Wolfenstein-type parametrization of the quark mixing matrix

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We define two Wolfenstein-type parametrizations of the quark mixing matrix, where the parameters are easily related to directly measurable quantities. For the first parametrization, the expansion in $\lambda = |V_{us}|$ may be easily carried out to any desired order, and we give the result of this expansion up to order λ^8. As an illustration of the advantages of our second parametrization, we use it in the discussion of how to accommodate the measured value of B_d-$\bar{B}_d$ mixing within the three-generation standard model.

I. INTRODUCTION

One of the most convenient parametrizations of the Cabibbo-Kobayashi-Maskawa[1] (CKM) matrix is the one suggested by Wolfenstein.[2] In this parametrization, the quark mixing matrix is expanded in terms of the small parameter $\lambda = \sin\theta_C \approx 0.22$. The attractiveness of Wolfenstein's parametrization stems from the fact that all of its parameters, with the exception of λ, are of order unity, and therefore one readily obtains the order of magnitude of the CKM matrix elements by taking the leading terms in their expansions in powers of λ. Wolfenstein has chosen a phase convention where V_{ud}, V_{us}, V_{cd}, V_{ts}, and V_{tb} are real and positive and demanded that the imaginary part of the unitary relations be satisfied to order λ^5, while the real part be satisfied only to order λ^3. He thus obtained

$$V \approx \begin{bmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta + i\eta\lambda^2/2) \\ -\lambda & 1-\lambda^2/2 - i\eta A^2\eta^4 & A\lambda^2(1+i\eta\lambda^2) \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{bmatrix} . \quad (1)$$

It is clear that in this parametrization the parameters A,ρ,η are not related in a simple way to directly measurable quantities.

In this paper we suggest two different parametrizations of the CKM matrix, each one of them having in common with Wolfenstein's parametrization the fact that there is only one small parameter, which can be used as an expansion parameter. In both parametrizations that we suggest, all the parameters are simply related to directly measurable quantities, to all orders in the expansion parameter.

WOLFENSTEIN-TYPE PARAMETRIZATION THROUGH THREE MODULI AND ONE PHASE

In the parametrization that we will first present, V coincides up to order λ^3 with the form given in Eq. (1), but differs from it in higher orders. These higher-order corrections can be easily calculated to any desired order. We define the parametrization in the following way:

$$V_{us} = \lambda, \quad V_{cb} = A\lambda^2, \quad V_{ub} = A\mu\lambda^3 \exp(i\phi) ,$$
$$V_{ud}, V_{us}, V_{cs}, V_{cb}, V_{tb} \text{ real positive} . \quad (2)$$

Thus, $\lambda = |V_{us}|$, $A = |V_{cb}/V_{us}^2|$, $\mu = |V_{ub}/(V_{us}V_{cb})|$ are simply related to measurable quantities and limits on them are readily deducible from the experimental input:[3]

$$|V_{us}| = 0.220 \pm 0.003 ,$$
$$|V_{cb}|^2 = (1.85 \pm 0.65) \times 10^{-3} , \quad (3)$$
$$|V_{ub}/V_{cb}| < 0.22 .$$

Furthermore, $\phi = \arg(V_{ub}V_{cs}V_{us}^*V_{cb}^*)$, and therefore ϕ is simply related to $\mathrm{Im}(V_{ub}V_{cs}V_{us}^*V_{cb}^*)$, the invariant which controls the strength of CP violation in the standard model.[4] Our choice of independent parameters and our phase convention are inspired by the availability of experimental data, as well as by a recent parametrization proposed by Bjorken and Dunietz.[5] We proceed to show that Eq. (2) completely defines a parametrization of the CKM matrix. From normalization of the first row and last column of V and taking into account that we have defined V_{ud}, V_{tb} to be real positive, it follows that

$$V_{ud} = [1-\lambda^2-(A\mu\lambda^3)^2]^{1/2} ,$$
$$V_{tb} = [1-A^2\lambda^4-(A\mu\lambda^3)^2]^{1/2} . \quad (4)$$

Furthermore, from the orthogonality of the first and second rows, one deduces

$$2\,\mathrm{Re}(V_{us}V_{cb}V_{ub}^*V_{cs}^*) = U_{ud}U_{cd} - U_{us}U_{cs} - U_{ub}U_{cb} , \quad (5)$$

where $U_{ij} = |V_{ij}|^2$. Taking now into account that $U_{ud} = 1 - U_{us} - U_{ub}$, $U_{cd} = 1 - U_{cs} - U_{cb}$, Eq. (5) leads to a quadratic equation for $|V_{cs}|$. Since in our convention V_{cs} is real, positive, one obtains

$$V_{cs}=\{[1-\lambda^2-A^2\lambda^4+A^2(1-2\mu^2)\lambda^6+A^2\mu^2\lambda^8 + A^4\mu^2\lambda^{10}+A^4\mu^2(\mu^2-\sin^2\phi)\lambda^{12}]^{1/2} - A^2\mu^2\lambda^6\cos\phi\}/(1-A^2\mu^2\lambda^6) . \quad (6)$$

