg M_0^2$ (M_0 is the scale parameter of the theory), and use the expressions

$$K(q,k)_{\alpha\beta\alpha'\beta'} = C_2[ig(q,k)]^2(\gamma^\mu)_{\beta\beta'}(\gamma^\nu)_{\alpha\alpha'} \times D_{\mu\nu}(q-k) ,$$
$$Z(k^2)=1, \quad S(k) = -\frac{i}{\hat{k}} , \tag{4.10}$$

where all the quantities were defined in the previous section.

Let us spend some words about these choices. According to the renormalization-group analysis and thanks to the asymptotic freedom of the gauge interaction, it is meaningful to approximate the kernel K in the large momentum region with its lowest perturbative order, but one has to insert the running coupling constant which takes automatically into account the vertex perturbative corrections at least in the leading-logarithmic approximation. Remember that we have used the same arguments in the calculation of Γ_2 [see Eq. (3.11)]. In this way the expression (4.10) for the kernel is not merely the asymptotic limit of the ladder graph, but, with the insertion of the running coupling constant, faithfully represents the complete (relevant) kernel. Also, since we are working in the Landau gauge, there is no wave-function renormalization at this order $[Z_\Psi(\mu,\Lambda)=1]$. This means that the anomalous dimension γ_Ψ is equal to zero in this approximation and so there are no logarithmic corrections to the lowest perturbative order of the proper four-fermion scattering amplitude.

Substituting (4.10) in (4.9), we find in the deep Euclidean region

$$\Sigma'(q^2) = m_0(\Lambda) + 3C_2 \int^\Lambda \frac{d^4k}{(2\pi)^4} g^2(q,k) \frac{\Sigma'(k^2)}{k^2(q-k)^2} . \tag{4.11}$$

This equation is nothing but the linearized form of the SD equation we obtain by varying the effective action. To see this, consider the variation of the action when the bilocal source $J(x,y)$ is off, that is, at the physical point. We get

$$\mathbf{S}^{-1}(p) = \mathbf{S}_0^{-1}(p) + \frac{\delta\Gamma_2}{\delta \mathbf{S}} = i\hat{p} - \mathbf{m}_0(\Lambda) - \mathbf{\Sigma}(p^2)$$
$$= i\hat{p} - \mathbf{\Sigma}'(p^2) \tag{4.12}$$

with the self-energy function $\mathbf{\Sigma}$ evaluated at the minimum of the effective potential. Recalling our parametrization (3.15) for $\mathbf{S}(p)$, we get at the extremum

$$\mathbf{S}(p) = i\,\mathbf{A}(p^2)\hat{p} + \mathbf{B}(p^2) + i\gamma_5\mathbf{C}(p^2)$$
$$= [i\hat{p} - \mathbf{\Sigma}_s'(p^2) - i\gamma_5\mathbf{\Sigma}_p'(p^2)]^{-1} . \tag{4.13}$$

From Eq. (4.13) we get (after diagonalization in flavor space)

$$\mathbf{A}(p^2) = -\frac{1}{p^2 + \mathbf{\Sigma}_s'^2(p^2) + \mathbf{\Sigma}_p'^2(p^2)} ,$$
$$\mathbf{B}(p^2) = -\frac{\mathbf{\Sigma}_s'(p^2)}{p^2 + \mathbf{\Sigma}_s'^2(p^2) + \mathbf{\Sigma}_p'^2(p^2)} , \tag{4.14}$$
$$\mathbf{C}(p^2) = \frac{\mathbf{\Sigma}_p'(p^2)}{p^2 + \mathbf{\Sigma}_s'^2(p^2) + \mathbf{\Sigma}_p'^2(p^2)} .$$

Taking into account Eqs. (3.22) and (3.23) and using (4.14) we get the nonlinear SD equations for $\mathbf{\Sigma}_s'$ and $\mathbf{\Sigma}_p'$:

$$\mathbf{\Sigma}_s'(q^2) = \mathbf{m}_0(\Lambda) + 3C_2 \int \frac{d^4p}{(2\pi)^4} \frac{\mathbf{\Sigma}_s'(p^2)}{p^2 + \mathbf{\Sigma}_s'^2(p^2) + \mathbf{\Sigma}_p'^2(p^2)} \times \frac{g^2(p,q)}{(p-q)^2} , \tag{4.15}$$

$$\mathbf{\Sigma}_p'(q^2) = 3C_2 \int \frac{d^4p}{(2\pi)^4} \frac{\mathbf{\Sigma}_p'(p^2)}{p^2 + \mathbf{\Sigma}_s'^2(p^2) + \mathbf{\Sigma}_p'^2(p^2)} \times \frac{g^2(p,q)}{(p-q)^2} , \tag{4.16}$$

which, in the ultraviolet regime, where we can neglect the self-energy contribution in the denominators [and for $\mathbf{m}_0(\Lambda) = m_0(\Lambda)\mathbf{1}$], coincide with (4.11).

In the region $k^2,(q-k)^2 \gg M_0^2$ we are considering, we will assume for the function $g^2(q,k)$ the behavior expected from Eqs. (3.13) and (3.14) in the large momentum range.[30] Substituting in (4.11) we get

$$\Sigma'(q^2) = m_0(\Lambda) + 3C_2 g^2(q) \int_0^{q^2} \frac{d^4k}{(2\pi)^4} \frac{\Sigma'(k^2)}{k^2(q-k)^2} + 3C_2 \int_{q^2}^{\Lambda^2} \frac{d^4k}{(2\pi)^4} g^2(k) \frac{\Sigma'(k^2)}{k^2(q-k)^2} . \tag{4.17}$$

According to our assumption of ultraviolet dominance, we will use a regularized expression for $\Sigma'(k^2)$ in the $k \to 0$ limit (see the next section) and so we will not be faced by infrared divergences in (4.17). Finally we integrate over the angles to obtain

$$\Sigma'(q^2) = m_0(\Lambda) + \frac{3C_2}{16\pi^2} \left[\frac{g^2(q)}{q^2} \int_0^{q^2} dk^2 \Sigma'(k^2) + \int_{q^2}^{\Lambda^2} dk^2 g^2(k) \frac{\Sigma'(k^2)}{k^2} \right] \tag{4.18}$$

showing that the chiral-symmetry-violating part of the fermion self-energy satisfies an homogeneous integral equation.

For the determination of the ultraviolet asymptotics of the dynamical mass function $\Sigma'(q^2)$, one has to solve (4.18) with a finite Λ and, only afterwards perform the $\Lambda \to \infty$ limit. Equation (4.18) can be transformed into a differential equation by following the same procedure we used in deriving Eqs. (3.27) and (3.28). In this way we get

$$\frac{16\pi^2}{3C_2}\frac{d}{dq^2}\left[\frac{1}{\frac{d}{dq^2}\left[\frac{g^2(q)}{q^2}\right]}\frac{d}{dq^2}\Sigma'(q^2)\right]=\Sigma'(q^2) \quad (4.19)$$

with the boundary condition

$$\left[q^2\frac{d}{dq^2}\Sigma'(q^2)+\left[1-\frac{q^2}{g^2(q)}\frac{d}{dq^2}g^2(q)\right]\times[\Sigma'(q^2)-m_0(\Lambda)]\right]_{q^2=\Lambda^2}=0 . \quad (4.20)$$

The general solution of Eq. (4.19) takes the form

$$\Sigma'(q^2)=a_1\Sigma_1(q^2)+a_2\Sigma_2(q^2) \quad (4.21)$$

with

$$\Sigma_1(q^2)\underset{q^2\to\infty}{\sim}\left[\ln\frac{q^2}{M_0^2}\right]^{-d}, \quad (4.22)$$

$$\Sigma_2(q^2)\underset{q^2\to\infty}{\sim}\frac{1}{q^2}\left[\ln\frac{q^2}{M_0^2}\right]^{d-1}, \quad (4.23)$$

and

$$d=\frac{3C_2b}{8\pi^2} .$$

In the literature these two solutions are commonly referred to as the irregular and the regular solution, respectively. By substituting (4.21) in the boundary condition (4.20) and retaining only the leading contributions for large values of Λ we get

$$a_1=-a_2d\frac{1}{\Lambda^2}\left[\ln\frac{\Lambda^2}{M_0^2}\right]^{2d-2}+m_0(\Lambda)\left[\ln\frac{\Lambda^2}{M_0^2}\right]^{d} . \quad (4.24)$$

We can now remove the cutoff Λ. Remembering that, in the leading-logarithmic approximation, the relation between the bare mass and the mass renormalized at the point μ reads

$$m(\mu)=m_0(\Lambda)Z_m^{-1}(\mu,\Lambda) ,$$
$$Z_m(\mu,\Lambda)=\left[\frac{\ln(\mu^2/M_0^2)}{\ln(\Lambda^2/M_0^2)}\right]^{d}, \quad (4.25)$$

we obtain

$$a_1=m(\mu)\left[\ln\frac{\mu^2}{M_0^2}\right]^{d} . \quad (4.26)$$

The result is that the constant a_1 is proportional to the explicit chiral-symmetry-breaking parameter and so it vanishes in the chiral limit. In this way we have found that the asymptotic behavior of the irregular solution $\Sigma_1(q^2)$ exactly corresponds to the result of a straightforward renormalization-group analysis for nonvanishing bare fermion mass. Hence we expect that the solution which actually represents chiral symmetry realized in the Goldstone mode has the softer asymptotic behavior of $\Sigma_2(q^2)$.

It is possible to express the constant a_2 through the phenomenological parameter $\langle\bar{\Psi}\Psi\rangle_\mu$ (a summation over spinor and color indices is understood; however no summation is performed on the flavor indices). Indeed, in the theory with cutoff we have

$$\begin{aligned}\langle\bar{\Psi}\Psi\rangle_\Lambda&=-\lim_{x\to 0}\langle 0\,|\,T\Psi(x)\bar{\Psi}(0)\,|\,0\rangle_\Lambda\\&=-\lim_{k^2\to\infty}\mathrm{Tr}\int^\Lambda\frac{d^4k}{(2\pi)^4}S^{(\Lambda)}(k)\\&=\lim_{k^2\to\infty}a_2 4N\int^\Lambda\frac{d^4k}{(2\pi)^4}\frac{\Sigma_2(k^2)}{k^2}\\&=a_2\frac{2N}{3C_2b}\left[\ln\frac{\Lambda^2}{M_0^2}\right]^{d},\end{aligned} \quad (4.27)$$

where we have only considered the Σ_2 contribution to the dynamical mass because the explicit symmetry-breaking term does not contribute due to the definition of the T product. Also, in deriving (4.27), we have used $\Sigma_2^{(\Lambda)}=\Sigma_2$ since in the Landau gauge there is no wave-function renormalization at this order. From the relation between the bare and the renormalized condensate

$$\langle\bar{\Psi}\Psi\rangle_\Lambda=\left[\frac{\ln(\Lambda^2/M_0^2)}{\ln(\mu^2/M_0^2)}\right]^{d}\langle\bar{\Psi}\Psi\rangle_\mu , \quad (4.28)$$

we finally determine a_2:

$$a_2=\frac{3C_2b}{2N}\left[\ln\frac{\mu^2}{M_0^2}\right]^{-d}\langle\bar{\Psi}\Psi\rangle_\mu \quad (4.29)$$

(notice that both the constants a_1 and a_2 are renormalization-group invariant, i.e., independent of μ).

Summing up, the ultraviolet asymptotics of the fermion mass function, in the case in which both spontaneous and explicit chiral-symmetry breaking take place, is given by

$$\Sigma'(q^2)\underset{q^2\to\infty}{\sim}m(\mu)\left[\frac{\ln(q^2/M_0^2)}{\ln(\mu^2/M_0^2)}\right]^{-d}+\frac{3C_2}{4N}\langle\bar{\Psi}\Psi\rangle_\mu\frac{g^2(q)}{q^2}\left[\frac{\ln(q^2/M_0^2)}{\ln(\mu^2/M_0^2)}\right]^{d} . \quad (4.30)$$

Let us notice that, if one wants to take into account also the pseudoscalar contribution in Σ', the previous considerations are still true and one obtains an extra term in (4.30) proportional to the pseudoscalar condensate. So, separating the scalar from the pseudoscalar contribution in Σ' we get

$$\Sigma_s'(q^2)\underset{q^2\to\infty}{\sim}m(\mu)\left[\frac{\ln(q^2/M_0^2)}{\ln(\mu^2/M_0^2)}\right]^{-d}+\frac{3C_2}{4N}\langle\bar{\Psi}\Psi\rangle_\mu\frac{g^2(q)}{q^2}\left[\frac{\ln(q^2/M_0^2)}{\ln(\mu^2/M_0^2)}\right]^{d}, \quad (4.31)$$

$$\Sigma_p'(q^2) \underset{q^2\to\infty}{\sim} \frac{3C_2}{4N}\langle\bar{\Psi}i\gamma_5\Psi\rangle_\mu \frac{g^2(q)}{q^2}\left[\frac{\ln(q^2/M_0^2)}{\ln(\mu^2/M_0^2)}\right]^d . \tag{4.32}$$

These results are consistent with those obtained by means of the Wilson operator-product expansion (OPE) analysis.[33] In the OPE evaluation of $\Sigma'(q^2)$ the factors $g^2(q)$ and $[\ln(q^2/M_0^2)]^d$ are due to the renormalization-group improvement of the Wilson coefficients and, in particular, $[\ln(q^2/M_0^2)]^d$ comes from the anomalous dimension of $(\bar{\Psi}\Psi)$, or equivalently of $(\bar{\Psi}i\gamma_5\Psi)$.

It is surprising that, although the first papers concerning the ultraviolet asymptotics of the fermion self-energy in QCD appeared in the mid 1970s (Refs. 32 and 33), until now there is no common opinion about the form of these asymptotics. For example, in recent studies of the spontaneous chiral-symmetry breaking of massless QCD in the framework of variational approaches, some authors[34] agree with us in using the regular solution $\Sigma_2(q^2)$ as given in (4.23), while other authors[35] use the irregular solution $\Sigma_1(q^2)$ given in (4.22). Also, it was affirmed in a recent paper[36] that the ultraviolet asymptotics of the dynamical quark mass has the irregular form (4.22) while the regular solution is only an artifact of the Hartree-Fock approximation. This conclusion has been criticized by Miransky[37] and we agree with him in stating that only the regular solution can ensure the conservation of the anomaly-free axial-vector currents $J^i_{\mu 5}=\bar{\Psi}\gamma_\mu\gamma_5(\lambda_i/2)\Psi$ in massless QCD [this condition is of course necessary to guarantee the spontaneous character of chiral $SU(n)_L\otimes SU(n)_R$ symmetry breaking].

In fact, let us specialize the previous analysis of the ultraviolet asymptotics of the fermion self-energy to the chiral case. In the theory with ultraviolet cutoff Λ and bare mass $m_0(\Lambda)$, the axial-vector currents satisfy the equations

$$\partial^\mu J^i_{\mu 5}=2m_0(\Lambda)\left[\bar{\Psi}\gamma_5\frac{\lambda_i}{2}\Psi\right]_\Lambda . \tag{4.33}$$

Because of asymptotic freedom, the dependence on Λ of the composite operator $[\bar{\Psi}\gamma_5(\lambda_i/2)\Psi]_\Lambda$ is well known [see Eq. (4.28)]:

$$\left[\bar{\Psi}\gamma_5\frac{\lambda_i}{2}\Psi\right]_\Lambda = Z_m^{-1}(\mu,\Lambda)\left[\bar{\Psi}\gamma_5\frac{\lambda_i}{2}\Psi\right]_\mu \tag{4.34}$$

with Z_m given in (4.25). So the condition ensuring the conservation of the axial-vector currents is a rapid decrease of the bare mass $m_0(\Lambda)$ as $\Lambda\to\infty$:

$$\lim_{\Lambda\to\infty} m_0(\Lambda)Z_m^{-1}(\mu,\Lambda)=0 . \tag{4.35}$$

The condition (4.35) is necessary and sufficient to determine uniquely the asymptotics of the fermion self-energy. In fact, by using it in the boundary condition (4.24), we find that the coefficient a_1 is equal to zero in the $\Lambda\to\infty$ limit, and therefore only the regular solution $\Sigma_2(q^2)$ does contribute in (4.21).

In a paper subsequent to Ref. 36, Reinders and Stam[38] discuss the dynamical quark mass function in the frame of the OPE. They find that, asymptotically, the regular solution is consistent with the OPE, while the result from analytic continuation to lower values of p^2 leads to a freezing of the quark self-energy at its threshold value, reached for $p^2=m^2_{\rm dyn}$ where $m_{\rm dyn}$ is a sort of constituent-quark mass. As we will see in the next section, the *Ansatz* for the dynamical quark mass function we will use agrees with this result.

As a check of this procedure, we want to show that the gap equations (4.15) and (4.16) are UV finite if we use for $\mathbf{m}_0(\Lambda)$ the dependence from the cutoff Λ implied by Eq. (4.23). As far as the pseudoscalar part is concerned, the integral representation (4.16) is clearly ultraviolet convergent due to the behavior (4.32) for Σ_p'. As far as Σ_s' we have

$$\begin{aligned}
&\mathbf{m}_0(\Lambda)+3C_2\int\frac{d^4p}{(2\pi)^4}\frac{\Sigma_s'(p^2)}{p^2+\Sigma_s'^2(p^2)+\Sigma_p'^2(p^2)}\frac{g^2(p,q)}{(p-q)^2}\Bigg|_{p^2\to\infty}\\
&\sim \mathbf{m}_0(\Lambda)+\frac{3C_2}{16\pi^2}\int_0^{\Lambda^2}dp^2\Sigma_s'(p^2)\frac{g^2(p)}{p^2}\Bigg|_{p^2\to\infty}\\
&\sim \mathbf{m}_0(\Lambda)+\frac{3C_2}{16\pi^2}\int_0^{\Lambda^2}\frac{dp^2}{p^2}\mathbf{m}(\mu)\left[\frac{\ln(p^2/M_0^2)}{\ln(\mu^2/M_0^2)}\right]^{-d}\frac{2b}{\ln(p^2/M_0^2)}=\mathbf{m}_0(\Lambda)-\mathbf{m}(\mu)\left[\frac{\ln(\Lambda^2/M_0^2)}{\ln(\mu^2/M_0^2)}\right]^{-d}=0 ,
\end{aligned}\tag{4.36}$$

where we have used Eq. (4.25). In other words, the two types of asymptotic behavior (4.31) and (4.32) give us a perfectly UV regularized theory in the leading-logarithmic approximation.

V. VARIATIONAL *ANSATZ* FOR THE FERMION SELF-ENERGY AND RENORMALIZATION CONDITION FOR THE EFFECTIVE ACTION

In order to derive a variational *Ansatz* for the self-energy to be used in the effective potential, we will assume a momentum dependence of $\Sigma(p^2)$ outside the extremum as the one suggested by the solution of the SD equation. We know already the UV behavior of this solution and we need to study the IR regime. This can be done by using the gap equations (4.15) and (4.16).

Let us suppose that

$$\Sigma'(q^2)\underset{q^2\to 0}{\sim}(q^2)^{-\alpha},\quad \alpha>0 .$$

By looking at Eqs. (4.15) and (4.16) in the $q^2\to 0$ limit, it is easy to see that, with this assumption, one has a finite contribution at the lower limit of integration on the right-hand side, while, on the left-hand side one has an infrared divergence. It follows that, for $q^2 \to 0$, $\mathbf{\Sigma}'(q^2)$ must go to a finite constant.

