Multimode resonant gravitational-wave antennas: How many modes is enough?

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A model for multimode gravitational-wave antennas is presented. A complete sensitivity analysis is carried out for a simplified, but realistic degenerate case. We discuss the optimum configuration, proving that it is set by the two requirements that the last oscillator be light enough to achieve a sufficiently large energy coupling factor β and the mass ratio μ satisfy a simple optimization condition. We show that the constraints imposed by current technology set an optimal number of modes. It is not advantageous to increase the degrees of freedom beyond this value, as this would add thermal noise to the total noise figure, without substantial gain in the efficiency of signal conversion.

I. INTRODUCTION

The search for gravitational radiation[1] demands more and more sensitive detectors to improve the chances of capturing impulsive events generated in nearby galaxies. The main efforts to increase the sensitivity of resonant detectors have focused, in recent years, on reducing the thermal background noise, with the adoption of cryogenic antennas and high Q materials, on reducing the losses in the readout circuitry and the amplifier noise, with the use of superconductive electronics, and finally on improving the efficiency of the conversion of vibrational energy to a measurable electrical signal. Much improvement is still possible in the first two fields, especially with the construction of detectors that will operate at a temperature below 0.1 K and the development of nearly quantum limited SQUID amplifiers, but the problem of the electromechanical coupling (i.e., the transducer) is the one that can in principle allow a very large gain.

The efficiency of the electromechanical conversion is customarily described by the coupling coefficient β (Ref. 2), defined as the fraction of mechanical energy transformed into electrical energy in one period of oscillation (a more formal definition is given in Sec. II). This parameter is invariably, at present, much smaller than the limiting value of unity that is in principle achievable in passive devices.

A major milestone in the evolution of high-coupling transducers has been the development of resonant devices.[2,3] In this scheme the signal energy deposited in the antenna is transferred to a light oscillator incorporated in the transducer design and tuned to the antenna resonant frequency ω_0, so that the signal is transformed to a larger amplitude of vibration. The second oscillator acts as an impedance-matching stage between the massive antenna ($M_a = 1-5$ tons) and the mechanical input of the transducer. As only this second oscillator takes part in the energy conversion, the coupling factor β, that is inversely proportional to the oscillator mass M_t, can be made a factor $\mu = M_t / M_a$ larger. It would seem that, by making M_t arbitrarily small, one could easily achieve $\beta \approx 1$. However, by decreasing M_t, thermal noise generated in the second oscillator eventually becomes the dominant noise source in the whole detector. The present state of the art of other parameters (temperature, Q's, amplifier noise, electrical matching, etc.) sets an optimum value of $\mu \approx 2\times 10^{-4}$ (Ref. 4) and, with values close to this, the best linear transducers realized so far have achieved a β of a few 10^{-2} (Ref. 5).

Furthermore, the detector behaves as a tightly coupled two-mode system, and its intrinsic bandwidth is limited by the frequency splitting of the two normal modes, i.e., $\Delta\omega = \omega_0\sqrt{\mu}$. The choice of the value of μ implies therefore a tradeoff between detector bandwidth and efficiency of the energy conversion.

In order to overcome this problem, Richard[6] has proposed and pursued a scheme that uses resonators of progressively decreasing mass to transfer the signal from the antenna down to the transducer in several steps. As the mass ratio between each resonator and the next is now closer to unity, the energy transfer is much faster and the impedance matching can be more effective.

This paper will address the issue of whether it is advantageous to increase the number of intermediate oscillators to an arbitrary value, or rather if there exists a point of diminishing return or even an optimal value for the number of modes in the detector.

A sensitivity analysis of a multimode system is in principle quite complex, because with the number of modes it increases the number of noise sources to be considered and the complexity of their frequency dependence, as well as the dimensionality of the "parameter space." Numerical calculations[7,8] and approximate analytical relations[9] for three modes have been reported so far.

We will show here how the analysis of bandwidth and matching conditions developed for one- and two-mode detectors[4,9,10] can be extended, under general simplifying assumptions, to more complex systems in order to infer the potentials and limitations of such devices.

II. A MODEL FOR MULTIMODE DETECTORS

We are concerned here with the analysis of an electromechanical system like the one shown in Fig. 1, each resonator is modeled as a simple mass M_j on a spring. The spring constant K_j is chosen in such a way to obtain the desired uncoupled resonant frequency $\omega=(K_j/M_j)^{1/2}$. Damping can be neglected in the response function,[10] but will be considered later as a noise source, and the various oscillators are arranged on a linear chain. We identify the first one with the antenna: its mass M_1 is the mass of the quadrupole mode that interacts with the gravitational wave (GW) and must always be as large as it is feasible because it determines the GW absorption cross section. It is generally of the order of 10^3 kg. We shall neglect in the following the interaction of GW's with any of the oscillators except the first. The last resonator is the transducer, and its mass must be set by the above considerations. The position of each mass M_j is described by a coordinate x_j measured with respect to an inertial reference frame.