So far, we have exact expressions for six of the CKM matrix elements. Using the unitarity relations

$$V_{cd}=-(V_{cs}V_{us}^*+V_{cb}V_{ub}^*)/V_{ud}^* , \quad (7a)$$

$$V_{ts}=-(V_{us}V_{ub}^*+V_{cs}V_{cb}^*)/V_{tb}^* , \quad (7b)$$

$$V_{td}=-(V_{us}^*V_{ts}+V_{ub}^*V_{tb})/V_{ud}^* , \quad (7c)$$

one can then obtain exact expressions for the remaining three matrix elements.

For practical purposes, it is clearly convenient to have approximate expressions for the V_{ij}. From Eqs. (4), (6), and (7) one obtains, up to order λ^8,

$$V_{ud}\approx 1-\lambda^2/2-\lambda^4/8-(1+8A^2\mu^2)\lambda^6/16 -(5+32A^2\mu^2)\lambda^8/128 , \quad (8a)$$

$$V_{cs}\approx 1-\lambda^2/2-(1+4A^2)\lambda^4/8 +(-1+4A^2-16A^2-16A^2\mu\cos\phi)\lambda^6/16 +(-5+8A^2-64A^2\mu^2-16A^4)\lambda^8/128 , \quad (8b)$$

$$V_{tb}\approx 1-A^2\lambda^4/2-A^2\mu^2\lambda^6/2-A^4\lambda^8/8 , \quad (8c)$$

$$V_{cd}\approx -\lambda+A^2(1/2-\mu e^{-i\phi})\lambda^5 + A^2\mu(-\mu+\theta^{i\phi})\lambda^7/2 , \quad (8d)$$

$$V_{td}\approx A\lambda^3+A\mu\lambda^3e^{-i\phi}[-1+\lambda^2/2 +(1+4A^2)\lambda^4/8] , \quad (8e)$$

$$V_{ts}\approx -A\lambda^2+A(\tfrac{1}{2}-\mu e^{-i\phi})\lambda^4+A\lambda^6/8 + A(1-8A^2\mu^2+8A^2\mu e^{i\phi})\lambda^8/16 . \quad (8f)$$

Obviously, in specific calculations, one may find it sufficient to keep only a few powers in λ. Notice that if, for example, one keeps terms up to order λ^3 in the real parts and up to order λ^5 in the imaginary parts, one obtains a mixing matrix V which differs from that given in Eq. (1).

Let us now consider CP violation. For three generations, the strength of CP violation is proportional to the imaginary parts of the "quartets" $V_{ij}V_{kl}V_{il}^*V_{kj}^*$ $(i\neq k, j\neq 1)$. For the 3×3 CKM matrix, all such imaginary parts are equal up to the sign, as a result of unitarity,[4] and therefore it is sufficient to evaluate $I=\mathrm{Im}(V_{ub}V_{cs}V_{us}^*V_{cb}^*)$. In the present parametrization, the exact expression for I is

$$I=A^2\mu(\sin\phi)\lambda^6V_{cs} \quad (9)$$

with V_{cs} given by Eq. (6). The value of I may thus be easily calculated to any desired accuracy, by simply expanding V_{cs} in powers of λ, as done in Eq. (8b).

WOLFENSTEIN-TYPE PARAMETRIZATION THROUGH FOUR MODULI

Next we give a rephasing-invariant parametrization of the CKM matrix, where the parameters are simply related to four independent moduli. The parametrization is defined by

$$|V_{us}|^2=\epsilon,\quad |V_{cb}|^2=a\epsilon^2 , \quad |V_{ub}|^2=ap\epsilon^3,\quad |V_{td}|^2=aq\epsilon^3 . \quad (10)$$

In the choice of parameters, we took into account that $|V_{cb}|\approx|V_{us}|^2$, $|V_{ub}|\approx|V_{us}|^3$, and thus we kept the main feature of Wolfenstein's parametrization, of having one small parameter (in the present case ϵ) and three other parameters, a,p,q, of order unity. Indeed, from the experimental limits of Eq. (3) it follows that

$$\epsilon=0.048\pm0.0013,\quad a=0.79\pm0.32,\quad p<1.03 . \quad (11)$$

The matrix of squared moduli is written in terms of our parameters as

$$U_{ij}\equiv|V_{ij}|^2=\begin{pmatrix} 1-\epsilon-ap\epsilon^3 & \epsilon & ap\epsilon^3 \\ \epsilon+a(p-q)\epsilon^3 & 1-\epsilon-a\epsilon^2-a(p-q)\epsilon^3 & a\epsilon^2 \\ aq\epsilon^3 & a\epsilon^2+a(p-q)\epsilon^3 & 1-a\epsilon^2-ap\epsilon^3 \end{pmatrix} . \quad (12)$$