Recalling that at the extremum $\mathbf{\Sigma}(p^2)$ is given by

$$\mathbf{\Sigma}(p^2)=\mathbf{\Sigma}'(p^2)-\mathbf{m}_0(\Lambda) \tag{5.1}$$

we can make an *Ansatz* for $\mathbf{\Sigma}(p^2)$ interpolating between the UV and the IR behavior of the solutions of the SD equation. The simplest choice is

$$\mathbf{\Sigma}(p^2)=\mathbf{m}(\mu)f_1(p^2)-\mathbf{m}_0(\Lambda)+(\mathbf{s}+i\gamma_5\mathbf{p})f_2(p^2) \tag{5.2}$$

with

$$f_1(p^2)=\Theta(\mu^2-p^2)+\Theta(p^2-\mu^2)f(p^2)^{-d}\ , \tag{5.3}$$

$$f_2(p^2)=\mu\left[\Theta(\mu^2-p^2)+\Theta(p^2-\mu^2)\frac{\mu^2}{p^2}f(p^2)^{d-1}\right]\ , \tag{5.4}$$

$$f(p^2)=\frac{\ln(p^2/M_0^2)}{\ln(\mu^2/M_0^2)}\ , \tag{5.5}$$

where μ is the parameter separating the UV and the IR regions already introduced in Eq. (3.14). We will use the parameter-dependent test function (5.2) for the fermion self-energy in our effective potential formalism to investigate the stability of the theory.

The fields $\mathbf{s}_{ab}$ and $\mathbf{p}_{ab}$, $a,b=1,\ldots,n$, which here are constant fields because we are only interested in the evaluation of the effective potential, will be our variational parameters. The minimum of the effective potential will determine the values of these parameters corresponding to the optimal form of the test function for $\mathbf{\Sigma}(p^2)$. The matrices $\mathbf{s}$ and $\mathbf{p}$ evaluated at the extremum of the effective potential, which will be called $\langle\mathbf{s}\rangle$ and $\langle\mathbf{p}\rangle$, can be related to the fermionic condensates.

This can be seen directly by the comparison of our *Ansatz* with the expressions given in Eqs. (4.31) and (4.32). We obtain in this way

$$\langle\mathbf{s}_{ab}\rangle=\frac{3C_2}{4N}\frac{g^2(\mu)}{\mu^3}\langle\overline{\Psi}_a\Psi_b\rangle_\mu\ , \tag{5.6}$$

$$\langle\mathbf{p}_{ab}\rangle=\frac{3C_2}{4N}\frac{g^2(\mu)}{\mu^3}\langle\overline{\Psi}_a i\gamma_5\Psi_b\rangle_\mu,\quad a,b=1,\ldots,n\ . \tag{5.7}$$

If we had not chosen the renormalization point of the theory to coincide with the scale μ, and if we had instead renormalized at $p^2=\bar{\mu}^2$, then an extra factor

$$\left[\frac{\ln(\mu^2/M_0^2)}{\ln(\bar{\mu}^2/M_0^2)}\right]^d$$

would have been present in Eqs. (5.6) and (5.7).

Substituting the *Ansatz* (5.2) in (3.40), one can show that the effective potential is UV finite (see Appendix A). Notice that, in order for the UV divergences to cancel out, the dependence of $\mathbf{m}_0(\Lambda)$ from the cutoff following from Eq. (4.25) is crucial. Of course, we have again the freedom of adding to $\mathbf{m}_0(\Lambda)$ a finite counterterm and we need to specify a suitable renormalization condition to remove this ambiguity (this is equivalent to specify the renormalization of the composite fields $\mathbf{s}$). We can derive such a condition by looking at the expression for the effective action in the limit of small masses. We can think to add the mass term by the following replacement in the bilocal source:

$$\mathbf{J}(x,y)\to\mathbf{J}(x,y)+\mathbf{m}\delta^4(x-y)\ . \tag{5.8}$$

Then the generating functional of the connected Green's functions in the presence of a small-mass term can be written as

$$W[\mathbf{m},\mathbf{J}]\underset{\mathbf{m}\to 0}{\sim}W[0,\mathbf{J}]+\int d^4x\ \mathrm{tr}\left[\mathbf{m}\frac{\delta W[0,\mathbf{J}]}{\delta\mathbf{J}(x,x)}\right]\ , \tag{5.9}$$

where the trace over the spinor and the color indices is understood. We can now calculate the effective action at its extremum $\mathbf{J}=0$, obtaining

$$\begin{aligned}\Gamma(\mathbf{m})\,|_{\mathbf{J}=0;\mathbf{m}\to 0}&=W(\mathbf{m},0)\\ &\underset{\mathbf{m}\to 0}{\sim}W(0,0)-\mathrm{tr}\left[\mathbf{m}\int d^4x\ \mathbf{S}(x,x)\right]_{\mathbf{J}=0}\\ &\sim\Gamma(0)\,|_{\mathbf{J}=0}+\Omega\ \mathrm{tr}(\mathbf{m}\langle\overline{\Psi}\Psi\rangle)\ .\end{aligned} \tag{5.10}$$

In fact $\mathbf{S}(x,x)$ is nothing but minus the scalar condensate. Therefore we have to require

$$\lim_{\mathbf{m}\to 0}\frac{1}{\Omega}\mathrm{tr}\left[\mathbf{m}^{-1}\frac{\delta\Gamma}{\delta\langle\overline{\Psi}\Psi\rangle}\right]\Bigg|_{\mathrm{extr}}=1 \tag{5.11}$$

or equivalently, by using the relation between the scalar condensate and the scalar field $\mathbf{s}$ (5.6),

$$\lim_{\mathbf{m}\to 0}\mathrm{tr}\left[\mathbf{m}^{-1}\frac{\partial V(\mathbf{s}^2+\mathbf{p}^2,\mathbf{s})}{\partial\mathbf{s}}\right]\Bigg|_{\mathrm{extr}}=\frac{4N\mu^3}{3C_2g^2(\mu)}\ . \tag{5.12}$$

Finally, we give the expression of the effective potential $V(\Gamma=\Omega V)$ as a function of the variational parameters of the theory, $\mathbf{s}$ and $\mathbf{p}$:

$$V=\frac{N\mu^4}{4\pi^2}\left[cA_1\mathrm{tr}(\mathbf{s}^2+\mathbf{p}^2)+(A_2+\delta z_f)\mathrm{tr}[\mathbf{m}(\mu)\cdot\mathbf{s}]-\tfrac{1}{2}\int_0^\infty dy\ y\ln\det(y+\mathbf{x}^2+\mathbf{z}^2+i[\mathbf{z},\mathbf{x}])\right]\ , \tag{5.13}$$

where

$$c=\frac{8\pi^2}{3C_2g^2(\mu)},\quad A_1=1+\frac{d^2}{2a}\int_0^1 du\,\frac{F(u)^{2d-2}}{1+2aF(u)},$$

$$A_2=\frac{2a}{\mu}\left[\int_0^1\frac{du}{u}\frac{1}{1+2aF(u)}-(d-1)\ln\frac{1+2a}{2a}\right],$$

$$F(u)=1-\frac{1}{2a}\ln u,\quad \mathbf{x}=\mathbf{m}\frac{f_1(y)}{\mu}+\mathbf{s}\frac{f_2(y)}{\mu},$$

$$\mathbf{z}=\mathbf{p}\frac{f_2(y)}{\mu}, \tag{5.14}$$

$$f_1(y)=\Theta(1-y)+\Theta(y-1)\left[1+\frac{1}{2a}\ln y\right]^{-d},$$

$$f_2(y)=\mu\left[\Theta(1-y)+\Theta(y-1)\frac{1}{y}\left[1+\frac{1}{2a}\ln y\right]^{d-1}\right],$$

$$a=\ln\frac{\mu}{M_0},$$

and we have introduced the finite counterterm δz_f to be determined such as to satisfy the normalization condition (5.12).

VI. MASSLESS EFFECTIVE POTENTIAL

Let us consider the logarithm term in Γ in the massless case

$$\begin{aligned}\Gamma_{\log}&=-\mathrm{Tr}\ln\left[\mathbf{S}_0^{-1}+\frac{\delta\Gamma_2}{\delta\mathbf{S}}\right]\\&=-\mathrm{Tr}\ln[i\hat{p}-\boldsymbol{\Sigma}_s(p^2)-i\gamma_5\boldsymbol{\Sigma}_p(p^2)]\\&=-N\Omega\int\frac{d^4p}{(2\pi)^4}\ln\mathrm{Det}[i\hat{p}-(\mathbf{s}+i\gamma_5\mathbf{p})f_2(p^2)],\end{aligned} \tag{6.1}$$

where we have used Eq. (5.2) in the massless case and $f_2(p^2)$ is given in (5.4). The matrix

$$\mathbf{A}=\mathbf{s}+i\gamma_5\mathbf{p} \tag{6.2}$$

considered as an $n\times n$ complex matrix, can be diagonalized by a chiral rotation using two unitary matrices $\mathbf{U}$ and $\mathbf{V}$:

$$\mathbf{A}_d=\mathbf{U}\,\mathbf{A}\mathbf{V}^\dagger. \tag{6.3}$$

Since this transformation leaves the Γ_2 contribution in the effective action invariant, we can simply insert $\mathbf{A}_d$ in the evaluation of the determinant

$$\ln\mathrm{Det}[i\hat{p}-\mathbf{A}_d f_2(p^2)]=\ln\mathrm{Det}\left\{i\hat{p}-\left[\left[s_0+\sum_{i=1}^{n-1}s_ih_i\right]+i\gamma_5\left[p_0+\sum_{i=1}^{n-1}p_ih_i\right]\right]f_2(p^2)\right\}, \tag{6.4}$$

where we have expanded $\mathbf{A}_d$ in terms of the generators of the Cartan subalgebra of U(n), **1**, and h_i, $i=1,\ldots,n-1$, normalized as $\mathrm{tr}(h_ih_j)=\delta_{ij}$. Substituting (6.4) in (6.1), one obtains

$$\begin{aligned}\Gamma_{\log}&=-N\Omega\int\frac{d^4p}{(2\pi)^4}\ln\mathrm{Det}\{[i\hat{p}-(\chi_1+i\gamma_5\pi_1)f_2(p^2)][i\hat{p}-(\chi_2+i\gamma_5\pi_2)f_2(p^2)]\cdots[i\hat{p}-(\chi_n+i\gamma_5\pi_n)f_2(p^2)]\}\\&=-N\Omega\sum_{a=1}^{n}\int\frac{d^4p}{(2\pi)^4}\ln\mathrm{Det}[i\hat{p}-(\chi_a+i\gamma_5\pi_a)f_2(p^2)],\end{aligned} \tag{6.5}$$

where

$$\chi_a=s_{aa}=s_0+\sum_{i=1}^{n-1}s_i(h_i)_{aa},\quad \pi_a=p_{aa}=p_0+\sum_{i=1}^{n-1}p_i(h_i)_{aa},\quad a=1,\ldots,n, \tag{6.6}$$

that is, χ_a and π_a are the eigenvalues of the matrices **s** and **p**, respectively. On the other hand,

$$\mathrm{tr}(\mathbf{s}^2+\mathbf{p}^2)=\sum_{a=1}^{n}(\chi_a^2+\pi_a^2). \tag{6.7}$$

Therefore, evaluating as before the determinant over the γ matrices in $\Gamma_{\log}$ and performing the integration over the angular variables, we obtain the following expression for the effective potential in the massless case which we will indicate as $V^{(0)}$ (we omit an additive infinite constant):

$$\begin{aligned}V^{(0)}&=\frac{N\mu^4}{4\pi^2}\sum_{a=1}^{n}V_1^{(0)}(\chi_a,\pi_a),\\V_1^{(0)}(\chi,\pi)&=\left[c\left[1+\frac{d^2}{2a}\int_0^1du\,\frac{F(u)^{2d-2}}{1+2aF(u)}\right](\chi^2+\pi^2)-\tfrac{1}{2}\int_0^1dy\,y\ln\left[1+\frac{\chi^2+\pi^2}{y}\right]\right.\\&\qquad\left.-\frac{1}{2}\int_0^1\frac{du}{u^3}\ln[1+(\chi^2+\pi^2)u^3F(u)^{2d-2}]\right],\end{aligned} \tag{6.8}$$

where c, $F(u)$, and a are defined in (5.14).

As expected, $V^{(0)}$ is a completely finite quantity both in the ultraviolet and in the infrared regimes. The theory is in fact regularized in the infrared by the assumed constant behavior of the self-energy in the $p \to 0$ limit, whereas the convergence in the ultraviolet follows in the massless case from the physical meaning of χ and π. In fact, from the relations (5.6), (5.7), and (6.6) it follows that χ and π have operator dimension equal to three. However, because of the chiral invariance, linear terms in χ and π are forbidden so that the effective potential $V^{(0)}$ must start at least with $(\chi^2+\pi^2)$ in an expansion in the composite fields. Therefore, because of the absence of operators of dimension lower or equal to four, no ultraviolet divergences are expected in $V^{(0)}$.

We will now verify that the effective potential $V^{(0)}$ of Eq. (6.8) is bounded from below. To this end, let us introduce a function $\bar{V}^{(0)}$ obtained by $V^{(0)}$ setting $d=0$ and $F(u)=1$. This would correspond to consider a situation with a rigid coupling constant and zero anomalous dimensions. In this case one can perform exactly the integrations obtaining an analytical expression for $\bar{V}^{(0)}$:

$$\bar{V}^{(0)}=\frac{N\mu^4}{4\pi^2}\sum_{a=1}^{n}\bar{V}_1^{(0)}(\phi_a^2),\quad \phi_a^2=\chi_a^2+\pi_a^2\ ,$$
$$\bar{V}_1^{(0)}(\phi^2)=(c-\tfrac{1}{4})\phi^2+\tfrac{1}{4}\phi^4\ln\frac{1+\phi^2}{\phi^2}+\tfrac{1}{8}\phi^{4/3}\ln\frac{1-\phi^{2/3}+\phi^{4/3}}{(1+\phi^{2/3})^2}-\tfrac{1}{4}\sqrt{3}\phi^{4/3}\left[\frac{\pi}{2}-\arctan\frac{2-\phi^{2/3}}{\sqrt{3}\phi^{2/3}}\right]\ , \tag{6.9}$$

which shows the asymptotic behavior for large fields

$$\bar{V}_1^{(0)}(\phi^2)\underset{\phi^2\to\infty}{\sim} c\phi^2 \tag{6.10}$$

and we see that $\bar{V}^{(0)}$ is bounded from below. Starting from Eqs. (6.8) and (6.9), we can easily derive the expression

$$V_1^{(0)}(\phi^2)-\bar{V}_1^{(0)}(\phi^2)=\left[\frac{cd^2}{2a}\int_0^1 du\frac{F(u)^{2d-2}}{1+2aF(u)}\phi^2-\tfrac{1}{2}\int_0^1\frac{du}{u^3}\ln\frac{1+\phi^2u^3F(u)^{2d-2}}{1+\phi^2u^3}\right]. \tag{6.11}$$

In Eq. (6.11) the first term is positive definite because $F(u)>0$ for $0<u\le 1$. As far as the second term is concerned, one has to consider the quantity

$$F(u)^{d-1}=\left[1-\frac{1}{2a}\ln u\right]^{d-1}.$$

The expression (6.11) is positive definite when $F(u)^{d-1}\le 1$. Because $u\le 1$, it follows that $F(u)^{d-1}\le 1$ according to $(d-1)\le 0$. We recall that

$$d=\frac{3bC_2}{8\pi^2}=\frac{9C_2}{11N-2n}\ .$$

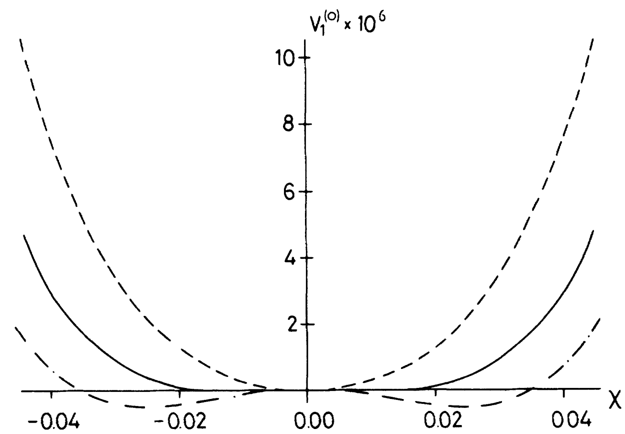

FIG. 2. Behavior of $V_1^{(0)}(\chi)$ near the critical point in the case of QCD with three flavors. The curves shown correspond to the values $\mu/\Lambda_{\rm QCD}=1.354$ (dash-dotted line), 1.355 (solid line), 1.357 (dashed line).

Thus, in QCD with color-triplet fermions ($N=3$, $C_2=\frac{4}{3}$), one has $d=12/(33-2n)$ and $d\le 1$ for $n\le\frac{21}{2}$. This means that, if we have less than six families, $V_1^{(0)}(\phi^2)-\bar{V}_1^{(0)}(\phi^2)>0$ for any value of ϕ^2. We have derived in this way a rigorous lower bound for $V_1^{(0)}(\phi^2)$. This bound clearly shows that also $V^{(0)}$ is bounded from below.

Without loss of generality, we can suppose that the vacuum expectation value (VEV) of π_a is equal to zero (it can be chirally rotated away). Since the effective potential has the additive form given in Eq. (6.8), every χ_a has the same VEV, $\langle\chi_a\rangle=u$. Therefore if χSB occurs, its pattern must be of the type $SU(n)_L\otimes SU(n)_R\to SU(n)_{L+R}$ because from the expression of χ_a in Eq. (6.6) follows

$$\langle s_0\rangle=u,\quad\langle s_i\rangle=0,\quad i=1,\ldots,n-1\ . \tag{6.12}$$

Performing the numerical analysis $V^{(0)}$, we find that, in QCD with three flavors ($M_0=\Lambda_{\rm QCD}$), the theory undergoes chiral-symmetry breaking for $\mu/\Lambda_{\rm QCD}<1.355$ which corresponds to

$$\alpha_s=\frac{g^2(\mu)}{4\pi}>0.73\pi\ . \tag{6.13}$$

The behavior of the effective potential near the critical point is showed in Fig. 2.

In this way we have proved that the quark-antiquark-gluon interaction provides the Goldstone realization of the chiral symmetry due to the spontaneous breaking of $SU(n)_L\otimes SU(n)_R$ into $SU(n)_{L+R}$.

VII. COMPARISON WITH OTHER STUDIES OF THE DYNAMICAL CHIRAL-SYMMETRY-BREAKING PHENOMENON

The method of the effective action for composite operators proves to be a convenient tool for studying field theories with dynamical symmetry breaking. The application of the direct variational method to the problem of

DSB in QCD-like gauge theories, has allowed us to take into account the nonlinear aspects of the problem and to justify the results first obtained in the framework of the linearized approximation (see, for example, Refs. 39 and 40). In fact, a way to investigate the dynamical mechanism of the spontaneous breakdown of chiral invariance in massless gauge theories is based on the exact solution of the linearized Bethe-Salpeter (BS) equations for the pseudoscalar Goldstone bosons. The main hypothesis of this approach is that the mechanism of the condensates formation, responsible for the spontaneous symmetry breaking, comes from the strong gauge forces acting at distances of the order of the size of the Goldstone bosons and a crucial point is the assumption that these distances are smaller than those at which the confinement forces dominate. In the model considered in Ref. 39 the dynamics of condensation is described by BS equations for the fermion-antifermion tightly bound states, in which parameters for infrared and ultraviolet cutoffs are introduced in order to pick out the momentum range responsible for binding. The infrared cutoff is identified with the confinement scale. Since the result of the analysis gives a critical value $(g)_{\rm crit}$ for the coupling constant at which chiral-symmetry breaking occurs and, since in non-Abelian gauge theories the domain of strong coupling $[g(p)>(g)_{\rm crit}]$ is the region of small momenta, the ultraviolet cutoff is identified with the value of p at which $g(p)\sim(g)_{\rm crit}$.