We shall consider a system composed of N resonators with effective masses (the mass taking part in the vibrational mode considered) increasingly small:

$$M_1 > M_2 > \cdots > M_{N-1} > M_N \ .$$

The action of a transducer is customarily described by two equations relating force and velocity at the mechanical end with voltage and current at the electrical one:[11]

$$F = Z_{11} u + Z_{12} I, \quad V = Z_{21} u + Z_{22} I \ , \tag{1}$$

where $u=(d/dt)(x_N - x_{N-1})$. We have assumed, for definiteness, that the transducer senses the relative motion between the last two masses, as shown schematically in Fig. 1. Other solutions are possible[7] but this does not change the qualitative features of our model, as the vibration sensed is mainly determined by the motion of the smallest mass.

In passive devices, to which we limit this analysis, the four coefficients Z_{ij} are linear differential operators. The energy coupling coefficient β is defined, in terms of this description, by the relation

$$\beta = \frac{|Z_{12}|\,|Z_{21}|}{M_N \omega\, |Z_{22}|} \equiv m_r / M_N \ , \tag{2}$$

the second definition is useful to define a "figure of merit" m_r for the transducer, independent of mass M_N of the last oscillator. The coupling strength m_r [sometimes referred to as the electromagnetic (em) mass of the transducer] is limited by current transducer technology.

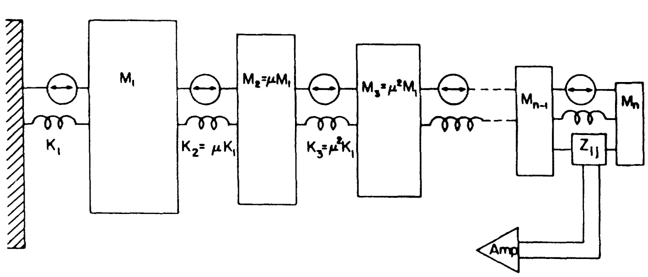

FIG. 1. A schematic diagram of the degenerate multimode gravitational-wave antenna considered for analysis.

We can estimate the energy stored in the transducer by averaging over a period the power available in it:

$$\begin{aligned} U_{\rm em} &= \int dt (F u^* + V I^*) \\ &= \int dt (Z_{11} |u|^2 + Z_{12} u^* I + Z_{21} I^* u + Z_{22} |I|^2) \ , \end{aligned} \tag{3}$$

where an asterisk indicates complex conjugation.

Note that the first term represents the mechanical energy stored in the last oscillator, and therefore will be considered in the mechanical part of the Lagrangian. Analogously, the last term is the dc electromagnetic energy stored in the transducer and is to first approximation a constant. We will assume in the following a lossless, linear transducer for which $Z_{12} = Z_{21}$.

We can now write the Lagrangian for this system:

$$\begin{aligned} L_m &= L_{\rm mech} - U_{\rm em} \\ &= \frac{1}{2} \left[\sum_j M_j \dot{x}_j^2 - \sum_j K_j (x_j - x_{j-1})^2 \right] \\ &\quad - \int dt [\, |Z_{12}| (\dot{x}_N - \dot{x}_{N-1})^* + I + Z_{22} I^2] \ , \\ &\qquad j = 1, \ldots, N \end{aligned} \tag{4}$$

so that, after Fourier transformation, and defining as usual $\omega_{0j}^2 \equiv K_j / M_j$ and $\mu_j \equiv M_{j+1}/M_j$, the equation of motion for the jth mass can be arranged to read

$$\begin{aligned} -\omega_{0j}^2 x_{j-1} + (-\omega^2 + \omega_{0j}^2 + \mu_j \omega_{0j+1}^2) x_j - \mu_j \omega_{0j+1}^2 x_{j+1} \\ = [F_j(\omega) - F_{j+1}(\omega)]/M_j \ . \end{aligned} \tag{5}$$

The Nyquist driving term $F_j(\omega)$ is the Fourier transform of a stochastic process with Gaussian distribution and zero mean (F_1 will in addition contain a term due the GW pseudoforce), so that the system will exhibit small oscillations about the equilibrium position.

Readout schemes that make use of electrical resonant circuits have been proposed[12] and realized.[13] Because of the low quality factor Q of such devices in kHz region, it is not advantageous to tune them close to the higher-Q mechanical resonances. For this reason we have neglected "kinetic" terms from the electrical variables. Should a tightly coupled electrical resonator be considered, it is always possible to make use of classical electrical-mechanical analogies[14] to model it with an additional equivalent mechanical oscillator.[15] When the resonant features of the transducer, if any, are separated and dealt with in the outlined fashion, it can be proven[16] that the presence of the additional electrical variable I (or V if a different amplifier model is used) does not affect the dynamics of the mechanical, resonant part of the detector. The only effect one needs to consider is a possible tuning of the last oscillator resonant frequency due to the additional restoring force exerted by the em field.[2,17] In other words, the eigenvalue equation of the resonant system is unaffected by the presence of the energy conversion process, except for adjusting one parameter. For this reason we shall simplify the analysis by leaving out the electrical variables from the dynamics of the system, and by taking into account the effect of amplifier and other electrical dissipations only in the noise analysis,[18] by referring them to the last oscillator as additional mechanical driving terms (see Sec. IV). This procedure has the advantage of making the analysis independent of the characteristics of the transducer used: for a given device, however, the Lagrangian (4) can give a complete description of the dynamics of all the variables, both mechanical and electrical.