The basic difference with respect to the previous parametrization is the fact that we have introduced $q=|V_{td}/(V_{us}V_{cb})|^2$ as a basic parameter, instead of $\phi=\arg(V_{ub}V_{cs}V_{us}^*V_{cb}^*)$. This is motivated by the relevance of $|V_{td}|^2$ to the analysis of B_d-$\bar{B}_d$ mixing. Furthermore, $|V_{td}|^2$ has the advantage of being directly measurable in the future through the decay and production rates of top-flavored mesons.

As has been previously shown,[6] the knowledge of four independent moduli completely determines the CKM matrix, apart from an ambiguity in the sign of I. Therefore, the parametrization that we have presented is complete and in particular we may express $|I|$ in terms of ϵ,a,p,q:

$$I^2=(a^2\epsilon^6/4)\{(-1+2p+2q-p^2-q^2+2pq)+2p(-1+p-q)\epsilon-p[p-2a(-1+p-q)]\epsilon^2 -2ap(2p-q+2pq-p^2-q^2)\epsilon^3-ap^2(a+2p-2q)\epsilon^4-2a^2p^2(p-q)\epsilon^5-a^2p^2(p-q)^2\epsilon^6\} . \quad (13)$$

The expression for I^2 given in Eq. (13) is exact, but rather complicated. However, given the fact that ϵ is small, I^2 is well approximated by the leading term in ϵ.

Next we consider unitarity constraints. Since in this parametrization the CKM matrix is not manifestly unitary, particular attention should be given to those constraints. It has been previously shown[6] that in a parametrization of the 3×3 CKM matrix through independent moduli, the only nontrivial unitarity constraint corresponds to the requirement $I^2 \geq 0$. Given the experimental data, it is sensible to look at unitarity as a constraint on the allowed values of q. Noting from Eq. (13) that I^2 is a quadratic form in q, the condition $I^2 \geq 0$ leads to the constraint

$$(1-\sqrt{px})/(1-ap\epsilon^3) \leq (q)^{1/2} \leq (1+\sqrt{px})/(1-ap\epsilon^3) , \quad (14)$$

where

$$x \equiv (1-\epsilon-ap\epsilon^3)(1-a\epsilon^2-ap\epsilon^3) = U_{ud}U_{tb} .$$

In Eq. (14), terms of order ϵ^3 and higher can be safely neglected and one then obtains the simpler constraint

$$1-[p(1-\epsilon-a\epsilon^2)]^{1/2} \leq (q)^{1/2} \leq 1+[p(1-\epsilon-a\epsilon^2)]^{1/2} . \quad (15)$$

In Fig. 1 we give the region in the (p,q) plane allowed by unitarity. It should be noted that the rephasing-invariant I, which measures the strength of CP violation, vanishes in the boundaries of the unitarity domain

B_j-$\bar{B}_d$ MIXING

Next we will do the analysis of B_d-$\bar{B}_d$ mixing, taking into account the recent results of the Argus Collaboration.[7] This analysis has previously been done by many authors.[8] We present it here in order to illustrate the usefulness of the second parametrization that we have introduced. The Argus Collaboration observed B_d-$\bar{B}_d$ mixing at the level[7]

$$r_d = 0.21 \pm 0.08 , \quad (16)$$

where $r_d = x_d^2/(2+x_d^2)$ and $x_d = \Delta M/\Gamma$. In the standard model, x_d is dominated by the top-quark contribution to the box diagram which gives[9]

$$x_d = (G_F^2/6\pi^2)\eta M_B(B_B f_B^2)\tau_B \,|\, V_{td}\,|^2 \,|\, V_{tb}\,|^2 \times m_t^2 f_1(m_t^2/m_w^2) , \quad (17)$$

where

$$f_1(y) = 1-\tfrac{3}{4}[y(1+y)/(1-y)^2][1+2y(\ln y)/(1-y^2)] \quad (18)$$

and the notation is standard. The function $f_1(y)$ is slowly decreasing, and has limits 1, $\frac{3}{4}$, and $\frac{1}{4}$ as one lets y approach 0, 1, and ∞, respectively. In our parametrization one has

$$|\, V_{td}\,|^2 = aq\epsilon^3 = |\, V_{cb}\,|^2 \,|\, V_{us}\,|^2 q . \quad (19)$$