The kernels of the BS equations are taken in the ladder approximation, with values of the coupling constant g and of the fermion dynamical mass m equal to the values of the running coupling constant $g(p)$ and of the fermion mass function $\Sigma(p^2)/Z(p^2)$, respectively, averaged in the appropriate momentum range $[S^{-1}(p)=iZ(p^2)\hat{p}-\Sigma(p^2)]$. With these approximations, it happens that the BS equations have the solutions for the tightly bound states with $M^2\le 0$ provided the coupling α

$$\alpha=\frac{g^2}{4\pi}\frac{N^2-1}{2N}\quad \text{for SU}(N)$$

exceeds its critical value

$$(\alpha)_{\rm crit}=\frac{\pi}{3}\ . \tag{7.1}$$

In QCD this means

$$(\alpha_s)_{\rm crit}=\left[\frac{g^2}{4\pi}\right]_{\rm crit}=\frac{\pi}{4}\ . \tag{7.2}$$

The main results are the following. In the symmetric unstable phase there exist n^2 pseudoscalar tachyons, while, in the phase in which the vacuum rearrangement results in spontaneous breakdown of the chiral $SU(n)_L\otimes SU(n)_R$ symmetry, a fermion mass appears. The spectrum of the fermion dynamical masses can be determined by requiring that the tachyons disappear in the stable phase and, instead of them, the (n^2-1)-plet of pseudoscalar Goldstone bosons appears [the singlet under $SU(n)_{L+R}$ acquiring mass through the Adler-Bell-Jackiw anomaly of the $U(1)_{L-R}$ current]. Also, when there is spontaneous and explicit breakdown of chiral symmetry, the mass formula for the pseudoscalar mesons can be derived, and, once compared with the current-algebra mass formula, it provides a dynamical realization of the PCAC hypothesis.

From the Ward identities for the axial-vector currents, it follows that such a way of determining the dynamical fermion mass is equivalent to looking for nontrivial solutions for the linearized Schwinger-Dyson equation for the self-energy $\Sigma(p^2)$ in the ladder approximation. An improved form of the Schwinger-Dyson equation for the quark propagator in QCD has been numerically studied by Higashijima.[34] He also assumes that the short-range force, rather than the confining force, is responsible for chiral-symmetry breaking in QCD, and so he approximates the kernel of the SD equation by one-gluon exchange, but improving with the running coupling constant. In order to tame the infrared singularities of

$$\lambda(t)=\frac{3C_2g^2(t)}{4\pi^2},\quad t=\ln\frac{p}{\Lambda_{\rm QCD}}\ ,$$

he defines a nonconfining QCD-like model by

$$\lambda(t)=\begin{cases}A/t_c & \text{if } t\le t_c\ ,\\ A/t & \text{if } t>t_c\ ,\end{cases} \tag{7.3}$$

with

$$A=\frac{3C_2}{11-2n/3}$$

(which exactly corresponds to our choice for the running coupling constant). The numerical result is that, in the case of triplet quarks and $n=3$, for $t_c<0.88$ the constituent-quark mass does not vanish in the chiral limit, that is, chiral symmetry is spontaneously broken. Since the parameter t_c corresponds to our $\ln(\mu/\Lambda_{\rm QCD})$, the critical point results for

$$\left[\frac{\mu}{\Lambda_{\rm QCD}}\right]_{\rm crit}\simeq 2.4 \tag{7.4}$$

which corresponds to a broken phase for

$$\alpha_s=\frac{g^2(\mu)}{4\pi}>\frac{\pi}{4}\ . \tag{7.5}$$

This is completely consistent with the critical value (7.2) of the coupling constant of Ref. 39, in which the linearized SD equation is considered. This is not surprising, since the dynamical mass function goes to zero as a power for large values of momenta, and so its contribution at the denominator of SD equation is quite inessential in the range of momenta at which chiral-symmetry breaking occurs.

As far as variational methods are concerned, let us mention the result obtained by Peskin[15] with a simple stability analysis of the CJT effective action. In order to compute whether chiral-symmetry breaking can be induced by one-gluon exchange in a SU(N) gauge theory of massless fermions (lowest-order approximation), he works in the following simplified framework: a fixed coupling constant g and $\Gamma_{\rm CJT}$ to quadratic order in the fermion self-energy Σ. Expecting that for g^2 sufficiently small the vacuum is chirally symmetric, he restricts the attention to the stability of the symmetric vacuum. He finds that the kinematical terms in $\Gamma_{\rm CJT}$ stabilize the chirally symmetric state $\mathbf{\Sigma}=0$, and so the interactions must counteract this effect. The explicit calculation of Γ_2, truncated at second order in $\mathbf{\Sigma}$, shows that a criterion for instability is

$$\frac{3C_2 g^2}{4\pi^2} > 1 \tag{7.6}$$

which, for quarks in the fundamental representation of SU(3) gives

$$\alpha_s > \frac{\pi}{4} .$$

This result represents only a criterion for the vacuum instability and it depends on the crude approximations done.

In general, other computations based on the CJT effective action formalism give a higher value for the critical coupling constant. For example, Gusynin and Sitenko in Ref. 34 make an analysis completely equivalent to ours but using the CJT functional. Their numerical analysis for QCD with three flavors gives for the coupling constant the critical value

$$(\alpha_s)_{\rm crit} \simeq \frac{\pi}{2} . \tag{7.7}$$

Similar results have been obtained by Castorina and Pi.[34] They use the original CJT variational principle and reach our same conclusions on the chiral-symmetry breaking. However, their potential is not bounded from below, since, as we have shown in Sec. II, all the stationary points corresponding to chiral-symmetry-breaking solutions are actually saddle points. Their statement of boundness from below of the $V_{\rm CJT}$ effective potential reflects the fact that they have not analyzed the behavior of the potential in the complete range of parameters. This conclusion has been proved directly by Gusynin and Sitenko in Ref. 34 where computer calculations show the monotone decreasing of $V_{\rm CJT}(\chi)$ for large values of χ. The unboundness from below of this function is maintained if any finite number of loops is taken into account in evaluating the potential, since the contribution of the multiloop diagrams vanishes in the limit of large dynamical mass.

As a final remark, it is interesting to notice that the variational methods with the specific *Ansätzes* for the fermion self-energy give a higher value for the critical coupling constant as compared to the methods based on the exact solution of the linearized equation or on the numerical solution of the nonlinear SD equation. This is a general feature of the variational calculations and clearly indicates that the true form of the self-energy is more complicated than that of the type (5.2).

VIII. EFFECTIVE POTENTIAL IN QCD WITH THREE MASSIVE FLAVORS

We can now discuss the general case of massive fermions and examine the particular predictions of the formalism for QCD. We will consider the case of three flavors u,d,s by introducing a diagonal mass matrix

$$\mathbf{m} = \begin{bmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{bmatrix} . \tag{8.1}$$

Let us calculate explicitly the determinant in (5.13) by using the standard relation holding for 3×3 matrices:

$$\det \mathbf{A} = \tfrac{1}{3}[(\operatorname{tr}\mathbf{A}^3) - \tfrac{3}{2}(\operatorname{tr}\mathbf{A})(\operatorname{tr}\mathbf{A}^2) + \tfrac{1}{2}(\operatorname{tr}\mathbf{A})^3] . \tag{8.2}$$

We get, in this way,[23]

$$V = \frac{N\mu^4}{4\pi^2} \left\{ c A_1 \operatorname{tr}(\mathbf{s}^2+\mathbf{p}^2) + (A_2 + \delta z_f)\operatorname{tr}[\mathbf{m}(\mu)\cdot\mathbf{s}] - \tfrac{1}{2}\int_0^\infty dy\, y \ln A_3 \right\} , \tag{8.3}$$

where

$$\mathbf{s}_a^b = \left[s_\alpha \frac{\lambda_\alpha}{\sqrt{2}} \right]_a^b , \tag{8.4}$$

$$\mathbf{p}_a^b = \left[p_\alpha \frac{\lambda_\alpha}{\sqrt{2}} \right]_a^b , \tag{8.5}$$

$\lambda_i =$ Gell-Mann matrices ,

$\lambda_0 = (\tfrac{2}{3})^{1/2}, \quad i=1,\ldots,8, \quad \alpha=0,\ldots,8$,

and

$$\begin{aligned} A_3 &= y^3 + 3y^2E + y(3E^2 - C^2) + E^3 - EC^2 + \tfrac{2}{3}D , \\ C^2 &= C_k C_k, \quad D = d_{klm} C_k C_l C_m , \\ C_k &= (\tfrac{2}{3})^{1/2}(x_0 x_k + z_0 z_k) + \tfrac{1}{2} d_{ijk}(x_i x_j + z_i z_j) \\ &\quad + f_{ijk} x_i z_j , \\ E &= \tfrac{1}{3}\operatorname{tr}(\mathbf{x}^2 + \mathbf{z}^2), \quad i,j,k,l,m, = 1,\ldots,8 . \end{aligned} \tag{8.6}$$

The quantities c, A_1, A_2, $\mathbf{x}$, and $\mathbf{z}$ are the same we have previously introduced in (5.14) and δz_f is the finite counterterm which we will determine by imposing the normalization condition (5.12).

It is possible to prove that, for a symmetric mass matrix $\mathbf{m}_a^b = m\,\delta_a^b$, $a,b=1,\ldots,n$, the search for the minima of the effective potential corresponding to vanishing charged fields can be performed on the null charged fields surface (see Appendix B). Of course, one cannot certainly exclude the possibility that V has some other minima. However, one can notice that, in the zero-mass case, the potential V does not show any other isolated minimum and therefore, for small m, the zero charged condensates minimum is certainly the absolute one.

This result is in agreement with the general result by Vafa and Witten.[40] They have shown that in vector gauge theories with $\theta=0$ (θ is the parameter connected with the axial anomaly), the global vector symmetries are not spontaneously broken and that, in the case of a symmetric mass matrix, all the condensates are equal [the theory breaks down to $SU(n)_V$]. Let us notice that for a nonsymmetric mass matrix, and small values of the fermionic masses, we do expect the same conclusions to hold, because of the continuity of the effective potential function. We will then assume that, also for the nonsymmetric case, the matrix of the condensate is diagonal at the physical point. This leads again to an effective potential at the minimum which is a sum of independent terms.

In the case of QCD with $n=3$, by taking the charged fields equal to zero in (8.3), and using the definition (6.6) for χ_a and π_a with $a=u,d,s$ and $(\lambda_0/\sqrt{2},\lambda_3/\sqrt{2},\lambda_8/\sqrt{2})$ as basis of the Cartan subalgebra of U(3), we get

$$V=\frac{3\mu^4}{4\pi^2}\sum_{a=u,d,s}V_1(\chi_a,\pi_a,\alpha_a)\,,$$
$$V_1(\chi,\pi,\alpha)=cA_1(\chi^2+\pi^2)+\mu\alpha\chi(A_2+\delta z_f)-\tfrac{1}{2}\int_0^1 dy\,y\ln\left[1+\frac{2\alpha\chi+(\chi^2+\pi^2)}{y+\alpha^2}\right]$$
$$-\tfrac{1}{2}\int_0^1\frac{du}{u^3}\ln\left[1+\frac{2\alpha\chi u^2F(u)^{-1}+(\chi^2+\pi^2)u^3F(u)^{2d-2}}{1+\alpha^2uF(u)^{-2d}}\right], \tag{8.7}$$

with $\alpha_a=m_a/\mu$ and $d=\frac{4}{9}$.

We can now determine δz_f by imposing the normalization condition (5.12) which in this case reads

$$\lim_{m_a\to 0}\frac{1}{m_a}\frac{\partial V}{\partial\chi_a}\bigg|_{\rm extr}=\frac{4N\mu^3}{3C_2g^2(\mu)}\,, \tag{8.8}$$

where the partial derivative is made with respect to the explicit dependence of V on χ_a (symmetry-breaking part). The result is

$$A_2+\delta z_f=\frac{1}{\mu}\left[2c+\int_0^1 du\frac{u}{u+\langle\chi^0\rangle^2}+\int_0^1\frac{du}{u}\frac{F(u)^{-1}}{1+\langle\chi^0\rangle^2u^3F(u)^{2d-2}}\right], \tag{8.9}$$

where $\langle\chi^0\rangle$ is the value of the field χ at the minimum of the potential in the massless case.

By substituting (8.9) in (8.7) we get the final form of the effective potential (with charged fields equal to zero) for QCD with three massive flavors:

$$V_1(\chi,\pi,\alpha)=c\left[1+\frac{d^2}{2a}\int_0^1 du\frac{F(u)^{2d-2}}{1+2aF(u)}\right](\chi^2+\pi^2)+\alpha\chi\left[2c+\int_0^1 du\frac{u}{u+\langle\chi^0\rangle^2}+\int_0^1\frac{du}{u}\frac{F(u)^{-1}}{1+\langle\chi^0\rangle^2u^3F(u)^{2d-2}}\right]$$
$$-\tfrac{1}{2}\int_0^1 dy\,y\ln\left[1+\frac{2\alpha\chi+(\chi^2+\pi^2)}{y+\alpha^2}\right]-\tfrac{1}{2}\int_0^1\frac{du}{u^3}\ln\left[1+\frac{2\alpha\chi u^2F(u)^{-1}+(\chi^2+\pi^2)u^3F(u)^{2d-2}}{1+\alpha^2uF(u)^{-2d}}\right], \tag{8.10}$$

where we recall that [see Eq. (5.14) with $M_0=\Lambda_{\rm QCD}$]

$$F(u)=1-\frac{1}{2a}\ln u,\quad a=\ln\frac{\mu}{\Lambda_{\rm QCD}}\,.$$

Remember that the following relations hold:

$$\langle\chi_u\rangle=\langle s_{uu}\rangle=\frac{1}{\sqrt{3}}\langle s_0\rangle+\frac{1}{\sqrt{2}}\langle s_3\rangle+\frac{1}{\sqrt{6}}\langle s_8\rangle=\frac{g^2(\mu)}{3\mu^3}\langle\bar{u}u\rangle_\mu\,,$$
$$\langle\chi_d\rangle=\langle s_{dd}\rangle=\frac{1}{\sqrt{3}}\langle s_0\rangle-\frac{1}{\sqrt{2}}\langle s_3\rangle+\frac{1}{\sqrt{6}}\langle s_8\rangle=\frac{g^2(\mu)}{3\mu^3}\langle\bar{d}d\rangle_\mu\,, \tag{8.11}$$
$$\langle\chi_s\rangle=\langle s_{ss}\rangle=\frac{1}{\sqrt{3}}\langle s_0\rangle-(\tfrac{2}{3})^{1/2}\langle s_8\rangle=\frac{g^2(\mu)}{3\mu^3}\langle\bar{s}s\rangle_\mu\,,$$

and analogously for $\langle\pi_a\rangle$ related to $\langle\bar{\Psi}_a i\gamma_5\Psi_a\rangle_\mu$.

We can now determine the values of the quark condensates from the stationary points of $V_1(\chi_a,\pi_a,\alpha_a)$ given in Eq. (8.10). In general, one should study the extrema of V_1 in the (χ,π) plane. However, we have proved in Appendix B that, if there is a minimum on the $\pi=0$ line, there are no other minima at $\pi\neq 0$. In Sec. IX we will show that, in the physically interesting case, such a minimum does exist. This will be proved by minimizing the effective potential at $\pi=0$ and verifying that the curvature is positive through the calculation of the pseudo-Goldstone masses. Therefore we have no spontaneous violation of C and CP according to a general result by Vafa and Witten.[41]

As examples we have plotted $V_1(\chi,\alpha)\equiv V_1(\chi,0,\alpha)$ as given in Eq. (8.10) as a function of χ in the massless case and for a value of the quark mass equal to 5.8 MeV (see Fig. 3). We see that in the massive case the degeneracy is removed and we have a minimum for a negative value of χ. In Fig. 4 we can see how the shape of the effective potential changes for an increasing value of the quark mass (118 MeV). The values 5.8 and 118 MeV correpond to the masses, renormalized at 1 GeV, of the quark u and s,

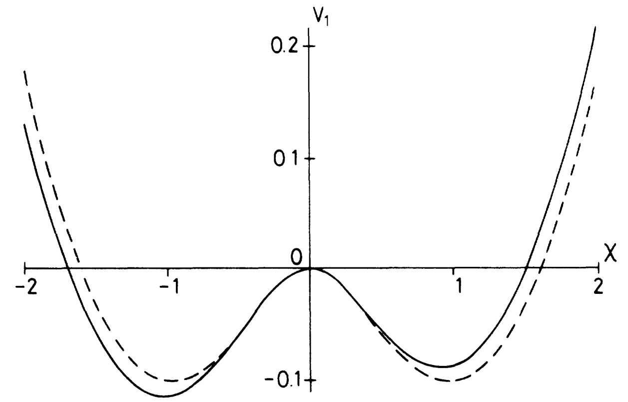

FIG. 3. Effective potential V_1 for QCD with three flavors as a function of χ for $m=0$ (dashed line) and for $m=5.8$ MeV (solid line). The curves are for a value of $\mu/\Lambda_{QCD}=1.11$.

respectively, as we will obtain in Sec. XI from the fit of the octet-meson masses. The boundness from below of our effective potential is evident in these graphs.

XI. PSEUDOSCALAR-MESON MASSES

We have seen in the previous sections that, if the coupling constant in the infrared region exceeds a critical value, dynamical breakdown of the chiral symmetry occurs in the massless case. According to the Goldstone theorem, we expect n^2-1 massless Goldstone bosons relative to the breaking of $SU(n)_L \otimes SU(n)_R$ to $SU(n)_{L+R}$ [as stated before we are neglecting the $U(1)_A$ problem]. These particles are represented by the composite fields $\mathbf{p}_{ab}$, $a,b=1,\ldots,n$, whose vacuum expectation values are related to the pseudoscalar condensates $\langle \bar{\Psi}_a i\gamma_5 \Psi_b \rangle_\mu$ by Eq. (5.7). When a mass matrix $\mathbf{m}$ for the quarks is allowed, the Goldstone bosons acquire mass and, in QCD with three flavors, they are the octet-pseudoscalar mesons. We will now calculate the masses of the pseudoscalar-octet mesons in our model.

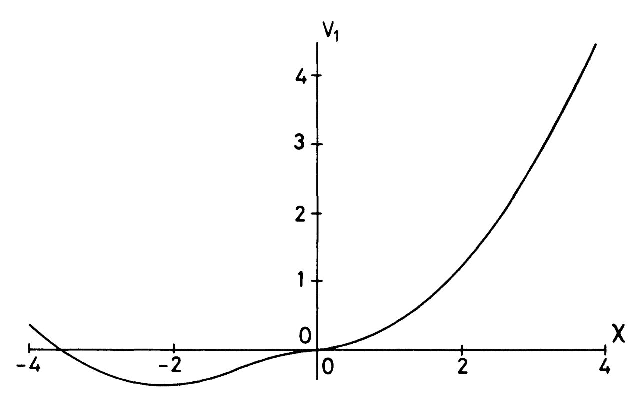

FIG. 4. Effective potential V_1 for QCD with three flavors as a function of χ for an $m=118$ MeV and $\mu/\Lambda_{QCD}=1.11$.