III. SIMPLIFYING THE MODEL: TOTAL DEGENERACY

Before writing down the equations of motion for the multimode detector, we shall make some simplifying assumptions on the parameters. First, we shall assume that the ratio between the mass of each resonator and the one next to it is constant:

$$\mu_j = M_{j+1}/M_j \equiv \mu < 1 .$$

The mass ratio μ is therefore set by $\mu^{N-1} = (M_N/M_1)$ and gets closer and closer to 1 as N increases. The preceding assumption is not a very restricting one: although other schemes are possible, a geometric progression is a very natural choice, based on the symmetry of the problem, and it is likely to be an optimal one. If we consider the linear chain of oscillators that constitute the detector as an impedance matching network between the antenna and the transducer, it is desirable to have the maximum homogeneity along the line, in order to reduce reflections of the sound wave.

On the same ground we shall also assume total degeneracy: all oscillators have the same resonant frequency $K_{ij}/M_j = \omega_{0j}^2 \equiv \omega_0^2$; this also implies $K_j/M_{j+1} = \mu\omega_0^2$.

The spring constant of the last oscillator will be chosen in such a way as to give a resonance at ω_0 when tuned up by the transducer

$$(K_N + \Delta K_{\text{em}})/M_N = \omega_0^2 . \tag{6}$$

We can now write the N mechanical equations of motion in a convenient form by dividing the jth equation by $M_j\omega_0^2$ and normalizing from now on all frequencies to ω_0^2:

$$\begin{aligned} -x_{j-1} + (-\omega^2/\omega_0^2 + 1 + \mu)x_j - \mu x_{j+1} &= [F_j(\omega) - F_{j+1}(\omega)]/M_j\omega_0^2 \quad j = 1, \ldots, N-1 , \\ -x_{N-1} + (-\omega^2/\omega_0^2 + 1)x_N &= F_N(\omega)/M_n\omega_0^2 . \end{aligned} \tag{7}$$

The Nth equation is different from the others because $K_{N+1} = 0$, the last mass floats on its spring without a restoring force from the other side. We will show that, apart from this feature, the problem is formally identical to a classical textbook problem of classical mechanics.[14,19]

It is worth repeating that this set of equations only defines the eigenvalue problem, i.e., the mechanical part of the detector. For a full sensitivity analysis we shall add the appropriate terms describing the coupling to the motion detector and the amplifier.

For simplicity, we introduce the shorthand notation

$$z = 1 + \mu - (\omega^2/\omega_0^2) , \tag{8}$$

z is a parameter similar to the mechanical impedance of an oscillator; it is here a dimensionless quantity relating force and displacement.

IV. EIGENFREQUENCIES AND AVAILABLE BANDWIDTH

With the notation introduced above, the matrix of coefficients of our system assumes a simple tridiagonal form:

$$\begin{bmatrix} z & -\mu & 0 & 0 & \cdots & & & 0 \\ -1 & z & -\mu & 0 & \cdots & & & 0 \\ 0 & -1 & z & -\mu & \cdots & & & 0 \\ \vdots & & & & & & & \vdots \\ 0 & & & & \cdots & -1 & z & -\mu \\ 0 & & & & \cdots & 0 & -1 & (z-\mu) \end{bmatrix} \equiv D_N . \tag{9}$$

The general, nondegenerate case can be rederived by letting $\mu \to \mu_j' = \mu_j(\omega_{0j+1}/\omega_{0j})^2$ and $z \to z_j = (1 + \mu_j' - x^2)$ in each jth equation.

