Effect of finite mass on gravitational transit time

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The dependence of the gravitational transit time on the mass of the particle is derived in this paper. The result confirms, as might be expected, that for highly relativistic particles the effect of a finite mass on the gravitational time delay is negligible compared to special-relativistic effects. As an application, the travel times of photons and the relativistic neutrinos from SN1987A are calculated to show explicitly the negligible effect a finite neutrino mass has on the arguments of Longo and Krauss and Tremaine who used the nearly simultaneous arrival of photons and neutrinos as a test of the Einstein equivalence principle.

The gravitational-time-delay effect for massless particles is one of the most remarkable predictions of the principle of equivalence.[1] Recently data regarding the transit time between photons and neutrinos from SN1987A (Refs. 2 and 3) have been used to test the principle of equivalence.[4,5] It is desirable to know explicitly the effect of a finite neutrino mass in these estimates, particularly so since a bound on the neutrino mass has been obtained from the dispersion in energy of the neutrino transit times from the same source.[6] In this paper we derive a formula for the transit time, which includes the effect of the metric, for a particle of finite mass and discuss its possible significance.

We use geometrical units $c=1$ and $G=1$ and we consider the motion of the particle in a spherically symmetric metric

$$ds^2=B(r)dt^2-A(r)dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2) \tag{1}$$

and assume B and A are given by the Robertson expansion

$$B=1-\frac{2m}{r}+\frac{2\beta M^2}{r^2}+\cdots, \quad A=1+\frac{2\gamma M}{r}+\cdots . \tag{2}$$

General relativity gives $\beta=0$ and $\gamma=1$.

The radial component of the freely falling equation of motion is[7]

$$\left[\frac{dr}{dt}\right]^2=B^2\left[\frac{1}{AB}-\frac{c_1}{r^2A}-\frac{c_2}{A}\right], \tag{3}$$

where c_1 and c_2 are two constants of motion. The constant c_2 can be obtained directly as

$$c_2=B^{-1}(r)[1-v_L^2(r)] , \tag{4}$$

where v_L is the proper velocity defined by $v_L=dl/d\tau$ with the proper length given by $(dl)^2\equiv A\,dr^2+r^2d\phi^2$ and the proper time given by $d\tau\equiv B^{1/2}dt$. Note that for a zero-mass particle $c_2=0$. The constant c_1 may be determined in terms of the impact parameter r_0; dr/dt vanishes at $r=r_0$. This gives

$$c_1=[B^{-1}(r_0)-c_2]r_0^2 . \tag{5}$$

The transit time from r to r_0 can be calculated from (3), (4), and (5). To first order in the small expansion parameter M/r_0 (except as it occurs in c_2), we arrive at

$$t(r,r_0)=\frac{1}{\sqrt{1-c_2}}\int_{r_0}^{r}dr\left[1-\frac{r_0^2}{r^2}\right]^{-1/2}\left[1+\frac{M}{r}\frac{1+\gamma-(2+\gamma)c_2}{1-c_2}+\frac{Mr_0}{r(r+r_0)}\frac{1}{1-c_2}\right] \tag{6}$$

and thus finally

$$t(r,r_0)=\frac{1}{\sqrt{1-c_2}}\left[(r^2-r_0^2)^{1/2}+\frac{1+\gamma-(2+\gamma)c_2}{1-c_2}M\ln\frac{r+(r^2-r_0^2)^{1/2}}{r_0}+\frac{M}{1-c_2}\left[\frac{r-r_0}{r+r_0}\right]^{1/2}\right] . \tag{7}$$

The effect of finite mass of the particle is contained in the parameter c_2, which, as we have noted, has value 0 for a zero mass particle and for which case (7) reduces to the well-known expression for the transit time of massless particles. The total transit time for travel from $r=R$ to $r=r_e$ is

$$t(r_e,R)=t(r_e,r_0)+t(r_0,R) . \tag{8}$$

Here R denotes the location of the source of particle emission and r_e that of observation (Earth). It is now convenient to parametrize c_2 in terms of its value at $r=r_e$:

$$c_2=B^{-1}(r_e)[1-v_L^2(r_e)]=B^{-1}(r_e)\frac{m^2}{E^2} , \tag{9}$$

where m is the particle mass and E the received energy. Treating M/r_e as small, collecting (7), (8), and (9) we obtain

$$t(r_e,R)=\frac{1}{(1-m^2/E^2)^{1/2}}\Bigg\{(R^2-r_0^2)^{1/2}+(r_e^2-r_0^2)^{1/2} + \frac{M}{1-m^2/E^2}\left[\left[\frac{R-r_0}{R+r_0}\right]^{1/2}+\left[\frac{r_e-r_0}{r_e+r_0}\right]^{1/2}\right] + \frac{M}{1-m^2/E^2}\left[1+\gamma-(\gamma+2)\frac{m^2}{E^2}\right]\ln\frac{[R+(R^2-r_0^2)^{1/2}][r_e+(r_e^2-r_0^2)^{1/2}]}{r_0^2} + \frac{M}{r_e}\frac{m^2/E^2}{1-m^2/E^2}[(R^2-r_0^2)^{1/2}+(r_e^2-r_0^2)^{1/2}]\Bigg\} \tag{10}$$

as our final expression. Of course the limit $m=0$ can be taken in this expression.