To better visualize our program let us concentrate, as an illustration, on the case of a single flavor, in which the potential V is a function of the scalar field χ, the pseudoscalar field π, and the quark mass m. First of all, one has to properly normalize the field π with respect to the canonical pseudo-Goldstone field ϕ_π:

$$\phi_\pi = b_\pi \pi \ . \tag{9.1}$$

The constant b_π can be obtained in terms of the pion decay constant f_π in the limit of zero four-momentum, using the soft pion theorems. Therefore, both the pseudoscalar masses and the decay coupling constants (see the next section) are evaluated in this approximation. Performing a chiral rotation in (9.1) and taking the vacuum expectation we get

$$b_\pi = -\frac{1}{\sqrt{2}} \frac{f_\pi}{\langle \chi \rangle} \ , \tag{9.2}$$

where, as usual, $\langle \chi \rangle$ stands for the value of χ at the minimum of V. To compute the mass of the pseudoscalar meson (the pseudo-Goldstone boson described by the field ϕ_π), one has to take the second derivative of the effective potential with respect to the field π, evaluate it at the minimum, and opportunately normalize it

$$M_\pi^2 = \left.\frac{d^2V}{d\phi_\pi^2}\right|_{\text{extr}} = \left.\frac{1}{b_\pi^2}\frac{d^2V}{d\pi^2}\right|_{\text{extr}} = \left. 2\frac{\langle \chi \rangle^2}{f_\pi^2}\frac{d^2V}{d\pi^2}\right|_{\text{extr}} \ . \tag{9.3}$$

Since $\langle \pi \rangle = 0$, the following relation holds:

$$\left.\frac{d^2V}{d\pi^2}\right|_{\text{extr}} = -\left.\frac{1}{\langle \chi \rangle}\frac{\partial V}{\partial \chi}\right|_{\text{extr}} \ , \tag{9.4}$$

where the derivative of V on the right-hand side means the derivative with respect to the explicit dependence on χ ($\partial V/\partial \chi$ is proportional to the explicit chiral-symmetry-breaking term). Equation (9.4) represents the Goldstone theorem. Substituting it in Eq. (9.3) we get

$$\begin{aligned} M_\pi^2 &= -\left.\frac{2}{f_\pi^2}\langle \chi \rangle \frac{\partial V}{\partial \chi}\right|_{\text{extr}} \\ &= -\left.\frac{2}{f_\pi^2}\langle \bar{\psi}\psi \rangle_\mu \frac{g^2(\mu)}{3\mu^3}\frac{\partial V}{\partial \chi}\right|_{\text{extr}} \ , \end{aligned} \tag{9.5}$$

where we have used the relation between the scalar field at the minimum and the scalar condensate. Taking into account the normalization condition (8.8), Eq. (9.5) reproduces the Adler-Dashen formula in the small-mass limit

$$M_\pi^2 \mid_{m\to 0} \sim -2m\frac{1}{f_\pi^2}\langle \bar{\psi}\psi \rangle_\mu \ . \tag{9.6}$$

This justifies our renormalization condition for the effective potential.

Let us now apply this procedure to the general case of QCD with three flavors in order to obtain an expression for the masses of the pseudoscalar-octet mesons.[23] We recall that the effective potential we have calculated depends on the standard parameters of QCD, Λ_{QCD}, m_u, m_d, m_s, and on the mass scale μ. So we will proceed in the following way. We will derive a convenient generalization of Eq. (9.5) which will allow us to compute directly the pseudo-Goldstone masses in terms of $\Lambda_{\rm QCD}$, m_u, m_d, m_s, and μ and of the decay coupling constants f_{ij}, $i,j=1,\dots,8$. Then we will derive within our formalism an expression for f_{ij} and, finally, we will determine the parameters of our theory from the experimental data. Having a system of coupled equations, this determination will be performed numerically.

First of all we need to normalize our dynamical variables p_i, $i=1,\dots,8$ [remember that we are neglecting the $U(1)_A$ problem] defined in (8.5). Let us call $\bar{p}_i$ the fields which diagonalize the matrix of the second derivatives of the effective potential given in Eq. (8.3). Then

$$\bar{p}_i=\sum_{j=1}^{8} p_j \mathbf{a}_{ji} \ , \tag{9.7}$$

where $\mathbf{a}$ is an orthogonal 8×8 matrix which is nondiagonal only in the 3-8 sector. The coefficients b_i which relate the fields $\bar{p}_i$ to the physical fields ϕ_i,

$$\phi_i=b_i\bar{p}_i=b_i\sum_{j=1}^{8} p_j \mathbf{a}_{ji} \ , \tag{9.8}$$

can be determined through standard current-algebra arguments. In fact, let us remind the standard relations of pion physics, for three quark flavors:

$$if_{ij}=\langle 0\,|\,[Q_5^i,\phi_j(0)]\,|\,0\rangle, \quad i,j=1,\dots,8 \ , \tag{9.9}$$

where ϕ_j are the canonical pseudo-Goldstone boson fields related to the p_j by Eq. (9.8),

$$Q_5^i=\int d^3\mathbf{x}\,\Psi^\dagger(x)\gamma_5\frac{\lambda_i}{2}\Psi(x) \tag{9.10}$$

are the axial charges, and f_{ij} are the decay coupling constants for the meson octet defined by

$$\langle 0\,|\,J^i_{\mu 5}(x)\,|\,\pi_j(p)\rangle=ip_\mu f_{ij}e^{-ipx}, \quad i,j=1,\dots,8 \ . \tag{9.11}$$

Here $J^i_{\mu 5}$ are the axial-vector currents and the $|\,\pi_j(p)\rangle$ are the meson-octet states with four-momentum p_μ which satisfy Eq. (1.6). Combining Eq. (8.5) with (5.7) we find that the fields p_i are related to the pseudoscalar fermion condensates by the relation

$$p_i=F\left|\bar{\Psi}i\gamma_5\frac{\lambda_i}{\sqrt{2}}\Psi\right| \ , \tag{9.12}$$

where

$$F=\frac{g^2(\mu)}{3\mu^3} \tag{9.13}$$

is the usual dimensional normalization factor. Substituting (9.12) in (9.8) and then in (9.9), we find

$$f_{ij}=-\frac{\sqrt{2}}{3}\mathbf{a}_{ij}b_jF\,{\rm tr}\langle\bar{\Psi}\Psi\rangle_\mu - \sum_{k,l=1}^{8}\mathbf{a}_{kj}b_jd_{ikl}F\,{\rm tr}\left\langle\bar{\Psi}\frac{\lambda_l}{\sqrt{2}}\Psi\right\rangle_\mu \ . \tag{9.14}$$

So, for the sector with charge or strangeness different from zero ($\mathbf{a}_{ij}=\delta_{ij}$), one gets (remember $\langle\chi_a\rangle=F\langle\bar{\Psi}_a\Psi_a\rangle_\mu$, $a=u,d,s$)

$$\begin{aligned} b_1=b_2&=-\frac{\sqrt{2}f_{\pi^\pm}}{\langle\chi_u\rangle+\langle\chi_d\rangle}=b_{\pi^\pm} \ , \\ b_4=b_5&=-\frac{\sqrt{2}f_{K^\pm}}{\langle\chi_u\rangle+\langle\chi_s\rangle}=b_{K^\pm} \ , \\ b_6=b_7&=-\frac{\sqrt{2}f_{K^0,\bar{K}^0}}{\langle\chi_d\rangle+\langle\chi_s\rangle}=b_{K^0,\bar{K}^0} \ , \end{aligned} \tag{9.15}$$

where, according to the usual conventions, we have set

$$f_{11}=f_{22}=f_{\pi^\pm}, \quad f_{44}=f_{55}=f_{K^\pm}, \quad f_{66}=f_{77}=f_{K^0,\bar{K}^0} \ . \tag{9.16}$$

For the sector with $Q=S=0$, one obtains

$$f_{\mu\nu}=\sum_{\rho=3,8}\mathcal{A}_{\mu\rho}\mathbf{a}_{\rho\nu}b_\nu, \quad \mu,\nu=3,8 \ , \tag{9.17}$$

with

$$\begin{aligned} \mathcal{A}_{33}&=-\frac{1}{\sqrt{2}}(\langle\chi_\mu\rangle+\langle\chi_d\rangle) \ , \\ \mathcal{A}_{38}=\mathcal{A}_{83}&=-\frac{1}{\sqrt{6}}(\langle\chi_u\rangle-\langle\chi_d\rangle) \ , \\ \mathcal{A}_{88}&=-\frac{1}{3\sqrt{2}}(\langle\chi_u\rangle+\langle\chi_d\rangle+4\langle\chi_s\rangle) \ . \end{aligned} \tag{9.18}$$

Summarizing, the coefficients b_i can be expressed in terms of the decay constants of the pseudoscalar mesons and the values of the fields χ_a ($a=u,d,s$), at the minimum of the effective potential, but, due to the mixing in the 3-8 sector, one has also to take into account the matrix $\mathbf{a}$ which transforms the fields p_i into the mass eigenstates [notice that in the SU(2)-symmetric case ($m_u=m_d$), $\mathbf{a}_{ij}=\delta_{ij}$].

Let us now derive an expression for the masses of the pseudoscalar-octet mesons by multiplying the second derivatives of the effective potential with respect to the pseudoscalar fields by the appropriate factors relating the physical fields ϕ_i to our variables p_i [we do not compute the η' mass since we have not considered the effects of the $U(1)_A$ anomaly]. The mass matrix is given by

$$\begin{aligned} M^2_{ij}=M^2_i\delta_{ij}&=\left.\frac{d^2V}{d\phi_i d\phi_j}\right|_{\rm extr} \\ &=\frac{1}{b_ib_j}\sum_{k,l=1}^{8}\mathbf{a}_{ki}\mathbf{a}_{lj}\left.\frac{d^2V}{dp_k dp_l}\right|_{\rm extr} \\ &\equiv\frac{1}{b_ib_j}\sum_{k,l=1}^{8}\mathbf{a}_{ki}\mathbf{a}_{lj}V_{kl} \quad i,j=1,\dots,8 \ . \end{aligned} \tag{9.19}$$

A direct calculation of the second derivatives, evaluated at the extremum of the effective potential, leads to the result

$$
\begin{aligned}
V_{11}=V_{22}&=\frac{3\mu^4}{4\pi^2}\left[2cA_1-\frac{1}{\mu^2}\int_0^\infty dy\,y\left[\frac{x_u}{x_u+x_d}\frac{1}{y+x_u^2}+\frac{x_d}{x_u+x_d}\frac{1}{y+x_d^2}\right]f_2(y)^2\right],\\
V_{33}&=\frac{3\mu^4}{4\pi^2}\left[2cA_1-\frac{1}{2\mu^2}\int_0^\infty dy\,y\left[\frac{1}{y+x_u^2}+\frac{1}{y+x_d^2}\right]f_2(y)^2\right],\\
V_{44}=V_{55}&=\frac{3\mu^4}{4\pi^2}\left[2cA_1-\frac{1}{\mu^2}\int_0^\infty dy\,y\left[\frac{x_u}{x_u+x_s}\frac{1}{y+x_u^2}+\frac{x_s}{x_u+x_s}\frac{1}{y+x_s^2}\right]f_2(y)^2\right],\\
V_{66}=V_{77}&=\frac{3\mu^4}{4\pi^2}\left[2cA_1-\frac{1}{\mu^2}\int_0^\infty dy\,y\left[\frac{x_d}{x_d+x_s}\frac{1}{y+x_d^2}+\frac{x_s}{x_d+x_s}\frac{1}{y+x_s^2}\right]f_2(y)^2\right],\\
V_{88}&=\frac{3\mu^4}{4\pi^2}\left[2cA_1-\frac{1}{\mu^2}\int_0^\infty dy\,y\left[\frac{1}{6}\frac{1}{y+x_u^2}+\frac{1}{6}\frac{1}{y+x_d^2}+\frac{2}{3}\frac{1}{y+x_s^2}\right]f_2(y)^2\right],\\
V_{38}=V_{83}&=\frac{3\mu^4}{4\pi^2}\left[\frac{1}{2\sqrt{3}}\frac{1}{\mu^2}\int_0^\infty dy\,y\left[\frac{1}{y+x_d^2}-\frac{1}{y+x_u^2}\right]f_2(y)^2\right],
\end{aligned}
\tag{9.20}
$$

where c, $A_1, f_2(y)$ are defined in Eq. (5.14) and x_a $(a=u,d,s)$ are the eigenvalues, evaluated at the extremum of V, of the matrix $\mathbf{x}$ defined in (5.14).

We shall now derive a more physically transparent expression for the elements V_{kl}. As stated before, when one sets the charged fields equal to zero, the effective potential decomposes in the sum of three contributions, one for each flavor, as showed in Eq. (8.7). For the further developments, it is convenient to rewrite Eq. (8.7) in the following form:

$$
\begin{aligned}
V&=\frac{3\mu^4}{4\pi^2}\sum_{a=u,d,s}V_1(\phi_a^2,\chi_a,m_a),\\
V_1(\phi^2,\chi,m)&=cA_1\phi^2+m\chi(A_2+\delta z_f)-\tfrac{1}{2}\int_0^\infty dy\,y\ln\left[y+\frac{m^2}{\mu^2}f_1^2(y)+\phi^2\frac{f_2^2(y)}{\mu^2}+2\chi\frac{m}{\mu}\frac{f_1(y)}{\mu}f_2(y)\right],
\end{aligned}
\tag{9.21}
$$

where $\phi^2=\chi^2+\pi^2$, $A_2+\delta z_f$ is given in (8.9) and c, $A_1, f_1(y), f_2(y)$ are given in (5.14). The extremum condition is

$$
\frac{dV_1}{d\chi}=2\chi\frac{\partial V_1}{\partial\phi^2}+\frac{\partial V_1}{\partial\chi}=0,
\tag{9.22}
$$

that is,

$$
\left.\frac{\partial V_1}{\partial\phi^2}\right|_{\rm extr}=-\frac{1}{2\langle\chi\rangle}\left.\frac{\partial V_1}{\partial\chi}\right|_{\rm extr}=cA_1-\frac{1}{2\mu^2}\int_0^\infty dy\,y\frac{f_2^2(y)}{y+[mf_1(y)/\mu+\langle\chi\rangle f_2(y)/\mu]^2}.
\tag{9.23}
$$

So, for the various flavors $a=u,d,s$ one gets

$$
-\frac{1}{\langle\chi_a\rangle}\left.\frac{\partial V}{\partial\chi_a}\right|_{\rm extr}=\frac{3\mu^4}{4\pi^2}\left[2cA_1-\frac{1}{\mu^2}\int_0^\infty dy\,y\frac{f_2^2(y)}{y+x_a^2}\right].
\tag{9.24}
$$

Let us substitute Eq. (9.24) in (9.20). For the 3-8 sector we get simply

$$
\begin{aligned}
V_{33}&=-\frac{1}{2\langle\chi_u\rangle}\left.\frac{\partial V}{\partial\chi_u}\right|_{\rm extr}-\frac{1}{2\langle\chi_d\rangle}\left.\frac{\partial V}{\partial\chi_d}\right|_{\rm extr},\\
V_{88}&=-\frac{1}{6\langle\chi_u\rangle}\left.\frac{\partial V}{\partial\chi_u}\right|_{\rm extr}-\frac{1}{6\langle\chi_d\rangle}\left.\frac{\partial V}{\partial\chi_d}\right|_{\rm extr}-\frac{2}{3\langle\chi_s\rangle}\left.\frac{\partial V}{\partial\chi_s}\right|_{\rm extr},\\
V_{38}=V_{83}&=\frac{1}{2\sqrt{3}}\frac{1}{\langle\chi_d\rangle}\left.\frac{\partial V}{\partial\chi_d}\right|_{\rm extr}-\frac{1}{2\sqrt{3}}\frac{1}{\langle\chi_u\rangle}\left.\frac{\partial V}{\partial\chi_u}\right|_{\rm extr}.
\end{aligned}
\tag{9.25}
$$

Let us now examine V_{11}, as given in Eq. (9.20). We can rewrite it in the form

$$
\begin{aligned}
V_{11}&=\frac{3\mu^4}{4\pi^2}\left[2cA_1-\frac{1}{\mu^2}\int_0^\infty dy\,y\left[\frac{1}{y+x_u^2}-\frac{x_d}{x_u+x_d}\frac{1}{y+x_u^2}+\frac{x_d}{x_u+x_d}\frac{1}{y+x_d^2}\right]f_2^2(y)\right]\\
&=-\frac{1}{\langle\chi_u\rangle}\left.\frac{\partial V}{\partial\chi_u}\right|_{\rm extr}-\frac{3\mu^2}{4\pi^2}\int_0^\infty dy\,y\frac{x_d(x_u-x_d)}{(y+x_u^2)(y+x_d^2)}f_2^2(y)
\end{aligned}
\tag{9.26}
$$

or, analogously, we can obtain

$$V_{11}=-\frac{1}{\langle\chi_d\rangle}\frac{\partial V}{\partial\chi_d}\bigg|_{\rm extr}-\frac{3\mu^2}{4\pi^2}\int_0^\infty dy\,y\frac{x_u(x_d-x_u)}{(y+x_u^2)(y+x_d^2)}f_2^2(y)\ . \tag{9.27}$$

So we can write an expression ud symmetric for V_{11} by summing Eqs. (9.26) and (9.27) and dividing by 2. The same arguments apply to V_{44} and V_{66} and the result is

$$\begin{aligned}V_{11}=V_{22}&=-\frac{1}{2\langle\chi_u\rangle}\frac{\partial V}{\partial\chi_u}\bigg|_{\rm extr}-\frac{1}{2\langle\chi_d\rangle}\frac{\partial V}{\partial\chi_d}\bigg|_{\rm extr}+V_{ud}\ ,\\ V_{44}=V_{55}&=-\frac{1}{2\langle\chi_u\rangle}\frac{\partial V}{\partial\chi_u}\bigg|_{\rm extr}-\frac{1}{2\langle\chi_s\rangle}\frac{\partial V}{\partial\chi_s}\bigg|_{\rm extr}+V_{us}\ ,\\ V_{66}=V_{77}&=-\frac{1}{2\langle\chi_d\rangle}\frac{\partial V}{\partial\chi_d}\bigg|_{\rm extra}-\frac{1}{2\langle\chi_s\rangle}\frac{\partial V}{\partial\chi_s}\bigg|_{\rm extra}+V_{ds}\ .\end{aligned} \tag{9.28}$$

We recall here that the partial derivatives are with respect to the explicit dependence on χ_a, and

$$V_{ab}=\frac{3\mu^2}{8\pi^2}\int_0^\infty dy\,y\frac{(x_a-x_b)^2}{(y+x_a^2)(y+x_b^2)}f_2^2(y)\ ,\quad a=u,d,s\ . \tag{9.29}$$

Inserting (9.28) in (9.19), we obtain the extension of the Goldstone theorem, as expressed in (9.4) to the case of three flavors. In fact in Eq. (9.28) the dependence on the explicit symmetry-breaking part of the effective potential has been isolated. This means that, in the $m\to 0$ limit, the V_{kl} are all naturally equal to zero independently of the values of the fields χ_a at the minimum. This is not clear if one uses directly Eq. (9.20) for evaluating V_{kl}, with obvious problems from a computational point of view.

As an example, let us consider the SU(2)-symmetric case given by $m_u=m_d=m$, and no strange quark. Clearly in this case $\langle\chi_u\rangle=\langle\chi_d\rangle=\langle\chi\rangle$ and $\mathbf{a}_{ij}=\delta_{ij}$. Then, using the first of Eq. (9.15), we obtain, for the pion

$$M_\pi^2=-\frac{1}{b_\pi^2}\frac{1}{\langle\chi\rangle}\frac{\partial V}{\partial\chi}\bigg|_{\rm extr}=-\frac{2}{f_\pi^2}\langle\chi\rangle\frac{\partial V}{\partial\chi}\bigg|_{\rm extr} \tag{9.30}$$

and, with the normalization condition (8.8), we get, for $m\to 0$,

$$M_\pi^2\,|_{m\to 0}\sim -2m\frac{1}{f_\pi^2}\langle\bar\psi\psi\rangle_\mu\ , \tag{9.31}$$

where $\langle\bar\psi\psi\rangle_\mu=\langle\bar u u\rangle_\mu=\langle\bar d d\rangle_\mu$. Equation (9.31) is the standard Adler-Dashen relation.