The determinant of the matrix (9) satisfies a simple recursion relation

$$D_N = zD_{N-1} - \mu D_{N-2} \tag{10}$$

with the initial conditions

$$D_0 = 1, \quad D_1 = z - \mu ,$$

and it can then be explicitly[16] written in terms of Tchebichev polynomials of the second kind U_n:

$$\begin{aligned} D_N &= (\sqrt{\mu})^N [U_N(z/2\sqrt{\mu}) - \sqrt{\mu}U_{N-1}(z/2\sqrt{\mu})] \\ &= (\sqrt{\mu})^N \{\sin[(N+1)\psi]/\sin\psi - \sqrt{\mu}\sin(N\psi)/\sin\psi\} , \end{aligned} \tag{11}$$

where $\psi \equiv \arccos(z/2\sqrt{\mu})$; to first order in $\sqrt{\mu}$ the roots of this equation are

$$\psi_k = \frac{k\pi - \sqrt{\mu}\sin\left[\dfrac{k\pi}{N+1}\right]}{N+1} . \tag{12}$$

Therefore, for $\sqrt{\mu} \ll 1$, the eigenfrequencies are simply given by

$$\omega_k^2 \approx \omega_0^2 \left[1 + \mu - 2\sqrt{\mu}\cos\frac{k\pi}{N+1}\right], \quad k = 1, \ldots, N , \tag{13}$$

so that all normal modes lie in what we define as the available bandwidth, given by

$$\Delta\omega/\omega_0 \approx 2\sqrt{\mu}\sin\frac{\pi}{2}\frac{N-1}{N+1} \equiv 2\alpha_N\sqrt{\mu} \tag{14}$$

with $\frac{1}{2} \le \alpha_N < 1$.

When the mass ratio μ is increased, the corrections due to the second term in Eq. (12) become more important and the frequencies of the normal modes deviate from the values of zero-order solution (13). It is easy to verify that even in the most extreme case considered, the available bandwidth never increases above the limits set by Eq. (14), although an overall shift of the mode pattern is noticeable. The approximations made in what follows will hold only if N and the ratio M_N/M_1 are chosen to satisfy the condition $\mu \ll 1$ which is not too strong a restriction, given the obvious experimental constraints.

The obvious advantage of a multimode detector is that it allows us to choose the parameter μ, and therefore the available bandwidth, by selecting N appropriately.

However, the available bandwidth is not necessarily the useful bandwidth, as the signal-to-noise ratio can drop to insignificant values in large sections of this frequency region, either between the modes[5(a)] or on the eigenfrequencies themselves.[8]

In order to derive the conditions that allow us to fully exploit the detector potential bandwidth we now have to compute the signal-to-noise ratio spectral density, and then find design criteria that will ensure its maximum flatness over the available bandwidth.

V. NOISE AND SIGNAL IN MULTIMODE ANTENNAS

Although the eigenmodes of the system can be easily computed,[16] they are not too useful to evaluate the antenna sensitivity, as the picture based on independent normal modes neglects important correlation effects.[20] Moreover, as discussed in Ref. 9 and verified by experimental evidence,[8] the signal-to-noise ratio tends to peak in between the normal modes when the system is dominated by thermal noise, while it concentrates about the eigenfrequencies only when amplifier noise is predominant. This behavior cannot be predicted on the basis of independent, noninteracting normal modes, and therefore we will take the cumbersome, but more accurate approach that consists of summing all the noise contributions due to the various sources as they appear at one point of the system, typically the output. As we have left out of the description of this system the electrical variable that describes the electromechanical transduction process, we will refer all measurements to the displacement of the last oscillator, i.e., x_N. The amplifier noise is folded into this picture by modeling[18] the wideband noise with a random error on the detected velocity, and the input noise with an additional noise force acting on the last mass plus a contribution to wideband noise. This is equivalent to referring the two amplifier equivalent noise sources to the input of the transducer (mechanical port) while preserving the white behavior of their spectrum. This implies two reasonable approximations: first, we neglect the frequency dependence of the transducer transimpedances, i.e., we consider the coupling parameter β a constant with respect to frequency; second, confound the transfer function of the amplifier input noise with that of the thermal noise force associated with the last oscillator; the error introduced is irrelevant unless the system is back-action limited.

We now proceed to construct an expression for the signal-to-noise ratio of such a system starting from the set of linear equations (7). We recall that, when optimum linear filtering is applied,[21] we have

$$S/N=\int \frac{d\omega}{2\pi}\frac{x^2(\omega)_{\rm signal}}{x^2(\omega)_{\rm noise}}=\int\frac{d\omega}{2\pi}\frac{|H_1(\omega)F_g(\omega)|^2}{\sum_j |H_j(\omega)|^2 S_{F_j}(\omega)+|H_N(\omega)|^2 S_{F_{\rm ba}}+S_{x_{\rm wb}}}\ , \tag{15}$$

where the $H_j(\omega)$ are the transfer functions relating the motion of the Nth mass to the jth driving force. As each force F_j (with the exception of F_1) appears in two consecutive equations, the transfer functions are

$$H_j(\omega)=\frac{x_N}{F_j}=(M_j\omega_0^2)^{-1}[A_{j,N}(\omega)+\mu A_{j-1,N}(\omega)]/D_N(\omega)\ , \tag{16}$$

where $A_{j,N}$ are the minors obtained from D_N by canceling the jth row and the Nth column. It is fairly straightforward to prove that

$$A_{j,N}=(-1)^{N-j}(\sqrt{\mu})^{j-1}U_{j-1}\ , \tag{17}$$

so that, recalling Eq. (11), we have simply

$$|H_j(\omega)=(M_1\omega_0^2)^{-1}(\sqrt{\mu})^{1-j}|D_{j-1}(\omega)/D_N(\omega)|\ . \tag{18}$$

We recall also that, for a short burst of gravitational radiation deposing an energy $\Delta E=k_BT_d$ (it is customary to express energies as temperatures times Boltzmann's constant k_B) in the antenna, one can write

$$|F_g(\omega)|^2=2k_BT_dM_1\ . \tag{19}$$

The detection noise temperature is defined as the value of T_d for which $S/N=1$ in Eq. (15).