If one assumes a highly relativistic particle $E \gg m$ and $r_e \approx r_0$ as is the case for the neutrinos from SN1987A, $t(r_e,R)$ can be written as $t_0+\delta t$ where $t_0=(1+m^2/2E^2)D$ and

$$\delta t = M\left[\left[1+\gamma+\frac{\gamma-1}{2}\frac{m^2}{E^2}\right]\ln\frac{2R}{r_0}+\left[\frac{R-r_0}{R+r_0}\right]^{1/2}\left[1+\frac{3}{2}\frac{m^2}{E^2}\right]+\frac{(R^2-r_0^2)^{1/2}}{r_0}\frac{m^2}{E^2}\right]. \tag{11}$$

t_0 is the dominant special-relativistic effect while δt is the gravitational-time-delay effect.

To compute the transit time from SN1987A, we take[8] $D=52$ kpc, $r_0=10$ kpc, and $M=10^{11}yM_O$, where y is expected[9] to lie in the range 1.4–6. The resulting transit time is then given by

$$t=2.68\times10^{12}\ \text{sec}\left[1+[1+(2.16\gamma_m+10.16)y\times10^{-7}]\frac{m^2}{E^2}+4.32y\gamma_m\times10^{-7}\right]. \tag{12}$$

Here the notation γ_m is used to admit the possibility that the metric parameter may vary with particle, a particular type of violation of the Einstein equivalence principle. From this we see the difference in transit time of a photon and a particle of mass m and energy E is

$$\Delta t_{\gamma m}=2.68\times10^{12}\ \text{sec}\left[[1+(2.16\gamma_m+10.16)y\times10^{-7}]\frac{m^2}{E^2}+4.32y(\gamma_m-\gamma_\gamma)\times10^{-7}\right], \tag{13}$$

whereas the difference in transit time of particles of the same mass but different energies E_1 and E_2 is given by

$$\Delta t_{12}=2.68\times10^{12}\ \text{sec}[1+(2.16\gamma_m+10.16)y\times10^{-7}]m^2\left[\frac{1}{E_1^2}-\frac{1}{E_2^2}\right]. \tag{14}$$

It is clear in (14) that the effect of gravitation (term containing y) on this dispersion time is down by a factor between 10^{-5} and 10^{-6} compared to the dominant term which is the special-relativistic time dispersion and, as was done in Ref. 6, can be safely ignored in estimating the mass m leading to the bound $m \le 15$ eV for the neutrino mass from the input $\Delta_{12} \le 10$ sec for $E_1=7.5$ MeV and $E_2=40$ MeV. In (13) the term involving ym^2/E^2 is the gravitational effect on the difference in transit time between a photon and a relativistic particle of mass m due to the mass of the particle. It is clear that for "reasonable" values of γ_m that this term is down by a factor between 10^{-5} and 10^{-6} from the special-relativistic effect on this time difference and thus can be ignored as was argued in Refs. 4 and 5. Equation (13) becomes

$$\Delta t_{\gamma m}=2.68\times10^{12}\ \text{sec}\left[\frac{m^2}{E^2}+4.32y(\gamma_m-\gamma_\gamma)\times10^{-7}\right]. \tag{15}$$

From (15) we see that if $\Delta t_{\gamma m}<0$ then $\gamma_m<\gamma_\gamma$, a violation of the equivalence principle. On the other hand, if one assumes only that $|\Delta t_{\gamma m}|<T$ and if $T \gg 2.68\times10^{12}\ \text{sec}\, m^2/E^2$ for the observed neutrinos then one obtains a bound

$$|\gamma_m-\gamma_\gamma|<8.6\times10^{-7}\frac{T}{y}. \tag{16}$$

With $T=6$ h (Ref. 10), which indeed satisfies $T \gg 2.68\times10^{12}\ \text{sec}\, m^2/E^2$ for the observed neutrinos, and $y=6$ one obtains for $|\gamma_m-\gamma_\gamma|$ the value 3.1×10^{-3}, in agreement with Refs. 4 and 5.

In summary, we have calculated the explicit dependence, Eq. (10), of the gravitational transit time on the mass of the particle. This effect of a finite neutrino mass is found to be negligible on the test of the Einstein equivalence principle as already argued by Longo[4] and Krauss and Tremaine.[5]

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[6] See, for example, J. N. Bahcall and S. Glashow, Nature (London) **32**, 135 (1987).

[7] See, for example, S. K. Bose, *An Introduction to General Relativity* (Wiley Eastern Limited, New Delhi, 1980).

[8] There is significant uncertainty regarding the value of D. See E. Kolb, A. Stebbins, and M. Turner, Phys. Rev. D **35**, 3598 (1987).

[9] See, for example, J. V. Narlikar, *Introduction to Cosmology* (Jones and Bartlett, Boston, 1983). The larger value $y=6$ corresponds to the estimate in J. Bahcall, M. Schmidt, and R. N. Soneira, Astrophys. J. **265**, 730 (1983), in which most of the mass is in the halo.

[10] There is considerable uncertainty in the difference in transit times of the photons and neutrinos as is discussed in Refs. 4 and 5. The value $T=6$ h was adopted in Ref. 4.