As already observed, there is mixing between the components along the directions 3 and 8 [see (9.25)]. This is, of course, expected and we have to diagonalize the matrix of the second derivatives of the effective potential in order to get the masses of the physical π^0 and η_8. [The η_8 is not the true physical particle, but the result of undoing the mixing between the members of the octet and of the singlet of SU(3) which leads to the physical η and η'.] The explicit calculation, performed up to the order

$$\left[\frac{m_u-m_d}{m_s}\right]^2 ,$$

gives for the 3-8 sector

$$\begin{bmatrix}a_{33} & a_{38}\\ a_{83} & a_{88}\end{bmatrix}=\frac{1}{N}\begin{bmatrix}1 & -\dfrac{V_{38}}{V_{33}-V_{88}}\\ \dfrac{V_{38}}{V_{33}-V_{88}} & 1\end{bmatrix} \tag{9.32}$$

with

$$N=\left[1+\frac{V_{38}^2}{(V_{33}-V_{88})^2}\right]^{1/2} .$$

X. PSEUDOSCALAR-MESON DECAY CONSTANTS

In our context there are at least two procedures for evaluating the decay constants of the pseudoscalar mesons: (i) to evaluate the couplings of the fields ϕ_i to the axial-vector currents $j_{5\mu}^i$ and (ii) to evaluate the residue at the pole of the meson propagator. For (i) we need to know the meson-quark-antiquark vertex, while for (ii) the knowledge of the bound-state wave function of the pseudoscalar fields is required. These quantities can be directly determined from the effective action if we are able to extend our previous calculations from constant fields $\mathbf{s}$ and $\mathbf{p}$ to arbitrary functions of space-time. In fact from the effective potential one can extract amplitudes for composite operators of vanishing four-momentum, but, to derive amplitudes of nonzero momentum, one has to allow for space-time dependence in the composite fields.

Our effective action Γ [see (2.40)] consists of two terms: the Γ_2 term, which in our approximation is a quadratic expression in the composite fields [so the relation (3.39) holds], and the term with the logarithm, which can be interpreted as the sum of all the graphs with a fermionic loop. Therefore, in order to determine the meson-quark-antiquark vertex, we only need to generalize the logarithm term of Γ to the case of local fields.

Let us recall the structure of the self-energy for constant fields $\mathbf{s}$ and $\mathbf{p}$ [Eq. (5.2)]:

$$-\frac{\delta\Gamma_2}{\delta\mathbf{S}(p)}=\mathbf{\Sigma}(p^2)=\mathbf{m}(\mu)f_1(p^2)-\mathbf{m}_0(\Lambda)+(\mathbf{s}+i\gamma_5\mathbf{p})f_2(p^2) \tag{10.1}$$

with $f_1(p^2)$ and $f_2(p^2)$ given in (5.3) and (5.4), respectively. For the sake of simplicity, from now on we will omit the square in the arguments of Σ, f_1, and f_2. The operator we need to generalize is

$$\mathbf{S}_0^{-1}+\frac{\delta\Gamma_2}{\delta\mathbf{S}} \quad \text{with } \mathbf{S}_0^{-1}(p)=i\hat{p}-\mathbf{m}_0(\Lambda)$$

in the Landau gauge. We will use a Weyl symmetrization prescription,[24] that is,

$$\left\langle x\left|\mathbf{S}_0^{-1}+\frac{\delta\Gamma}{\delta\mathbf{S}}\right|y\right\rangle=\langle x\mid i\hat{p}-\mathbf{m}(\mu)f_1(p)-\tfrac{1}{2}[\mathbf{s}(\mathbf{x})+i\gamma_5\mathbf{p}(\mathbf{x}),f_2(\mathbf{p})]_+\mid y\rangle, \tag{10.2}$$

where x_μ and p_μ have the canonical definition

$$\mathbf{x}_\mu\mid x\rangle=x_\mu\mid x\rangle,\quad \mathbf{p}_\mu\mid p\rangle=p_\mu\mid p\rangle\ ,[\mathbf{x}_\mu,\mathbf{p}_\nu]=-ig_{\mu\nu}\ . \tag{10.3}$$

Notice that this prescription maintains the charge-conjugation property of the fermion propagator

$$\mathcal{C}\left[\left[\mathbf{S}_0^{-1}+\frac{\delta\Gamma_2}{\delta S}\right](x,y)\right]\mathcal{C}^{-1}=\left[\mathbf{S}_0^{-1}+\frac{\delta\Gamma_2}{\delta\mathbf{S}}\right]^T(y,x)\ . \tag{10.4}$$

We shall also see that the Weyl symmetrization prescription leads to a meson-quark-antiquark vertex which is consistent with the axial Ward identity.

In order to evaluate $\mathrm{Tr}\ln(\mathbf{S}_0^{-1}+\delta\Gamma_2/\delta\mathbf{S})$, we translate the composite fields s and **p** with respect to their value at the minimum of the effective potential. Let us define

$$\langle\mathbf{s}\rangle=\begin{bmatrix}\langle\chi_u\rangle & 0 & 0\\ 0 & \langle\chi_d\rangle & 0\\ 0 & 0 & \langle\chi_s\rangle\end{bmatrix} \tag{10.5}$$

and introduce the operators

$$(\bar{\mathbf{S}})^{-1}(p)=i\hat{p}-\mathbf{m}(\mu)f_1(p)-\langle\mathbf{s}\rangle f_2(p)\ ,$$
$$\mathbf{\Phi}(x)=\mathbf{s}(x)+i\gamma_5\mathbf{p}(x)-\langle\mathbf{s}\rangle\ , \tag{10.6}$$
$$\mathbf{R}=\tfrac{1}{2}[\mathbf{\Phi}(\mathbf{x}),f_2(\mathbf{p})]_+\ ,$$

where $\bar{\mathbf{S}}$ can be interpreted as the quark propagator with mass $\langle\mathbf{s}\rangle f_2(p)$, dynamically generated. We thus obtain

$$\mathrm{Tr}\ln\left[\mathbf{S}_0^{-1}+\frac{\delta\Gamma_2}{\delta\mathbf{S}}\right]=\mathrm{Tr}\ln(\bar{\mathbf{S}})^{-1}+\mathrm{Tr}\ln(1-\bar{\mathbf{S}}\mathbf{R})$$
$$=\mathrm{Tr}\ln(\bar{\mathbf{S}})^{-1}-\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}(\bar{\mathbf{S}}\mathbf{R})^n\ . \tag{10.7}$$

The last trace can be calculated by inserting intermediate eigenstates of x_μ:

$$\mathrm{Tr}(\bar{\mathbf{S}}\mathbf{R})^n=\int\prod_{i=1}^{2n}d^4x_i\,\mathrm{tr}[\langle x_1\mid\bar{\mathbf{S}}\mid x_2\rangle\langle x_2\mid\mathbf{R}\mid x_3\rangle\times\cdots\langle x_{2n-1}\mid\bar{\mathbf{S}}\mid x_{2n}\rangle\times\langle x_{2n}\mid\mathbf{R}\mid x_1\rangle]\ . \tag{10.8}$$

From the definition of the operator **R**, introducing

$$f_2(x-y)\equiv\langle x\mid f_2(\mathbf{p})\mid y\rangle\ , \tag{10.9}$$

we get

$$\langle x_{2i}\mid\mathbf{R}\mid x_{2i+1}\rangle=\tfrac{1}{2}[\mathbf{\Phi}(x_{2i})f_2(x_{2i}-x_{2i+1})+f_2(x_{2i}-x_{2i+1})\mathbf{\Phi}(x_{2i+1})]$$
$$=\tfrac{1}{2}\int d^4z_i[\delta^4(z_i-x_{2i})+\delta^4(z_i-x_{2i+1})]f_2(x_{2i}-x_{2i+1})\mathbf{\Phi}(z_i)$$
$$=\int d^4z_i\,V(x_{2i},x_{2i+1};z_i)\mathbf{\Phi}(z_i)\ , \tag{10.10}$$

where we have defined

$$V(x,y;z)\equiv\tfrac{1}{2}[\delta^4(z-x)+\delta^4(z-y)]f_2(x-y)\ . \tag{10.11}$$

So, using

$$\bar{\mathbf{S}}(x-y)\equiv\langle x\mid\bar{\mathbf{S}}(\mathbf{p})\mid y\rangle\ , \tag{10.12}$$

we finally obtain

$$\mathrm{Tr}(\bar{\mathbf{S}}\mathbf{R})^n=\int\prod_{i=1}^{2n}d^4x_i\prod_{j=1}^{n}d^4z_j\bar{\mathbf{S}}(x_1-x_2)V(x_2,x_3;z_1)\mathbf{\Phi}(z_1)\times\bar{\mathbf{S}}(x_3-x_4)V(x_4,x_5;z_2)\mathbf{\Phi}(z_2)\times\cdots\bar{\mathbf{S}}(x_{2n-1}-x_{2n})V(x_{2n},x_1;z_n)\mathbf{\Phi}(z_n)\ . \tag{10.13}$$

From Eq. (10.13) it is clear that the operator $\mathrm{Tr}(\bar{\mathbf{S}}\mathbf{R})^n$ corresponds to n bound-state fields $\mathbf{\Phi}$ emerging from a fermionic loop calculated with fermion propagator $\bar{\mathbf{S}}$ as illustrated in Fig. 5. The vertex $V(x,y;z)$ has the following expression in momentum space:

$$V(x,y;z)=\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}V(p,q)\times e^{-ipx+iqy+i(p-q)z} ,$$
$$V(p,q)=\tfrac{1}{2}[f_2(p)+f_2(q)] , \qquad (10.14)$$

where the momenta are as in Fig. 6.

Actually we are interested in the pseudoscalar bound-state vertices. We can read their expressions directly from Eqs. (10.13) and (10.14):

$$\mathbf{V}_5^j(p,q)=i\gamma_5\frac{\lambda^j}{2\sqrt{2}}[f_2(p)+f_2(q)] . \qquad (10.15)$$

This is the effective coupling $g_{p\bar{q}q}$. To get the effective coupling $g_{\phi\bar{q}q}$ for physical mesons, we must properly normalize the fields p_i by using Eq. (9.8). The result is

$$\mathbf{G}_5^i(p,q)=\frac{1}{b_i}\sum_{j=1}^{8}\mathbf{V}_5^j(p,q)\mathbf{a}_{ji}$$
$$=i\gamma_5\frac{1}{b_i}\sum_{j=1}^{8}\frac{\lambda^j}{2\sqrt{2}}\mathbf{a}_{ji}[f_2(p)+f_2(q)] . \qquad (10.16)$$

The G_5^i are the physical pseudoscalar-meson–quark–antiquark vertices.

We will now prove that the expression (10.16) is exactly what one needs in order to satisfy the axial-vector Ward identities relating the proper axial-vector vertex functions $\mathbf{\Gamma}_{5\mu}^i$ and the fermion propagator $\mathbf{S}$ (Ref. 42). Let us consider the case of massless quarks. Then, the axial-vector Ward identities have the following form [see Eq. (4.1)] :

$$iq^\mu\mathbf{\Gamma}_{5\mu}^i(p+q,p)=\gamma_5\frac{\lambda^i}{2}S^{-1}(p)+S^{-1}(p+q)\gamma_5\frac{\lambda^i}{2} \qquad (10.17)$$

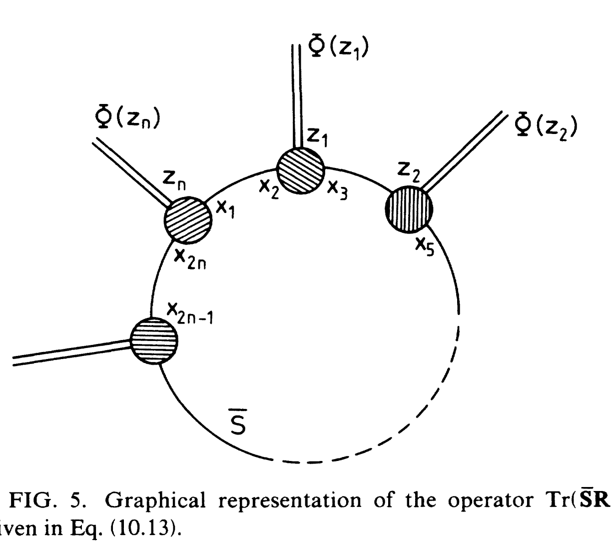

FIG. 5. Graphical representation of the operator $\mathrm{Tr}(\bar{\mathbf{S}}\mathbf{R})^n$ given in Eq. (10.13).

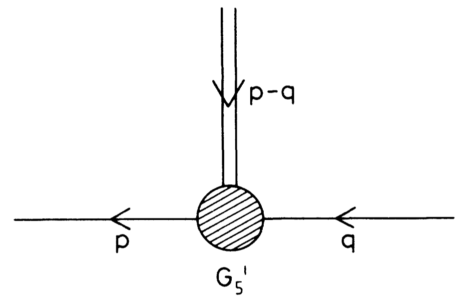

FIG. 6. The pseudoscalar bound-state vertex with two fermion lines of momenta p and q whereas $p-q$ is a bound-state line.

(remember that in the massless case the fermion propagator is proportional to the unit matrix in the flavor space, since the condensates in the self-energy all have the same value for each flavor). Let us substitute the expression for the fermion propagator as given by The Schwinger-Dyson equation in the massless case

$$S^{-1}(p)=i\hat{p}-\bar{\Sigma}(p) , \qquad (10.18)$$

where $\bar{\Sigma}$ is the self-energy evaluated at the minimum of the effective potential. We get

$$iq^\mu\mathbf{\Gamma}_{5\mu}^i(p+q,p)=i\hat{q}\gamma_5\frac{\lambda^i}{2}-\gamma_5\frac{\lambda^i}{2}[\bar{\Sigma}(p+q)+\bar{\Sigma}(p)] . \qquad (10.19)$$

From Eq. (10.19) it follows that one can have a nonzero dynamical quark mass if and only if $\mathbf{\Gamma}_{5\mu}^i$ has a pseudoscalar pole at $q^2=0$ (Goldstone pole) with residue proportional to the pion decay constant f_π. Then we can write

$$\mathbf{\Gamma}_{5\mu}^i(p+q,p)=\frac{\lambda^i}{2}\gamma_\mu\gamma_5-f_\pi\mathbf{G}_5^i(p+q,p)\frac{q_\mu}{q^2}+\tilde{\mathbf{\Gamma}}{}_{5\mu}^i(p+q,p) . \qquad (10.20)$$

Here $\mathbf{G}_5^i(p+q,p)$ represents the proper pseudoscalar-meson–quark–antiquark vertex function and $\tilde{\mathbf{\Gamma}}{}_{5\mu}^i(p+q,p)$ is a term regular at $q^2=0$, which can be ignored in the approximation we are considering, since it is of order g^2. Finally, the comparison of Eq. (10.19) with Eq. (10.20) gives the following expression for pion-quark-antiquark vertex function:

$$\mathbf{G}_5^i(p+q,p)=-i\gamma_5\frac{\lambda^i}{2}\frac{1}{f_\pi}[\bar{\Sigma}(p+q)+\bar{\Sigma}(p)] \qquad (10.21)$$

which is in complete agreement with (10.16). In fact, in the approximation of massless quarks, we have

$$\mathbf{a}_{ij}=\delta_{ij},\quad b_1=b_2=-\frac{1}{\sqrt{2}}\frac{f_\pi}{\langle\chi\rangle} ,\bar{\Sigma}(p)=\langle\chi\rangle f_2(p) . \qquad (10.22)$$

It is worthwhile to stress that the result (10.16) crucially depends on the symmetrization prescription used.

In practice, what we have shown here is that our effective action reproduces, at one-loop level, the results of the dynamical perturbation theory (DPT) introduced some time ago by Pagels and Stokar.[42]

At this point, having derived the meson-quark-antiquark vertex function, it is possible to obtain an expression for the meson decay constants f_{ij} by directly evaluating the coupling of the fields ϕ_j to the axial-vector currents $J^i_{5\mu}$. From Fig. 7 one gets (remember that we are working in the Euclidean space)

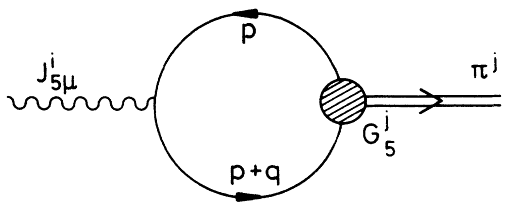

FIG. 7. Graphical representation of the matrix element $\langle 0 | J^i_{5\mu}(0) | \pi_j(q) \rangle$.

$$\langle 0 | J^i_{5\mu}(0) | \pi_j(q) \rangle = i q_\mu f_{ij} = 3 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[i\gamma_\mu \gamma_5 \frac{\lambda_i}{2} \mathbf{S}(p+q) \mathbf{G}^j_5(p+q,p) \mathbf{S}(p) \right] . \tag{10.23}$$

Substituting the expressions for the quark propagator and for the meson-quark-antiquark vertex, we get

$$i q_\mu f_{ij} = \frac{3}{\sqrt{2} b_j} \sum_{k=1}^{8} \mathbf{a}_{kj} \int \frac{d^4p}{(2\pi)^4} [f_1(p+q) + f_1(p)] \mathrm{Tr}\left[i\gamma_\mu \gamma_5 \frac{\lambda_i}{2} [i(\hat{p}+\hat{q}) - \bar{\Sigma}(p+q)]^{-1} i\gamma_5 \frac{\lambda_i}{2} [i\hat{p} - \bar{\Sigma}(p)]^{-1} \right] , \tag{10.24}$$

where

$$\bar{\Sigma}(p) = \begin{bmatrix} \bar{\Sigma}_u(p) & 0 & 0 \\ 0 & \bar{\Sigma}_d(p) & 0 \\ 0 & 0 & \bar{\Sigma}_s(p) \end{bmatrix} \tag{10.25}$$

with $\bar{\Sigma}_a(p) = m_a f_1(p) + \langle \chi_a \rangle f_2(p)$, $a = u,d,s$. Let us evaluate the traces over the spinor and the flavor indices in (10.24):

$$q_\mu f_{ij} = \frac{3}{\sqrt{2} b_j} \sum_{k=1}^{8} \mathbf{a}_{kj} \sum_{a,b=u,d,s} \mathbf{c}^{ik}_{ab} \int \frac{d^4p}{(2\pi)^4} [f_2(p+q) + f_2(p)] \frac{p_\mu [\bar{\Sigma}_a(p+q) - \bar{\Sigma}_b(p)] - q_\mu \bar{\Sigma}_b(p)}{[(p+q)^2 + \bar{\Sigma}^2_a(p+q)][p^2 + \bar{\Sigma}^2_b(p)]} , \tag{10.26}$$

where $\mathbf{c}^{ik}$ are 3×3 matrices

$$\mathbf{c}^{11} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{c}^{22}, \quad \mathbf{c}^{44} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \mathbf{c}^{55}, \quad \mathbf{c}^{66} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \mathbf{c}^{77} ,$$
$$\mathbf{c}^{33} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{c}^{88} = \frac{1}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad \mathbf{c}^{38} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{c}^{83} . \tag{10.27}$$

To obtain an expression for f_{ij}, we will differentiate with respect to q_ν in Eq. (10.26), and, according to the soft-pion limit, we will take $q=0$. In this way we get

$$f_{ij} = \frac{1}{b_j} \sum_{k=1}^{8} \mathbf{M}_{ik} \mathbf{a}_{kj} \tag{10.28}$$

with

$$g_{\mu\nu} \mathbf{M}_{ik} = \frac{3}{\sqrt{2}} \sum_{a,b=u,d,s} \mathbf{c}^{ik}_{ab} \int \frac{d^4p}{(2\pi)^4} \left[\frac{\partial f_2(p)}{\partial p_\nu} p_\mu \frac{\bar{\Sigma}_a(p) - \bar{\Sigma}_b(p)}{[p^2 + \bar{\Sigma}^2_a(p)][p^2 + \bar{\Sigma}^2_b(p)]} + 2 f_2(p) \frac{p_\mu \dfrac{\partial \bar{\Sigma}_a(p)}{\partial p_\nu} - g_{\mu\nu} \bar{\Sigma}_b(p)}{[p^2 + \bar{\Sigma}^2_a(p)][p^2 + \bar{\Sigma}^2_b(p)]} \right.$$
$$\left. - 2 f_2(p) \frac{\left[2 p_\mu p_\nu + \dfrac{\partial \bar{\Sigma}^2_a(p)}{\partial p_\nu} p_\mu \right] [\bar{\Sigma}_a(p) - \bar{\Sigma}_b(p)]}{[p^2 + \bar{\Sigma}^2_a(p)]^2 [p^2 + \bar{\Sigma}^2_b(p)]} \right] . \tag{10.29}$$

For the sector with charge and strangeness different from zero, inserting (9.15) in (10.28), one finds

$$f^2_{\pi^\pm} = -\frac{\langle \chi_u \rangle + \langle \chi_d \rangle}{\sqrt{2}} \mathbf{M}_{11} ,$$
$$f^2_{K^\pm} = -\frac{\langle \chi_u \rangle + \langle \chi_s \rangle}{\sqrt{2}} \mathbf{M}_{44} , \tag{10.30}$$
$$f^2_{K^0,\bar{K}^0} = -\frac{\langle \chi_d \rangle + \langle \chi_s \rangle}{\sqrt{2}} \mathbf{M}_{66}$$

[clearly from Eq. (10.29) we get $\mathbf{M}_{11}=\mathbf{M}_{22}$, $\mathbf{M}_{44}=\mathbf{M}_{55}$, and $\mathbf{M}_{66}=\mathbf{M}_{77}$].