The noise force spectral densities (bilateral) are related to the dissipation by Nyquist theorem and defined (in terms of the quality factor Q_j of the jth oscillator) as

$$S_{F_j}(\omega)=2k_BT_aM_j\omega_0/Q_j=2k_BT_aM_1\omega_0(\mu^{j-1}/Q_j)\ . \tag{20}$$

Finally we account for the electronics noise assuming for definiteness a voltage amplifier. The usual[22] amplifier model consists of an ideal (noiseless) amplifier plus an input impedance and two noise sources that we shall assume uncorrelated, voltage noise that is entirely transferred to the amplifier output (wideband noise) and a current source that generates circulating noise currents in the input circuit (and has therefore a resonant signature). It is customary to express the spectral densities of these two quantities in terms of an *amplifier noise temperature* T_n and a *noise match impedance* $R_{\rm opt}$, such that $S_v=k_BT_nR_{\rm opt}$ and $S_I=k_BT_n/R_{\rm opt}$.

The wideband noise source $S_{\rm wb}$ must include also the wideband contribution from the input noise,[4,16] i.e., the noise voltage arising from the circulating noise current

$$S_{V_{(\rm wb)}}=k_B T_n(R_{\rm opt}+|Z_{22}|^2/R_{\rm opt}) =k_B T_n|Z_{22}|(\lambda^{-1}+\lambda) , \quad (21)$$

where with λ we indicate as usual[23] the ratio between the overall electrical impedance and the amplifier noise match impedance. This provides an error on x_N that can be represented, using the relation $V=Z_{21}\dot{x}$, as

$$S_{x_{\rm wb}}=\frac{S_{V_{\rm wb}}}{\omega^2|Z_{21}|^2} =(\omega_0^3 M_1)^{-1}k_B T_n(1+\lambda^2)/(\beta\lambda\mu^{N-1}) , \quad (22)$$

where, Eq. (2) for β and $\omega\approx\omega_0$ have been used.

The input current noise $S_{I_{\rm ba}}=k_B T_n\lambda/Z_{22}$ is often called back action, referring to the fact that it is converted back into the system by the reverse action of the transducer and appears as an effective additional mechanical noise. With the approximations outlined above it is modeled as an additional force $F_N'=Z_{12}I_n$, such that

$$S_{F_{\rm ba}}=|Z_{12}|^2 S_{\rm ba}=M_N\omega_0\beta\lambda k_B T_n =M_1\omega k_B T_n(\beta\lambda\mu^{N-1}) . \quad (23)$$

Other noise sources, due to dissipating elements in the transducer and/or the readout circuit, are modeled as resistors in the input circuit, and taken into account by modifying accordingly Re Z_{22}.

Putting together all the terms in Eqs. (17)–(24) we can reexpress Eq. (15) as

$$S/N(\omega)d\omega=\frac{2T_d}{2\pi}\frac{d\omega}{\omega_0}\left[\sum_{j=0}^{N-1}(D_j)^2 2T_a(\mu^j Q_{j+1})^{-1}+(D_{N-1})^2 T_n\beta\lambda/\mu^{N-1}+D_N^2 T_n(1+\lambda^2)/(\beta\lambda\mu^{N-1})\right]^{-1} . \quad (24)$$

Note that the merely algebraic step of multiplying through by D_N^2 (that in practice corresponds to divide by the transfer function H_1 from the antenna input to the output), is in all respects equivalent to referring the measurement to the antenna input.

Equation (24) gives a closed-form solution for the signal-to-noise ratio spectral density of gravitational-wave antennas with any number of modes. The second term, representing the back-action force, is often included in the last term of the sum, as it has the same frequency dependence, by redefining an effective temperature $T_{\rm eff}=T_a(1+T_n Q_N\beta\lambda/2T_a)$ (Ref. 4). However, this effective temperature only applies to the last oscillator, while all other masses are at the same temperature T_a. In order to avoid this unphysical representation, we have found it convenient to define the noise parameters θ_j corresponding to dimensionless noise sources:

$$\theta_j=2T_a/T_n Q_j,\ \ j=1,\dots,N-1 ,$$
$$\theta_N=2T_a/T_n Q_n+\beta\lambda,\ \ \theta_{N+1}=\mu(\lambda+\lambda^{-1})/\beta . \quad (25)$$

The explicit linear dependence on μ of the last noise parameter θ_{N+1} plays a crucial role in the optimization procedure that follows. Using these parameters, and the definition (11) of $D_j(\omega)$, we can define the total noise spectral density

$$N(\omega)=\sum_{j=0}^{N}[U_j(y)-\sqrt{\mu}U_{j-1}(y)]^2\theta_{j+1} , \quad (26)$$

where

$$y=\cos\psi=(1+\mu-\omega^2/\omega_0^2)/2\sqrt{\mu} \quad (27)$$

and $U_{-1}\equiv 0$ should be intended. It can be seen from Eq. (13) that, for the frequencies of interest to our problem $-1<y<1$.

We can now cast Eq. (24) in a compact and suggestive form:

$$T_d=\frac{\pi T_n\omega_0}{\int\frac{d\omega}{N(\omega)}}\approx\frac{\pi T_n}{\sqrt{\mu}\int\frac{dy}{N(y)}} . \quad (28)$$

It is remarkable that such a simple equation accurately describes the sensitivity of a system in principle quite complex, and that it applies to any detector that can be described by the equations of motion (7) with driving forces (20). Note that the only frequency dependence in Eq. (28) is in the noise spectral density $N(\omega)$, so that it is sufficient, for all frequency considerations on the signal-to-noise ratio, to examine $1/N(\omega)$.

A famous example: the two-mode system. It is instructive to rederive results found in the well-studied case of a two-mode system[4,9,10] before trying to generalize them to any number of modes. From Eq. (13) we immediately find

$$\omega_\pm^2=\omega_0^2[1+\mu-2\sqrt{\mu}\cos(k\pi/3)]\quad(k=1,2) \approx\omega_0^2(1\pm\sqrt{\mu}) . \quad (29)$$

It has been proven[9] that the best noise temperature of a detector is obtained when the signal-to-noise ratio is uniformly distributed over the available bandwidth. The requirement of maximum useful bandwidth can be expressed by the condition $N(\omega_0)=N(\omega_\pm)$ where $\omega_\pm$ are the normal-mode frequencies ($y=\pm\frac{1}{2}$) limiting the available bandwidth. This gives, within our approximations, the same result as requiring the second derivative of $N(y)$ to vanish at $y=0$ ($\omega=\omega_0$), as it can be proven by direct computation. By applying this condition to Eq. (26) we obtain $\theta_3(2-\mu)=\theta_2$, i.e., to first order in μ,

$$\mu_{\rm opt}(N=2)=(\beta\lambda T_a/Q_L T_n+\beta^2\lambda^2)/2(1+\lambda^2) \quad (30)$$

as derived in Refs. 9 and 16 for two-mode antennas.

In this optimized case, when the signal-to-noise ratio is roughly constant over the bandwidth, we can substitute the integral over frequencies with the product of the maximum useful bandwidth times the function value at the center frequency ω_0, i.e.,

$$T_d \approx T_n N(\omega_0)\omega_0/\Delta(\omega) , \quad (31)$$

so that the noise temperature of the detector can be approximated by

$$T_d = (\mu_{\rm opt})^{-1/2}\{2T_a/Q_a + \mu_{\rm opt} T_n[\beta\lambda + (\lambda^2+1)/\beta\lambda]\} . \quad (32)$$

Note that once the identification $\sqrt{\mu} = \Delta\omega/\omega_0 = \pi/t_s\omega_0$ (t_s is the sampling time) is made, Eq. (32) is formally identical to the simple condition for energy sensitivity first derived by Giffard.[18] While in that derivation the bandwidth was set by the electronics with the choice of an optimum t_s, here we are adjusting the antenna hardware parameters (masses, couplings, and impedances) in such a way that optimum filtering will take full advantage of the detector's available bandwidth.

VI. USEFUL BANDWIDTH AND OPTIMIZATION

We shall now generalize this criterion to a system with $N > 2$, requiring both first and second derivative of $N(\omega_0)$ to vanish, in order to ensure that $1/N(\omega_0)$ is a stationary point.

Using properties of the Tchebichev polynomials,[24] it can be proven that

$$[d^2N(\omega)/d^2\omega]_{\omega=\omega_0} = \sum_{j=0}^{N} (a_j + \mu a_{j-1})\theta_{j+1} , \quad (33)$$

$$a_j = -j(j+2), \quad j \text{ even} ,$$
$$a_j = (j+1)^2, \quad j \text{ odd} . \quad (34)$$

Recalling that θ_{N+1} is explicitly of order μ, we find that to first order in μ, Eq. (33) leads to the following generalization of (30):

$$\mu_{\rm opt}(N) = \frac{-\sum_j a_j\theta_{j+1}}{\sum_j a_{j-1}\theta_{j+1} + a_N(1+\lambda^2)/\beta\lambda} , \quad j = 0, 1, \ldots, N-1 . \quad (35)$$