As an illustration, let us derive a representation for f^2_π in the case of massless quarks. Then

$$g_{\mu\nu}\mathbf{M}_{11} = 6\sqrt{2}\int \frac{d^4p}{(2\pi)^4} f_2(p) \frac{p_\mu \dfrac{\partial \bar{\Sigma}(p)}{\partial p_\nu} - g_{\mu\nu}\bar{\Sigma}(p)}{[p^2+\bar{\Sigma}^2(p)]^2} . \tag{10.31}$$

So, using the relations (10.22) and performing the angular integration, we obtain

$$f^2_\pi = \frac{3}{(2\pi)^2}\int_0^\infty dp^2 p^2 \frac{\bar{\Sigma}^2(p) - \frac{1}{2}p^2\bar{\Sigma}(p)\dfrac{d\bar{\Sigma}(p)}{dp^2}}{[p^2+\bar{\Sigma}^2(p)]^2} \tag{10.32}$$

which is the expression given by Pagels and Stokar.[42] We recall that Euclidean variables have been used.

On the other hand, for the sector with $Q=S=0$ one obtains

$$f_{\mu\nu} = \frac{1}{b_\nu}\sum_{\rho=3,8} \mathbf{M}_{\mu\rho}\mathbf{a}_{\rho\nu} \tag{10.33}$$

which, together with Eq. (9.17), gives

$$f_{\mu\nu}f_{\mu\nu} = \sum_{\rho,\tau=3,8} \mathbf{M}_{\mu\rho}\mathbf{a}_{\rho\tau}\mathcal{A}_{\mu\tau}\mathbf{a}_{\tau\nu} , \tag{10.34}$$

$$b_3^2 = \frac{\sum_{\rho=3,8}\mathbf{M}_{3\rho}\mathbf{a}_{\rho 3}}{\sum_{\rho=3,8}\mathcal{A}_{3\rho}\mathbf{a}_{\rho 3}}, \quad b_8^2 = \frac{\sum_{\rho=3,8}\mathbf{M}_{8\rho}\mathbf{a}_{\rho 8}}{\sum_{\rho=3,8}\mathcal{A}_{8\rho}\mathbf{a}_{\rho 8}} . \tag{10.35}$$

Equation (10.34), in which there is no summation on the indices μ and ν, gives the square of the decay constants in the 3-8 sector. Here the matrices $\mathbf{M}$ and $\mathcal{A}$ depend only on the parameters of our model: μ, $\Lambda_{\rm QCD}$, m_u, m_d, and m_s. Therefore the f_{ij}'s are completely determined as functions of these quantities. On the other hand, with Eq. (10.35) we can calculate the coefficients b_3 and b_8 which enter in the expressions of $M^2_{\pi^0}$ and $M^2_{\eta_8}$ [see Eq. (9.19)].

As an example let us consider the SU(2)-symmetric case ($m_u=m_d$). Then $\mathbf{a}_{ij}=\delta_{ij}$ and also $\mathcal{A}_{38}=\mathcal{A}_{83}=0$; $\mathbf{M}_{38}=\mathbf{M}_{83}=0$, so

$$f^2_{33}=f^2_{\pi^0}=\mathbf{M}_{33}\mathcal{A}_{33}, \quad f^2_{88}=f^2_{\eta_8}=\mathbf{M}_{88}\mathcal{A}_{88} , \tag{10.36}$$

$$b_3^2=\frac{\mathbf{M}_{33}}{\mathcal{A}_{33}}, \quad b_8^2=\frac{\mathbf{M}_{88}}{\mathcal{A}_{88}} . \tag{10.37}$$

Using (10.36) in (10.37), and Eq. (9.18), we get

$$b_3^2 = \frac{f^2_{\pi^0}}{\mathcal{A}^2_{33}} = \frac{2f^2_{\pi^0}}{(\langle\chi_u\rangle+\langle\chi_d\rangle)^2} ,$$
$$b_8^2 = \frac{f^2_{\eta_8}}{\mathcal{A}^2_{88}} = \frac{18f^2_{\eta_8}}{(\langle\chi_u\rangle+\langle\chi_d\rangle+4\langle\chi_s\rangle)^2} . \tag{10.38}$$

Equation (9.19) for the 3-8 sector in the symmetric case reads

$$M_3^2=\frac{1}{b_3^2}V_{33}, \quad M_8^2=\frac{1}{b_8^2}V_{88} , \tag{10.39}$$

which means that, as expected, there is no mixing. So we can identify

$$M_3=M_{\pi^0}, \quad M_8=M_{\eta_8} , \tag{10.40}$$

and the following relations hold:

$$M^2_{\pi^0}=\frac{1}{2f^2_{\pi^0}}(\langle\chi_u\rangle+\langle\chi_d\rangle)^2\frac{d^2V}{dp_3dp_3} ,$$
$$M^2_{\eta_8}=\frac{1}{18f^2_{\eta_8}}(\langle\chi_u\rangle+\langle\chi_d\rangle+4\langle\chi_s\rangle)^2\frac{d^2V}{dp_8dp_8} . \tag{10.41}$$

It follows from the previous considerations that in the general case one has to correct Eq. (10.41) with terms of order $(m_u-m_d)^2/m_s^2$.

XI. NUMERICAL RESULTS

The expressions we have found in the previous sections for the decay coupling constants and for the pseudoscalar-meson masses are functions of the parameters of the model: μ, $\Lambda_{\rm QCD}$, m_u, m_d, and m_s. The problem is now to determine these parameters from the experimental data. We have a system of coupled equations given by (9.19) and (10.28), so this determination can only be done in an approximate way.

In order to realize this program, we start from the SU(2) sector with $m_u=m_d=0$. Then the representation for f^2_π [see Eq. (10.32)], gives $f_\pi/\Lambda_{\rm QCD}$ as a function of c

$$\frac{f_\pi}{\Lambda_{\rm QCD}}=F(c) \tag{11.1}$$

(remember that in QCD with three flavors [see Eq. (5.14)] $c=2\pi^2/g^2(\mu)$, or equivalently $\mu/\Lambda_{\rm QCD}=e^{4c/9}$). The numerical analysis of Eq. (11.1) shows that the function $F(c)$ has a maximum $F_{\rm max}$ when $c=0.23$. This means that in our model $\Lambda_{\rm QCD}\geq f_\pi/F_{\rm max}$. So, in order to reproduce the experimental value $f_\pi=93$ MeV with the possible smallest value of $\Lambda_{\rm QCD}$, we have to fix $c=0.23$ corresponding to the maximum for F (see Fig. 8). In this way, having fixed the value of c, we can determine μ (or $\Lambda_{\rm QCD}$)

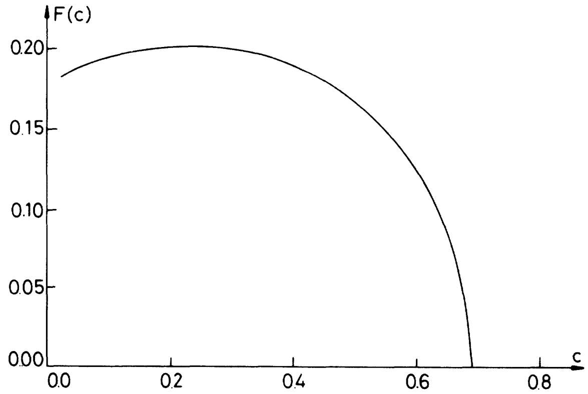

FIG. 8. Plot of $f_\pi/\Lambda_{\rm QCD}$ vs c in the massless case. The curve reaches a maximum value $F_{\max}$ for $c=0.23$.

from the experimental value of f_π. Furthermore, from the value of the minimum of the effective potential for massless quarks, one can extract the values of the condensates $\langle \bar{u}u^0\rangle_\mu=\langle \bar{d}d^0\rangle_\mu$ (the superscript 0 means $m=0$). Using these values in the Adler-Dashen relation (9.31), we obtain a first approximation for the quark mass $m_u=m_d$, given the experimental value of the pion mass. Substituting these first results in Eq. (10.28) with $i,j,k=1$ we redetermine μ, assuming again $f_\pi=93$ MeV. Then we iterate this procedure. From the minimum of the effective potential we obtain the values of the condensates for the quarks u and d and then, from the second derivative of the effective potential evaluated at the minimum, we extract the values of the quark masses. Schematically,

$$f_\pi=93\ \text{MeV}\to\mu\ ,$$

$$\frac{dV}{d\chi}=0\to\langle \bar{u}u\rangle_\mu=\langle \bar{d}d\rangle_\mu\ ,$$

$$M_\pi^2=(139\ \text{MeV})^2\to\hat{m}(\mu)=\frac{m_u(\mu)+m_d(\mu)}{2}\ ,$$

where we have explicitly indicated the dependence on the renormalization point μ of the quark masses and condensates. The iteration converges very quickly and the results we obtain are

$$\mu=497\ \text{MeV},\quad \Lambda_{\rm QCD}=449\ \text{MeV},\quad \hat{m}(\mu)=18\ \text{MeV}\ . \tag{11.2}$$

With these values, it is easy to determine m_s, given, for instance, $M_{K^\pm}=494$ MeV. The result is

$$m_s(\mu)=294\ \text{MeV}\ . \tag{11.3}$$

Finally, assuming that the electromagnetic mass difference between $K^\pm$ and $K^0,\bar{K}^0$ is of order of 1.5 MeV (Ref. 22), we can calculate the difference $m_u(\mu)-m_d(\mu)$ and, combining with the previous result (11.3), we find

$$m_u(\mu)=14.5\ \text{MeV},\quad m_d(\mu)=21\ \text{MeV}\ . \tag{11.4}$$

We are now ready to calculate the masses and the decay constants for the octet mesons. The values we get[23] are the following, to be compared with the experimental ones (the values in parentheses are current-algebra predictions):

	Evaluated	Experiment
$M_{\pi^\pm}$	139 MeV	139.6 MeV
M_{π^0}	138.7 MeV	135 MeV
$M_{K^\pm}$	492 MeV	494 MeV
$M_{K^0,\bar{K}^0}$	498 MeV	498 MeV
M_{η_8}	546 MeV	(566) MeV
f_π	93 MeV	93 MeV
f_K	105 MeV	$f_K/f_\pi=1.17$
f_η	111 MeV	$(f_\eta/f_\pi=1.3)$

The fit for the meson masses is very good (agreement within 3%) and the ratios $f_K/f_\pi=1.13$ and $f_\eta/f_\pi=1.19$ are in rather good agreement with the experimental results[43] and with the current-algebra calculations,[44] respectively.

Some observations are now in order. It is known that the main contribution to the $\pi^\pm-\pi^0$ mass difference is electromagnetic. The mass splitting we find in the framework of our model comes only from the explicit SU(2) breaking due to $m_u\neq m_d$ and therefore, it has to be compared with the current-algebra predictions.[22,45] For the fit we have reported, the mass difference is $(M_{\pi^\pm}-M_{\pi^0})=0.3$ MeV of which 0.11 MeV comes from the π^0-η_8 mixing.

Here M_{η_8} is the mass of the eight component of the octet. However, η_8 mixes with the singlet η_0 because of the SU(3) breaking, and the physical states are given by

$$\eta=\eta_8\cos\theta-\eta_0\sin\theta,\quad \eta'=\eta_8\sin\theta+\eta_0\cos\theta\ , \tag{11.5}$$

where θ must be determined so as to diagonalize the square mass matrix

$$\begin{bmatrix} M_{\eta^0}^2 & M_{08}^2 \\ M_{80}^2 & M_{\eta_8}^2 \end{bmatrix}\ . \tag{11.6}$$

Since we are ignoring the mixing with the SU(3) singlet, the output of our model is M_{η_8}, not to be compared with the experimental value of M_η but with the prediction of the modified Gell-Mann–Okubo mass formula which gives $M_{\eta_8}=566$ MeV (Ref. 45).

Let us also specify that in the determination of f_{π^0} and f_{η_8}, we have neglected the mixing terms, since their contribution is almost irrelevant. Our explicit calculation gives in fact $f_{38},f_{83}\simeq 1.5$ MeV. (See also Ref. 44, where the determination of the off-diagonal meson decay constants is performed in the context of chiral perturbation theory.)

Having determined the values of the quark masses, one can extract the corresponding numerical values of the quark condensates from the minima of the effective potential. Let us recall that the values we get for the masses and for the condensates must be interpreted as those for the renormalization point $\mu=497$ MeV. To compare our results with the values obtained by quite different methods, quoted in the literature, let us perform a rescaling at 1 GeV. From Eq. (4.25) we have that the current-quark masses scale as

$$m(\bar{\mu})=m(\mu)\left[\frac{\ln\bar{\mu}/\Lambda_{\rm QCD}}{\ln\mu/\Lambda_{\rm QCD}}\right]^{-d} \quad (11.7)$$

with $d=\frac{4}{9}$. Notice that this scaling behavior follows from the cutoff dependence of the bare mass which is essential for the UV finiteness of the effective potential. Since $m\langle\bar{\psi}\psi\rangle$ must be a renormalization-group-invariant quantity

$$m(\bar{\mu})\langle\bar{\psi}\psi\rangle_{\bar{\mu}}=m(\mu)\langle\bar{\psi}\psi\rangle_{\mu}\,, \quad (11.8)$$

we obtain

$$\begin{aligned} &m_u(1)=5.8\ \text{MeV}, \quad \langle\bar{u}u\rangle_1=(-223)^3\ \text{MeV}^3\,,\\ &m_d(1)=8.4\ \text{MeV}, \quad \langle\bar{d}d\rangle_1=(-225)^3\ \text{MeV}^3\,,\\ &m_s(1)=118\ \text{MeV}, \quad \langle\bar{s}s\rangle_1=(-284)^3\ \text{MeV}^3\,. \end{aligned} \quad (11.9)$$

XII. COMPARISON WITH CURRENT-ALGEBRA AND SUM-RULE PREDICTIONS FOR THE QUARK MASSES

The earliest information about quark masses was derived from current algebra, although, in that framework, the quark masses appeared only implicitly in the transformation properties, namely, in the commutation rules involving the currents and the energy-momentum tensor. There are convincing estimates of the quark mass ratios from the comparison of various current-algebra Ward identities at zero momentum transfer with physical parameters such as the masses and the decay constants of hadrons. The success of the current algebra predictions is mainly due to the fact that the ratio of the quark masses is defined unambiguously, as it is scale independent. For example, a standard way to extract information about the quark mass ratios is based on the chiral expansion.[22,46] In fact, since the masses of the light quarks u,d,s turn out to be small, the deviations from chiral symmetry may be studied by treating the quark mass term in the Hamiltonian as a perturbation, with massless QCD as the unperturbed system. The chiral symmetry implies a set of Ward identities which link the various Green's functions and therefore interrelate the expansion coefficients. In this way, by expanding the mass of the bound states in powers of the quark masses, one obtains the first-order mass formulas. For the pseudoscalar-meson octet they read

$$\begin{aligned} &M^2_{\pi^\pm}=(m_u+m_d)B+O(m_q\ln m_q)\,,\\ &M^2_{K^\pm}=(m_u+m_s)B+O(m_q\ln m_q)\,,\\ &M^2_{K^0,\bar{K}^0}=(m_d+m_s)B+O(m_q\ln m_q)\,, \end{aligned} \quad (12.1)$$

with the same constant

$$B=-\frac{2}{f_\pi^2}\langle\bar{u}u\rangle$$

(in fact, in the chiral limit, $f_\pi=f_K$ and $\langle\bar{u}u\rangle=\langle\bar{d}d\rangle=\langle\bar{s}s\rangle$), showing that the Gell-Mann–Okubo additive rule is well satisfied by the pseudoscalar octet. So, if one ignores the higher-order corrections, Eq. (12.1) gives the quark mass ratios in terms of the masses of the pseudo-Goldstone bosons.

The study of higher-order terms in the quark mass expansion cannot be done by power expansion around the chiral limit since the theory contains massless physical particles (Goldstone bosons) which generate infrared singularities.[47] To obtain a reliable approximation scheme, it is necessary to reorder the expansion and sum up the leading infrared singularities that occur at any order in the quark masses. With the estimates of the higher-order terms in the quark mass expansion, one can analyze the quark mass ratios m_u:m_d:m_s, given the experimental values of the meson and baryon masses. For example, in Ref. 22, the ratio $R=(m_s-\hat{m})/(m_d-m_u)$, with $\hat{m}=(m_u+m_d)/2$, is determined on the basis of five independent manifestations of isospin breaking (K^+-K^0, p-n, Σ^+-Σ^0, Ξ^0-Ξ^-, and ρ-ω mixing). Treating the values one obtains in this way as independent, one derives[22]

$$R=43.5\pm2.2\ (42.65)\,. \quad (12.2)$$

In (12.2), as well as in the following equations, we add in parentheses the values we get in our model. From the observed masses of π, K, and η, one can extract the value of the ratio $m_s/\hat{m}$ and, from the analysis of the corresponding higher-order terms in the quark mass expansion, one gets[22]

$$m_s/\hat{m}=25.0\pm2.5\ (16.62)\,. \quad (12.3)$$

The above results for R and $m_s/\hat{m}$ imply the following values for the related quark mass ratios:[22]

$$\begin{aligned} &m_d/m_u=1.76\pm0.13\ (1.45)\,,\\ &m_s/m_d=19.6\pm1.6\ (14.05)\,,\\ &m_s/m_u=34.5\pm5.1\ (20.34)\,. \end{aligned} \quad (12.4)$$

Let us remark that the discrepancies between the chiral perturbation theory results and ours (written in parentheses), are essentially due to the fact that in our calculations $f_\pi\neq f_K\neq f_\eta$ and also the values of the quark condensates are different for each flavor.

While it is possible to calculate the quark mass ratios on the basis of current algebra alone, the absolute values of the light-quark masses are not accurately known even if there has been substantial progress during the last few years. The framework which has been used are various QCD sum rules. The kind of objects one deals with are two-point functions like

$$\Pi(q^2)=i\int d^4x\,\langle 0\,|\,T\mathcal{O}(x)\mathcal{O}^\dagger(0)\,|\,0\rangle e^{iqx}\,, \quad (12.5)$$

where $\mathcal{O}(x)$ denotes a composite operator of quark and gluon fields with specified quantum numbers. They are assumed to satisfy dispersion relations of the type

$$\Pi(q^2)=\int_0^\infty ds\frac{1}{s-q^2-i\epsilon}\frac{1}{\pi}\text{Im}\Pi(s)+\text{subtractions}\ . \tag{12.6}$$

The weighted average of the hadronic spectral function $(1/\pi)\text{Im}\Pi(s)$ on the right-hand side, for sufficiently large spacelike q^2 values, must match $\Pi(q^2)$ on the left-hand side which, up to subtractions, is a calculable quantity in QCD. Following the work by Shifman, Vainshtein, and Zakharov[48] (SVZ), there has been a lot of effort to improve on the QCD evaluation of the left-hand side of equations such as (12.6). These authors have proposed to use the Wilson operator-product expansion of the time-ordered product in Eq. (12.6) to parametrize nonperturbative effects due to the confining nature of the QCD vacuum. In order to extract information about light-quark masses, the appropriate two-point functions are those involving the divergence of the axial-vector currents,

$$\psi_5(q^2)=i\int d^4x\langle 0\,|\,T\partial^\mu J_{5\mu}(x)\partial^\nu J^\dagger_{5\nu}(0)\,|\,0\rangle e^{iqx}\ , \tag{12.7}$$

and the corresponding two-point functions associated with the divergence of the vector currents,

$$\psi(q^2)=i\int d^4x\langle 0\,|\,T\partial^\mu J_\mu(x)\partial^\nu J_\nu(0)\,|\,0\rangle e^{iqx}\ . \tag{12.8}$$

The reason for considering these particular two-point functions is that in QCD the operators $\partial^\mu J_{5\mu}$ and $\partial^\mu J_\mu$, which are renormalization-group-invariant operators, are proportional to the sums and to the differences of quark masses, respectively. For example, there are two familiar Ward identities which relate $\psi_5(0)$ and $\psi(0)$ to products of quark masses and vacuum expectation values of quark-antiquark fields:

$$\begin{aligned}\psi_5(0)&=-(m_u+m_d)\langle\bar{u}u+\bar{d}d\rangle\ ,\\ \psi(0)&=-(m_u-m_d)\langle\bar{u}u-\bar{d}d\rangle\ .\end{aligned} \tag{12.9}$$

Although some consensus on the values of the light-quark masses has by now been reached among various group of authors, the errors remain rather large. For example, Gasser and Leutwyler[22] show that the sum rules they consider for the divergence of the axial-vector current, are consistent with

$$\hat{m}(1)=7\pm2\ \text{MeV},\quad m_s(1)=180\pm50\ \text{MeV}\ . \tag{12.10}$$

With the results for the ratios $m_u{:}m_d{:}m_s$ given in Eq. (12.4), the estimate $\hat{m}(1)=7\pm2$ MeV amounts to the following figures:

$$\begin{aligned}m_u(1)&=5.1\pm1.5\ \text{MeV}\ (5.8)\ ,\\ m_d(1)&=8.9\pm2.6\ \text{MeV}\ (8.4)\ ,\\ m_s(1)&=175\pm55\ \text{MeV}\ (118)\ .\end{aligned} \tag{12.11}$$

These absolute mass values are not known very accurately. If we focus on the central values, we see that our estimates (in parentheses) for the u- and d-quark masses at the same renormalization point compare very well, while the situation for the strange quark is more uncertain. Also, if one uses the value $\hat{m}(1)=7\pm2$ MeV in the Adler-Dashen relation, one can fix the order parameter $\langle\bar{u}u\rangle=\langle\bar{d}d\rangle$ from the experimental values of M_π and f_π. The result is

$$\langle\bar{u}u\rangle_1=(-225\pm25)^3\ \text{MeV}^3\ (-(223)^3) \tag{12.12}$$

and it agrees very well with our value.

In a recent paper, Narison[49] discussed in detail various determinations of chiral-symmetry-breaking parameters from the light-meson systems by using the SVZ-Laplace transform QCD sum rules and also calculated a weighted average of various estimates coming from different methods. For the u, d, and s masses he gets (for $100<\Lambda_{\text{QCD}}<150$ MeV)

$$\begin{aligned}m_u(1)&=5.1\pm0.9\ \text{MeV}\ (5.8)\ ,\\ m_d(1)&=9.0\pm1.6\ \text{MeV}\ (8.4)\ ,\\ m_s(1)&=148.4\pm15.3\ \text{MeV}\ (118)\ .\end{aligned} \tag{12.13}$$

For the ratios of the quark vacuum condensates, he gets

$$\begin{aligned}\langle\bar{d}d\rangle/\langle\bar{u}u\rangle&=1-(1\pm0.3)\times10^{-2}\ (1+3\times10^{-3})\ ,\\ \langle\bar{s}s\rangle/\langle\bar{u}u\rangle&=0.6\pm0.2\ (1.08)\ ,\end{aligned} \tag{12.14}$$

where, as before, we have reported in parentheses the values we get from Eq. (11.9).

Dominguez and De Rafael[50] have recently presented an improved determination of the light-quark masses in QCD by combining the information provided by the effective chiral Lagrangian of QCD at long distances (see, for example, Refs. 22 and 44) and the QCD behavior at short distances within the framework of Gaussian sum rules and finite-energy sum rules. A result of their work is the determination of the sum of the running u and d masses at 1 GeV:

$$m_u(1)+m_d(1)=15.5\pm2.0\ \text{MeV}\ (14.2)\ . \tag{12.15}$$

They also determine the strange-quark mass from the combination of the result (12.15) with the current-algebra determination of the ratio $m_s/\hat{m}$ [for example, the value in Ref. 22 is given in Eq. (12.3)] and they get

$$m_s(1)=199\pm33\ \text{MeV}\ (118)\ . \tag{12.16}$$

Also, the best determination in their framework of the down- and up-quark mass difference follows from the combination of the current-algebra determination of the ratio $(m_d-m_u)/(m_d+m_u)$ with Eq. (12.15). They obtain in this way the individual values

$$\begin{aligned}m_u(1)&=5.6\pm1.1\ \text{MeV}\ (5.8)\ ,\\ m_d(1)&=9.9\pm1.1\ \text{MeV}\ (8.4)\ .\end{aligned} \tag{12.17}$$

Finally we mention the results obtained by Reinders and Rubinstein[51] about the determination of the mass and the condensate of the strange quark. Their strategy consists in taking advantage of the constraints coming from heavy-quark physics and then in analyzing the light-quark meson channels with strange quarks. They find that the compatibility of all the sum rules they calculate specifies a very narrow window for $m_s\langle\bar{s}s\rangle$. As a consequence they establish that

$$m_s\langle \bar{s}s\rangle = -(210\pm 5)^4\ \text{MeV}^4\ (-(228)^4) \quad (12.18)$$

and

$$m_s(1) = 110\pm 10\ \text{MeV}\ (118)\ . \quad (12.19)$$

They also suggest the following value for the condensate:

$$\langle \bar{s}s\rangle_1 = (0.8\pm 0.1)\langle \bar{u}u\rangle_1\ , \quad (12.20)$$

which, however, does not agree with their previous results. In fact, taking literally the values given in Eqs. (12.18) and (12.19) one would get

$$\langle \bar{s}s\rangle_1 = -(260\pm 20)^3\ \text{MeV}^3\ (-(284)^3) \quad (12.21)$$

which is in very good agreement with our result.

Summing up, we can say that, as far as the value of $[m_u(1)+m_d(1)]$ is concerned, one finds in the literature values ranging between 10 and 19 MeV and we obtain 14 MeV. The situation for the mass of the strange quark is more confused. One finds evaluations varying from 100 up to 230 MeV. We find 118 MeV. At the same time, there is a general tendency within the sum-rule approach to indicate that the value of the condensate $\langle \bar{q}q\rangle$ decreases for heavier quarks. In our approach, the condensate increases with the mass simply because the extremum of the effective potential moves further away from the origin as the symmetry breaking increases. This is clearly shown in Fig. 9 where we have a plot of the cubic root of the quark-antiquark condensate as a function of $m_q(1)$. We see that the condensate varies very slowly for large values of the current-quark mass. For example, it varies by $\sim(0.3\ \text{GeV})^3$ when passing from a value of 20 GeV to a value of 40 GeV for the mass. This means that, for heavy quarks, the effects of the condensate become secondary as also expected from the asymptotic behavior of the quark self-energy.

To illustrate this phenomenon let us compute the "constituent-quark mass." Politzer[33] has proposed a definition which, in the Euclidean region, reads

$$m_{\text{const}} = \bar{\Sigma}'(p_E^2 = -4m_{\text{const}}^2)\ , \quad (12.22)$$

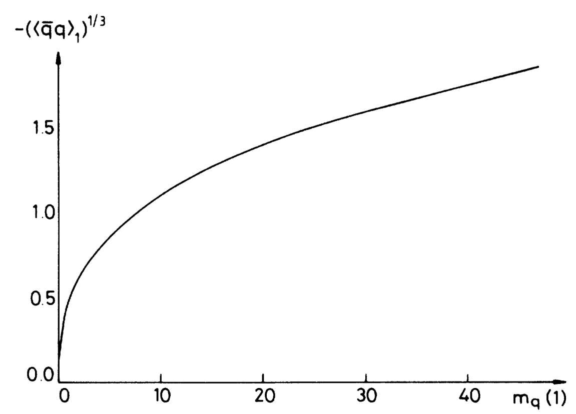

FIG. 9. Plot of $-(\langle \bar{q}q\rangle_1)^{1/3}$ vs $m_q(1)$. The values are in GeV.

where $\bar{\Sigma}'(p^2) = m_0(\Lambda)+\bar{\Sigma}(p^2)$ is evaluated at the extremum of the effective potential. Using our previous results, we get the following form for Politzer's equation:

$$m_{\text{const}} = m(\mu)\left[1+\frac{1}{2a}\ln\frac{4m_{\text{const}}^2}{\mu^2}\right]^{-4/9} + \mu\chi\frac{\mu^2}{4m_{\text{const}}^2}\left[1+\frac{1}{2a}\ln\frac{4m_{\text{const}}^2}{\mu^2}\right]^{-5/9} \quad (12.23)$$

with $a = \ln(\mu/\Lambda_{\text{QCD}})$. A numerical study of Eq. (12.23) with the values of μ, Λ_{QCD}, m_u, m_d, and m_s given in (11.2)–(11.4) and the values of the condensates renormalized at μ obtained by rescaling in (11.9), gives essentially the same constituent mass for all the three light quarks:

$$(m_u)_{\text{const}} = 275\ \text{MeV},\quad (m_d)_{\text{const}} = 276\ \text{MeV}\ ,$$
$$(m_s)_{\text{const}} = 288\ \text{MeV}\ . \quad (12.24)$$

Here the constituent mass is largely dominated by the condensate scale. In the case of heavy quarks the situation is quite different. If, for instance, we look at the charm quark, using the value $m_c(2m_c) = 1.01$ GeV for the charm mass, we find

$$(m_c)_{\text{const}} = 0.97\ \text{GeV}\ . \quad (12.25)$$

Hence, in spite of the large value of the charm condensate $[\langle \bar{c}c\rangle_{2m_c} \sim -(600)^3\ \text{MeV}^3]$, the constituent mass differs from the current mass by only 4%. This means that, already for the charm quark, the effects of the condensate are negligible in a quantity such as the constituent mass. Obviously this phenomenon will be stronger for much heavier quarks. Finally let us observe that the inclusion of the charm quark in this picture, as well as of other heavy quarks, would not significantly change all the previous results for the meson masses.

XIII. CONCLUSIONS AND OUTLOOK

We have analyzed chiral-symmetry breaking in QCD-like gauge theories with fermions. We have included renormalization-group corrections to the gauge coupling and the fermion self-energy. We have used a variational method based on an effective potential for composite operators which is a modified version of the one introduced by Cornwall, Jackiw, and Tomboulis. This formalism has a nonlocal order parameter for which a nonlocal source $J(x,y)$ is introduced. Since the source function must finally be taken to vanish, an ambiguity of adding an arbitrary polynomial of the source is present, satisfying some suitable conditions. In particular our effective potential corresponds to a choice which gives the same local extrema as the CJT potential but has the advantage of being bound from below. V_{CJT} does not enjoy this property, and this instability is reflected in the saddle-point character of its stationary points. We have shown that the different choices of the source term for the two cases correspond to different choices of the dynamical variables. Our effective potential comes out to be completely expressed in terms of the fermion self-energy Σ, and our variational method consists in making use of a parametrized test function for Σ. The parametrization of the self-energy is in terms of constant fields related to the fermionic condensates, evaluated at the minimum of the effective potential.

We have assumed that the main contribution for chiral-symmetry breaking comes from short distances. For this reason we have introduced a parameter μ as infrared cutoff and we have focused our attention on the short-distance dynamics. In this range it is sensible to perform a loop expansion of the effective action and to retain only the lowest nontrivial contribution. The effective action has thus been calculated at the two-loop order. This approximation has been improved according to the renormalization-group analysis (the calculations are performed in the Landau gauge). For the momentum dependence of the fermion self-energy we have assumed a constant behavior in the infrared region ($p < \mu$) and a fall down like $1/p^2$(logs) for $p > \mu$ as suggested by the operator-product expansion analysis.

In the case of massless fermions we find that the theory possesses two phases: the chirally symmetric phase and the phase broken to the diagonal flavor subgroup, depending on the value of the coupling constant renormalized at the point μ, which discriminates the IR from the UV region. For QCD with three flavors the numerical calculations show that the color gauge dynamics spontaneously breaks the initial chiral symmetry down to $SU(3)_{L+R}$ for $\alpha_s = g^2(\mu)/(4\pi) > 0.73\pi$, by giving equal vacuum expectation values to each scalar quark-antiquark pair $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = \langle \bar{s}s \rangle$. The value of α_s which agrees with that obtained by other variational methods with specific *Ansätze* for the fermion self-energy, is higher than the value usually found with methods based on the exact solution of the linearized Schwinger-Dyson equation or on the numerical solution of the nonlinear equation. We think this is due to the fact that the true form of the fermion self-energy is more complicated than the one we have used. In particular the constant behavior of Σ in the infrared region is a rather crude approximation and the low-momentum regions are also involved in the definition of the critical value of α_s.

The central part of this work is the extension of the effective potential formalism to realistic situations when both spontaneous and explicit breakdown of the global chiral symmetry take place. We have examined in particular the predictions of our formalism for QCD with three flavors. The minimum of the effective potential is found to correspond to vanishing pseudoscalar condensates (no spontaneous P and CP violation) and of scalar charged condensates. We have determined the values of the quark condensates $\langle \bar{u}u \rangle$, $\langle \bar{d}d \rangle$, and $\langle \bar{s}s \rangle$ from the stationary points of the effective potential. They depend on the parameters of the model: the renormalization-invariant mass Λ_{QCD}, the quark masses m_u, m_d, and m_s, and the scale μ. To determine these parameters we have derived explicit expressions for the masses and decay constants of the pseudoscalar mesons. The equations relating the meson masses to the second derivatives of the effective potential with respect to the pseudoscalar fields can be recast in a form which only depends on the explicit symmetry-breaking part of the effective potential, particularly useful for computation. Also, after imposing the renormalization of the effective potential in the small-quark mass limit, these equations appear as the Adler-Dashen relations for three flavors.

Allowing for general space-time dependence of the variational parameters, we have calculated the effective action at one loop by using the Weyl symmetrization prescription to solve the quantum-mechanical ordering problem. In this way it has been possible to extract, directly from our functional, the expression for the meson-quark-antiquark vertices. This is a necessary ingredient for the calculation of the pseudoscalar-meson decay constants together with the normalization factors relating our dynamical variables to the canonical pseudo-Goldstone fields.

The expressions for the masses and the decay constants of the pseudoscalar mesons represent a system of coupled equations. The determination of the parameters of our model has thus been carried out by iteration. Our experimental inputs are

$$f_\pi = 93 \text{ MeV}, \quad M_{\pi^\pm} = 139 \text{ MeV} ,$$

$$M_{K^\pm} = 494 \text{ MeV}, \quad (M_{K^\pm} - M_{K^0})^\gamma = 1.5 \text{ MeV} ,$$

and we get a very good fit for the octet-meson masses (agreement within 3%) with the following choices:

$$\mu = 497 \text{ MeV}, \quad \Lambda_{QCD} = 449 \text{ MeV} ,$$

$$m_u(1) = 5.8 \text{ MeV}, \quad m_d(1) = 8.4 \text{ MeV} ,$$

$$m_s(1) = 118 \text{ MeV} ,$$

where μ/Λ_{QCD} has been fixed in order to obtain a minimal value for Λ_{QCD} in the massless case for given f_π, and the quark masses are renormalized at 1 GeV. (In our calculations the mixing in the 3-8 sector has been taken into account.)

Comparison with current-algebra and sum rules predictions shows that our estimates for the u- and d-quark masses and condensates agree very well, while for the strange quark the agreement is not as good. However the indications given in the literature for $m_s(1)$ are uncertain. One finds estimations varying from 100 up to 230 MeV. So, within the large errors, we can conclude that our variational approach to dynamical mass generation had led to quantitative results which are essentially in agreement with those obtained by quite different, more phenomenological, methods. This proves the validity of the method and encourages us to use it in other contexts.

The most direct extension of the method goes in the line of reproducing the results of "chiral perturbation theory" within our dynamical setting. In fact we have already given (see Barducci *et al.*[24]) a general derivation of any amplitude among pseudoscalars including possible external photons and W,Z bosons. The formalism can thus be applied to any problem of electromagnetic and weak interactions of pions, kaons, eta, etc. At present we are studying the implications for the K-$\bar{K}$ system.

The composite operator method based on our modified effective action can be straightforwardly extended to finite temperature. The general formulation at finite temperature has already been written down.[52] The treatment follows the work by Dolan and Jackiw[53] and by Bernard[54] on spontaneous symmetry breaking at finite temperature. The introduction of composite operators and of their associated bilocal sources allows for the extension to cases of dynamical symmetry breaking. The practical aim of this type of work is a theoretical treatment of QCD at finite temperature, in view of the importance of the phase transitions that are expected to take place in such a system. So far the only application[52] of our finite temperature formalism has been to the O(N) scalar model at large N, to verify that one reproduces the known results of that model.

The composite operator method can in principle be applied to supersymmetric QCD. The expected appearance of other type of condensates, beside the quark-antiquark condensate, and the necessary verification of additional Ward identities lead in this case to a rather complex problem, in principle not insoluble but which will demand still more work before we may get to some conclusion.

We have also in mind the application of our formalism to technicolor-type models. Let us spend some word on this subject.

Recently, Appelquist and Wijewardhana[55] have proposed a modification of the technicolor (TC) dynamics leading to a higher value of the technifermion condensate $\langle \bar{T}T \rangle$ but leaving the Goldstone-boson decay constant F, which determines the W and Z masses, essentially unaltered. In their model the asymptotic freedom of the TC theory is maintained but it is assumed that the large number of fermions expected in a realistic TC theory substantially slows down the running of the coupling. They show that the slow running modifies the ultraviolet behavior of the theory. In particular they find that there is a critical coupling α_c that the running coupling $\alpha(p)$ must exceed before chiral condensation can set in, and that the solution of the Schwinger-Dyson equation for constant $\alpha > \alpha_c$ has a $1/p$ power behavior multiplied by an oscillatory function. They use a mass scale M associated with the additional interactions that generate quark and lepton masses, as an UV cutoff. Then, the relatively slow fall of the technifermion self-energy allows for a higher value for the cutoff M than naively expected for a given value of the fermion mass. This raising of the cutoff leads to a suppression of the flavor-changing neutral currents and also raises the pseudomasses above current accelerator bounds. They find that fermion masses of the order of 100 MeV and pseudomasses of 100 GeV can be obtained for $M \sim 300$ TeV. On the other hand, $W^{\pm}$ and Z^0 masses remain essentially unaltered since the condensate $\langle \bar{T}T \rangle_M$ is clearly much more sensitive to the high-momentum behavior of the self-energy than is the decay constant F. Because of the interesting properties of this model, one can think of applying our variational formalism to the case of chiral-symmetry breaking in asymptotically free theories with slowly running couplings. The *Ansatz* for the self-energy to be used in this case must clearly be different from the one used in QCD. In fact the test function could start from a constant value for momenta $p < \Lambda_{\rm TC}$, falling slowly like $1/p\Phi(\text{logs})$ for a significant range $p > \Lambda_{\rm TC}$ and then take the asymptotic form $1/p^2(\text{logs})$.