In detectors of the present generation ($T_a \approx 1$ K, $Q \approx 10^6$, $T_n \approx 10^{-5}$ K) the noise parameters θ_j have values of the order of 10^{-1}, while $(\beta\lambda)^{-1}$ can be quite large ($\approx 5\times 10^3$ or larger) due to the difficulties in properly matching a SQUID. It is then natural to approximate (35) with

$$\mu_{\rm opt}(N) = \frac{\beta\lambda}{1+\lambda^2} \sum_{j=0}^{N-1} (-a_j/a_N)\theta_{j+1} .$$

Note that a $\mu_{\rm opt} > 0$ does not necessarily always exist, e.g., when an intermediate mass has a very low Q. Insertion of intermediate resonators can degrade the system rather than improve it, if it is not done properly. The coefficients (a_j/a_N) show that the masses near the end of the chain (closer to the amplifier) contribute a little more x to fix $\mu_{\rm opt}$. In general these coefficients, of order unity if N is not too large, can be neglected with respect to the more relevant differences in Q's so that, defining a global thermal noise source

$$\theta_{\rm th} \equiv \theta_1 + \theta_2 + \cdots + (\theta_N - \beta\lambda) = 2(T_a/T_n)\sum_j Q_j^{-1} ,$$

we can write

$$\mu_{\rm opt}(N) = \frac{\beta\lambda}{1+\lambda^2}(\theta_{\rm th} + \beta\lambda) , \quad (36)$$

where the last term on the right-hand side is only relevant for back-action limited detectors. Note that Eq. (36) is formally identical to Eq. (30) that was derived for $N = 2$.

Again, we approximate the frequency integral with a rectangle of width given by Eq. (14) and height evaluated at any point in the bandwidth. It is easy to show that at the center of the band ($\omega = \omega_0$, $y = 0$), any other noise source (those in odd-numbered positions) contribute to $N(\omega)$ with weight 1, while the others are negligible to lowest order in $\sqrt{\mu}$. Similarly, at $y = \pm 0.5$ all terms except those in positions labeled with multiples of three add with unit coefficient. We shall then take the most conservative approach of assuming $\langle N(\omega)\rangle = \sum_j \theta_j$. This also satisfies the heuristic approach of requiring all noise sources to contribute equally for a well designed detector. We then have

$$T_d \simeq (T_n/2\alpha_N\sqrt{\mu_{\rm opt}})[\theta_{\rm th} + \beta\lambda + \mu_{\rm opt}(\lambda^2+1)/\beta\lambda] , \quad (37)$$

where α_N is a parameter of the order of unity given by Eq. (14). In the following we shall substitute for $1/\alpha_N$ its largest value, i.e., 2 with little error.

Equation (37) shows that once the mass ratio is optimized, every resonator contributes equally to the noise temperature (if the Q's are comparable), independently of its mass and of the specific location. It should be mentioned that Eq. (37) also shows why the model is inadequate to treat in a satisfactory way a continuum limit to this system. A transition to the continuum would require $N \to \infty$, $\mu \to 1$, and therefore an infinite number of noise terms that add with equal weight; however the GW signal, collected only in the first mass of the chain, does not increase: this, besides being inconsistent with the limit $\mu \to 1$, would produce a vanishing signal-to-noise ratio.

Optimization of Eq. (37) with respect to μ would again yield Eq. (36), confirming the consistency of the derivation. The optimized value of the detection noise temperature is then

$$T_d = 2T_n[(1+\lambda^2)(1+\theta_{\rm th}/\beta\lambda)]^{1/2} . \quad (38)$$

It is then evident from Eq. (37) that increasing the number of modes introduces more thermal noise in the detector (as $\theta_{\rm Th}$ is roughly proportional to N), while it increases the fractional bandwidth $\sqrt{\mu}$ and allows a larger value of β. Competition between these effects must give an optimum value for N.

Note that the condition on the ratio μ between successive masses depends very weakly on the number N of resonators used; it is mainly determined by the matching of the amplifier and transducer to the mass or masses with the lowest Q. It might seem that nothing can be gained by going toward more complex systems as one cannot increase $\sqrt{\mu}$, a parameter that determines both the bandwidth and the sensitivity [see Eq. (37)] of our detector. Fortunately we still have a degree of freedom, namely, the mass of the last oscillator $M_N = M_1\mu^{N-1}$, which strongly depends on N, and through which we can set the desired coupling coefficient β.

The proper optimization procedure is then as follows. First note from Eq. (38) that

$$\beta \gg \theta_{\text{th}}/\lambda \approx N\theta_1/\lambda \tag{39}$$

is required for the system to be back-action limited, which is necessary to achieve the Giffard limit $T_d = 2T_n$. N has actually yet to be determined, but we can safely assume (and prove *a posteriori*) that taking $\beta > 10\theta_1/\lambda$ will satisfy all practical needs. Obviously, for passive transducers, β is limited to a max value of unity. In some cases a lower value of β could be desirable, e.g., to prevent electrical losses to degrade the Q's of the mechanical oscillators; in any case, once β is chosen, Eq. (36) unambiguously gives the optimum value of μ.