Composite models for quarks and leptons are usually based on confining color-type forces at higher scales. In fact the lack of dynamical understanding of those conjectures was one of the original motivations for our work on composite operators.

Finally, we mention the work[56] on the possible strong Higgs sector, where the composite operator formalism has allowed us to derive an effective Lagrangian for scalar and longitudinally polarized weak bosons, giving in particular the (multiple) production amplitudes for W's and Z's, and showing how unitarity becomes restored at high energies when one goes beyond lowest-order perturbation theory.

All these topics are under study and represent some of the further developments of the present work.

APPENDIX A: CANCELLATION OF THE UV DIVERGENCES IN THE EFFECTIVE ACTION

Let us substitute our *Ansatz* (5.2) for $\Sigma(p^2)$ in the expression (3.40) for the effective action and let us start by analyzing Γ_2. By performing an integration by parts in (3.30) we obtain

$$\Gamma_2=\frac{2N}{3C_2}\Omega\left[\frac{1}{2}\left[\frac{1}{\dfrac{d}{dp^2}\left[\dfrac{g^2(p)}{p^2}\right]}\operatorname{tr}\left[\frac{d}{dp^2}[\boldsymbol{\Sigma}_s^2(p^2)+\boldsymbol{\Sigma}_p^2(p^2)]\right]\right]_0^{\Lambda^2} - \int_0^{\Lambda^2}dp^2\frac{1}{\dfrac{d}{dp^2}\left[\dfrac{g^2(p)}{p^2}\right]}\left[\operatorname{tr}\left[\frac{d}{dp^2}\boldsymbol{\Sigma}_s(p^2)\right]^2+\operatorname{tr}\left[\frac{d}{dp^2}\boldsymbol{\Sigma}_p(p^2)\right]^2\right]\right] \tag{A1}$$

with

$$\boldsymbol{\Sigma}_s(p^2)=\mathbf{m}(\mu)f_1(p^2)-\mathbf{m}_0(\Lambda)+\mathbf{s}f_2(p^2)\ , \tag{A2}$$

$$\boldsymbol{\Sigma}_p(p^2)=\mathbf{p}f_2(p^2)\ , \tag{A3}$$

and $f_1(p^2),f_2(p^2)$ given in Eqs. (5.3) and (5.4). The surface term in (A1) gives a vanishing contribution. In fact, for $p^2=0$ we get obviously zero. Let us calculate explicitly the contribution for $p^2=\Lambda^2$. Introducing

$$f_\Lambda=f(p^2)\,|_{\Lambda^2}=\frac{1}{2a}\ln\frac{\Lambda^2}{M_0^2}=1+\frac{1}{2a}\ln\frac{\Lambda^2}{\mu^2},\quad a=\ln\frac{\mu}{M_0} \tag{A4}$$

with $f(p^2)$ defined in (5.5) we have

$$\frac{1}{\dfrac{d}{dp^2}\left[\dfrac{g^2(p)}{p^2}\right]}\operatorname{tr}\left[\frac{d}{dp^2}[\boldsymbol{\Sigma}_s^2(p^2)+\boldsymbol{\Sigma}_p^2(p^2)]\right]\Bigg|_{p^2=\Lambda^2}$$

$$=\operatorname{tr}[m^2(\mu)]\left[\frac{2ad}{b}\Lambda^2\frac{f_\Lambda^{-2d+1}}{1+2af_\Lambda}\right]+\operatorname{tr}(\mathbf{s}^2+\mathbf{p}^2)\left[\frac{4a}{b}\frac{\mu^6}{\Lambda^2}\frac{f_\Lambda^{2d}}{1+2af_\Lambda}\right]-\operatorname{tr}[\mathbf{m}_0(\Lambda)\cdot\mathbf{s}]\left[\frac{4a^2}{b}\mu^3\frac{f_\Lambda^{d+1}}{1+2af_\Lambda}\right]$$

$$-\operatorname{tr}[\mathbf{m}(\mu)\cdot\mathbf{m}_0(\Lambda)]\left[\frac{2ad}{b}\Lambda^2\frac{f_\Lambda^{-d+1}}{1+2af_\Lambda}\right]+\operatorname{tr}[\mathbf{m}(\mu)\cdot\mathbf{s}]\left[\frac{4a^2}{b}\mu^3\frac{f_\Lambda}{1+2af_\Lambda}\right]. \tag{A5}$$

Remembering that, in the leading-logarithmic approximation

$$\mathbf{m}_0(\Lambda)=\mathbf{m}(\mu)Z_m(\mu,\Lambda)=\mathbf{m}(\mu)f_\Lambda^{-d} \tag{A6}$$

with $Z_m(\mu,\Lambda)$ defined in (4.25), and performing the $\Lambda\to\infty$ limit we get

$$\frac{1}{\dfrac{d}{dp^2}\left[\dfrac{g^2(p)}{p^2}\right]}\operatorname{tr}\left[\frac{d}{dp^2}[\boldsymbol{\Sigma}_s^2(p^2)+\boldsymbol{\Sigma}_p^2(p^2)]\right]\Bigg|_{p^2=\Lambda^2\to\infty}\sim\operatorname{tr}[\mathbf{m}^2(\mu)]\left[\frac{2ad}{b}\Lambda^2\frac{f_\Lambda^{-2d+1}}{1+2af_\Lambda}\right]-\operatorname{tr}[\mathbf{m}(\mu)\cdot\mathbf{s}]\left[\frac{4a^2}{b}\mu^3\frac{f_\Lambda}{1+2af_\Lambda}\right]$$

$$-\operatorname{tr}[\mathbf{m}^2(\mu)]\left[\frac{2ad}{b}\Lambda^2\frac{f_\Lambda^{-2d+1}}{1+2af_\Lambda}\right]$$

$$+\operatorname{tr}[\mathbf{m}(\mu)\cdot\mathbf{s}]\left[\frac{4a^2}{b}\mu^3\frac{f_\Lambda}{1+2af_\Lambda}\right]=0\,, \tag{A7}$$

showing that no contribution arises from the surface term of Γ_2 in virtue of the mass renormalization.

So we are left with

$$\Gamma_2=-\frac{2N}{3C_2}\Omega\int_0^{\Lambda^2}dp^2\frac{1}{\dfrac{d}{dp^2}\left[\dfrac{g^2(p)}{p^2}\right]}\operatorname{tr}\left[\frac{d}{dp^2}[\boldsymbol{\Sigma}_s^2(p^2)+\boldsymbol{\Sigma}_p^2(p^2)]\right]. \tag{A8}$$

For large values of momenta we have

$$\operatorname{tr}\left[\frac{d}{dp^2}[\boldsymbol{\Sigma}_s^2(p^2)+\boldsymbol{\Sigma}_p^2(p^2)]\right]\Bigg|_{p^2\to\infty}\sim\operatorname{tr}[\mathbf{m}^2(\mu)]\left[\frac{d^2}{4a^2}\frac{1}{p^4}f(p^2)^{-2d-2}\right]$$

$$+2\operatorname{tr}[\mathbf{m}(\mu)\cdot\mathbf{s}]\left[-\frac{d}{4a^2}[d-1-2af(p^2)]\frac{\mu^3}{p^6}f(p^2)^{-3}\right]$$

$$+\operatorname{tr}(\mathbf{s}^2+\mathbf{p}^2)\left[\frac{1}{4a^2}[d-1-2af(p^2)]^2\frac{\mu^6}{p^8}f(p^2)^{2d-4}\right] \tag{A9}$$

so

$$\Gamma_2\,|_{\mathrm{UV}}=-\frac{2N}{3C_2}\Omega\{\operatorname{tr}[\mathbf{m}^2(\mu)]\mathcal{A}+2\operatorname{tr}[\mathbf{m}(\mu)\cdot\mathbf{s}]\mathcal{B}+\operatorname{tr}(\mathbf{s}^2+\mathbf{p}^2)\mathcal{C}\} \tag{A10}$$

with

$$\mathcal{A}=-\frac{d^2}{2b}\int^{\Lambda^2}dp^2\frac{f(p^2)^{-2d}}{1+2af(p^2)}\ ,$$
$$\mathcal{B}=\frac{d}{2b}\mu^3\int^{\Lambda^2}\frac{dp^2}{p^2}[d-1-2af(p^2)]\frac{f(p^2)^{-1}}{1+2af(p^2)}\ , \qquad (A11)$$
$$\mathcal{C}=-\frac{1}{2b}\mu^6\int^{\Lambda^2}\frac{dp^2}{p^4}[d-1-2af(p^2)]^2\frac{f(p^2)^{2d-2}}{1+2af(p^2)}\ .$$

Notice that the first term in (A10) does not depend on the fields $\mathbf{s}$ and $\mathbf{p}$ and so it can be left out since it only represents an additive constant. Furthermore, the integral in $\mathcal{C}$ is convergent for large values of momenta. It remains to analyze the ultraviolet divergences arising from the term proportional to $\mathrm{tr}[\mathbf{m}(\mu)\cdot\mathbf{s}]$. Taking the $\Lambda\to\infty$ limit, we find a divergence in $\mathcal{B}$ of the form

$$\mathcal{B}\,|_{\mathrm{div}}=\lim_{\Lambda\to\infty}-\frac{d}{b}a\mu^3\ln\ln\frac{\Lambda^2}{\mu^2}\ , \qquad (A12)$$

which, substituted in Eq. (6.10) gives

$$\Gamma_2\,|_{\mathrm{div}}=\lim_{\Lambda\to\infty}\frac{N\Omega}{2\pi^2}a\mu^3\mathrm{tr}[\mathbf{m}(\mu)\cdot\mathbf{s}]\ln\ln\frac{\Lambda^2}{\mu^2}\ . \qquad (A13)$$

Let us now examine the ultraviolet divergences in the logarithmic term of Γ. Let us call it $\Gamma_{\log}$:

$$\Gamma_{\log}=-\frac{N\Omega}{8\pi^2}\int^{\Lambda^2}dp^2p^2\ln\det\{p^2+[\mathbf{m}_0(\Lambda)+\boldsymbol{\Sigma}_s(p^2)]^2+\boldsymbol{\Sigma}_p^2(p^2)+i[\boldsymbol{\Sigma}_p(p^2),\mathbf{m}_0(\Lambda)+\boldsymbol{\Sigma}_s(p^2)]\}\ . \qquad (A14)$$

Substituting our *Ansatz* for the large momenta behavior of $\boldsymbol{\Sigma}_s$ and $\boldsymbol{\Sigma}_p$, subtracting an infinite constant, and using the relation

$$\det(1+\mathbf{A})=1+\mathrm{tr}\,\mathbf{A} \qquad (A15)$$

for infinitesimal $\mathbf{A}$, we obtain

$$\Gamma_{\log}\sim-\frac{N\Omega}{8\pi^2}\int^{\Lambda^2}dp^2\{\mathrm{tr}[\mathbf{m}^2(\mu)]f_1^2(p^2)+2\,\mathrm{tr}[\mathbf{m}(\mu)\cdot\mathbf{s}]f_1(p^2)f_2(p^2)+\mathrm{tr}(\mathbf{s}^2+\mathbf{p}^2)f_2^2(p^2)\}\ . \qquad (A16)$$

In (A16) the first term does not depend on the fields $\mathbf{s}$ and $\mathbf{p}$ and can be left out. Substituting the expressions (5.3) and (5.4) for $f_1(p^2)$ and $f_2(p^2)$, one finds that an ultraviolet divergence arises only from the term proportional to $\mathrm{tr}[\mathbf{m}(\mu)\cdot\mathbf{s}]$. The explicit calculation gives

$$\Gamma_{\log}\,|_{\mathrm{div}}=\lim_{\Lambda\to\infty}\left[-\frac{N\Omega}{8\pi^2}\mu^3\int^{\Lambda^2}\frac{dp^2}{p^2}f(p^2)^{-1}\right]\times2\,\mathrm{tr}[\mathbf{m}(\mu)\cdot\mathbf{s}]\sim-\lim_{\Lambda\to\infty}\frac{N\Omega}{2\pi^2}a\mu^3\mathrm{tr}[\mathbf{m}(\mu)\cdot\mathbf{s}]\ln\ln\frac{\Lambda^2}{\mu^2} \qquad (A17)$$

which exactly cancels the divergent part of Γ_2 [Eq. (A13)].

In this way we have a completely UV regularized expression for Γ. This is due to the fact that we are taking into account the renormalization-group effects in the leading-logarithmic approximation. As a consequence we do not obtain the usual logarithmic mass divergence from $\Gamma_{\log}$ but there remains only a divergent term which goes to infinity like $\ln\ln(\Lambda^2/\mu^2)$. In other words, the insertion of the running mass $\mathbf{m}(\mu)f_1(p^2)$ in the scalar part of the fermion self-energy regularizes the theory at one-loop order, at least in the leading-logarithmic approximation, while the residual divergence is canceled by the two-loop contribution of Γ_2.

Remember that also in the discussion of the Schwinger-Dyson equations (see Sec. IV), the asymptotic behavior for large momentum of the mass term in $\boldsymbol{\Sigma}_s'$ is responsible for the regularization of the theory in the ultraviolet range.

APPENDIX B: PROPERTIES OF THE EXTREMA OF THE EFFECTIVE POTENTIAL IN THE MASSIVE CASE

We will describe some general properties of the extrema of the effective potential as given in Eq. (5.13). First of all, we will prove that for a symmetric mass matrix $\mathbf{m}_a^b=m\delta_a^b$, $a,b=1,\ldots,n$, the search of minima with vanishing charge condensates can be restricted to the surface in which these condensates are zero.[21]

It is convenient to expand the fields $\mathbf{s}_{ab}$ (and $\mathbf{p}_{ab}$) in terms of the Cartan basis of U(n):

$$\mathbf{s}=s_0+\sum_{i=1}^{n-1}s_ih_i+\sum_{\alpha=1}^{(n-1)n/2}(s_\alpha e_\alpha+s_{-\alpha}e_{-\alpha}) \qquad (B1)$$

and analogously for $\mathbf{p}$. We want to show that, if the restriction of the potential V to the surface defined by

$$s_\alpha = s_{-\alpha} = 0, \quad p_\alpha = p_{-\alpha} = 0 \tag{B2}$$

has a minimum, then it is a minimum also of the function V. On this surface, V decomposes, as in the massless case, in the sum of n contributions, one for each flavor

$$V = \frac{N\mu^4}{4\pi^2} \sum_{a=1}^{n} V_1(\chi_a, \pi_a, m) \tag{B3}$$

with χ_a and π_a defined in (6.6). Because of the structure of (B3), the minimum of V will be at the point $\langle \chi_a \rangle = v$ and $\langle \pi_a \rangle = w$ independent on a since we are considering the U(n) symmetric case. So, using Eq. (6.6) this means

$$\begin{aligned} &\langle s_0 \rangle = v, \quad \langle p_0 \rangle = w , \\ &\langle s_i \rangle = \langle p_i \rangle = 0, \quad i = 1, \ldots, n-1 . \end{aligned} \tag{B4}$$

This is a minimum on the surface defined by (B2). It remains to be shown that this point is a minimum also along the charged field direction. Because of the SU(n) invariance of the potential, the first derivatives of V will have the general form

$$\begin{aligned} \frac{\partial V}{\partial s_i} &= A s_i + B p_i , \\ \frac{\partial V}{\partial p_i} &= B s_i + C p_i , \end{aligned} \tag{B5}$$

$$\begin{aligned} \frac{\partial V}{\partial s_\alpha} &= A s_{-\alpha} + B p_{-\alpha} , \\ \frac{\partial V}{\partial p_\alpha} &= B s_{-\alpha} + C p_{-\alpha} , \end{aligned} \tag{B6}$$

where A, B, and C are SU(n) invariants. By hypothesis, at the point defined by Eqs. (B2) and (B4), Eq. (B5) implies

$$AC - B^2 \geq 0, \quad A \geq 0 . \tag{B7}$$

We see from (B6) that the eigenvalues of the second derivatives of the potential are positive also along the charged directions, and therefore this point is a minimum in the space of all variables.

Second, we want to show that only two possibilities arise for the function $V_1(\chi, \pi, m)$: either V_1 has its absolute minimum on the line $\pi = 0$, or V_1 has two degenerate minima at the points $(\langle \chi \rangle, \langle \pi \rangle)$ and $(\langle \chi \rangle, -\langle \pi \rangle)$. This follows from the properties: (i) V_1 is bounded from below; (ii) the Hessian matrix of V_1 is positive definite on the possible extrema outside the line $\pi = 0$. In fact, if V_1 has a minimum on the line $\pi = 0$, no extrema outside this line can exist due to the positivity of the Hessian. However, if no minima exist on the line $\pi = 0$, then necessarily two degenerate minima at $\pi \neq 0$ must exist due to the parity invariance of V_1, $\pi \to -\pi$.

Let us now show that property (ii) holds. By using the equation

$$\frac{dV_1(\chi, \pi)}{d\pi} = 2\pi \frac{\partial V_1(\chi, \pi)}{\partial(\pi^2 + \chi^2)} \tag{B8}$$

which implies

$$\frac{\partial V_1(\chi, \pi)}{\partial(\pi^2 + \chi^2)} = 0 \tag{B9}$$

on an extremum outside the line $\pi = 0$, we can write the second derivatives of V_1 [Eq. (8.10)] evaluated at this point in the form

$$\begin{aligned} \left.\frac{d^2 V_1(\chi, \pi)}{d\chi^2}\right|_{\text{extr}} &= 2\int_0^\infty d\mu (\alpha G_1 + \chi G_2)^2 , \\ \left.\frac{d^2 V_1(\chi, \pi)}{d\pi^2}\right|_{\text{extr}} &= 2\int_0^\infty d\mu\, \pi^2 G_2^2 , \\ \left.\frac{d^2 V_1(\chi, \pi)}{d\chi\, d\pi}\right|_{\text{extr}} &= 2\int_0^\infty d\mu\, \pi G_2 (\alpha G_1 + \chi G_2) , \end{aligned} \tag{B10}$$

where

$$\begin{aligned} d\mu &= \frac{y\, dy}{[y^3 + \alpha^2 y^2 F(y)^{-2d} + 2\alpha y \chi F(y)^{-1} + (\chi^2 + \pi^2) F(y)^{2d-2}]^2} , \\ F(y) &= 1 + \frac{1}{2a} \ln y, \quad a = \ln \frac{\mu}{M_0}, \quad G_1 = y F(y)^{-1}, \quad G_2 = F(y)^{2d-2} . \end{aligned} \tag{B11}$$

Because $d\mu$ is a positive measure, we can use it to introduce a scalar product. We define

$$\|G_i\|^2 = \int_0^\infty d\mu\, G_i^2, \quad i = 1,2, \quad \langle G_1, G_2 \rangle = \int_0^\infty d\mu\, G_1 G_2 . \tag{B12}$$

Then the Hessian at the extremum is

$$\mathcal{H}|_{\text{extr}} = 4\alpha^2 \pi^2 (\|G_1\|^2 \|G_2\|^2 - \langle G_1, G_2 \rangle^2) \tag{B13}$$

which is a positive-definite quantity due to the Schwartz inequality.

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