Finally, recalling from Eq. (2) that β scales linearly with $(M_N)^{-1}$,

$$\beta = m_r/M_N = m_r/M_1\mu^{N-1} ,$$

we can derive the number of oscillators needed to achieve proper electromechanical matching:

$$N = 1 + [\ln(m_r/M_1\beta)/\ln(\mu_{\text{opt}})] \tag{40}$$

obviously rounded to the closest integer. Equation (40) is a slowly varying function of the three arguments involved, and gives a numerical result between 2 and 4 for most cases of practical interest (see Table I). This proves that the number of oscillators required to match a GW antenna to its amplifier will seldom exceed three, and could even be limited to two for high coupling, ultralow-temperature systems such as the one suggested in the fourth column of Table I.

In general, Eqs. (36), (39), and (40), along with the constraint on β mentioned above, constitute a set of equations to be solved self-consistently for β, μ, and N, but the approximate solution given here is accurate for most cases of interest.

VII. CONCLUSIONS

The proper impedance matching between an antenna and the amplifier is still an outstanding problem in gravitational-wave experiments, and prevents current detectors from achieving the so-called "Giffard limit" in sensitivity $T_d = 2T_n$. Several researchers have addressed the problem,[26] both theoretically and experimentally, with different approaches. We have shown that a general sensitivity analysis of a multimode gravitational wave antenna can be set up and numerically evaluated. We have then introduced the definitions of available bandwidth and useful bandwidth and showed that the latter approaches the first in the optimal cases. The analysis has been explicitly carried out in a particular, degenerate case that, due to its regularities, allows a closed form solution for a system with an arbitrary number of oscillators. The relevance of such a degenerate detector is related to its likelihood to actually represent a real multimode detector. Although the noise temperature of this system cannot be evaluated analytically [a numerical integration of Eq. (28) is required in most cases of realistic interest], some simple approximations can be made for the optimized case of a system with flat response in the whole

TABLE I. Comparison of antenna parameters and predicted sensitivity improvements for a typical room-temperature detector (Ref. 25) (characterized by a high value of λ), for the cryogenic detectors (Ref. 5) at Stanford (high β) and Rome (high Q's) and for an antenna of the next generation for which a realistic estimate of noise parameters has been made, based on the results obtained so far by various GW groups. For cryogenics the antennas estimates (Ref. 27) are given for λ and T_n, as no measurement on the input noise of practical dc SQUID's has been performed to date. Note that for the third-generation antenna, where the use of ultralow temperature should further reduce the thermal noise, a high value of m_r as the one indicated would permit a match to the amplifier with just two modes.

Parameters	Room-temperature antennas (1 mode)	Stanford	Rome	Third-generation antenna
M_1 (kg)	1200	2400	1135	1135
m_r (kg)	1.8×10^{-5}	1.5×10^{-2}	10^{-4}	30
λ	0.6	0.01	0.01	0.1
T_n (K)	0.72	8×10^{-5}	3×10^{-5}	4×10^{-6}
θ_2	4×10^{-3}	0.12	5×10^{-2}	5×10^{-3}
T_d	≈ 10 K	13 mK	12 mK	15 μK
μ_{opt}	7.7×10^{-4}	1.2×10^{-3}	6×10^{-4}	10^{-3}
N_{opt}	3–4	3	3	2–3
Expected T_d	2 K	300 μK	80 μK	15 μK
$\Delta\omega$	50 Hz	70 Hz	44 Hz	57 Hz

available bandwidth, leading to two simple conditions on the optimum number of modes and on the mass ratio μ. For the current state of the art of detector parameters these conditions show that three-mode systems can perform better than present two-mode detectors, while little or no gain is to be expected by increasing N beyond 4.

The correct design strategy then calls for choosing the appropriate value of β [Eq. (39)] that will make the thermal noise of the mechanical oscillators negligible with respect to electronic noise, or minimum if other constraints apply; the optimum mass ratio $\mu_{\rm opt}$ is then determined [Eq. (37)]. Finally Eq. (40) yields the number N of modes needed to noise match the antenna to the amplifier. In this fashion the problem of matching the mechanical system to the amplifier is solved without requiring prohibitive Q values to the antenna.

This allows an increase in sensitivity, with respect to a two-mode system [where Eq. (30) uniquely determines M_2] that is proportional to $(\beta_N/\beta_2)^{1/2}$ for thermally dominated antennas.

Although this analysis is limited to linear, passive transducers, it indicates the way to achieve a back-action-limited system, and has therefore potential consequencies relevant to the programs to develop back-action-evading read-out schemes.

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