The semileptonic bottom decays can be used to determine $|\, V_{cb}\,|^2$ through[9]

$$\Gamma(b \to cl\bar{v}_l) = [B(b \to cl\bar{v}_l)/\tau_b] = (G_F^2 m_b^5/192\pi^3) \,|\, V_{cb}\,|^2 f_2(m_c^2/m_b^2) , \quad (20)$$

where $f_2(z) = 1-8z+8z^3-z^4-12z^2 \ln z$. Using Eqs. (17), (19), and (20) one obtains

$$q = (32\pi\eta M_B \,|\, V_{us}\,|^2 \,|\, V_{tb}\,|^2)^{-1}[2r_d/(1-r_d)]^{1/2} \times \{m_b^5 f_2(m_c^2/m_b^2)/[B_B f_B^2 B(b \to cl\bar{v}_l)]\} \times [m_t^2 f_1(m_t^2/m_w^2)]^{-1} . \quad (21)$$

In Table I we give the values of q which thus arise when one uses Eq. (16) and the following ranges of values for the parameters entering in Eq. (21) (Ref. 3):

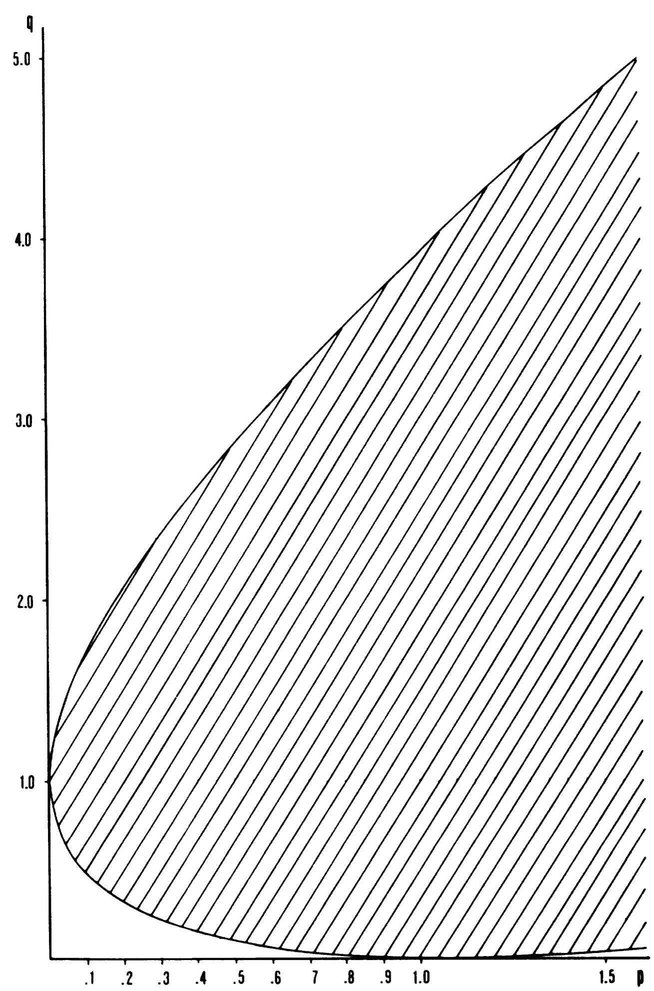

FIG. 1. Region in the (p,q) plane allowed by unitarity of the CKM matrix, obtained using Eq. (15); we took $\epsilon = 0.0484$, $(a\epsilon^2) = 1.85 \times 10^{-3}$.

TABLE I. Range of values for q required by the observed strength of B_d-$\bar{B}_d$ mixing, for different values of m_t. We indicate both the extreme and the central values of q.

m_t (GeV)	Minimum q	Central q	Maximum q
20	13.06	51.72	235.83
30	6.01	23.80	108.54
40	3.52	13.93	63.53
50	2.35	9.30	42.41
60	1.70	6.74	30.74
80	1.04	4.12	18.80
100	0.72	2.86	13.02
120	0.54	2.13	9.73
140	0.42	1.67	7.64
160	0.34	1.36	6.21
180	0.29	1.14	5.19
200	0.24	0.97	4.42

$$\begin{aligned}&\eta=0.85,\quad M_B=5.28\ \text{GeV},\quad |V_{us}|^2=0.220\ ,\\&|V_{tb}|^2\approx 1,\quad m_b=5.0\pm0.3\ \text{GeV}\ ,\\&(m_c/m_b)=0.30\pm0.03,\quad B_Bf_B^2=(0.15\pm0.05\ \text{GeV})^2\ ,\\&B(b\to c)l\bar{\nu}_l)=0.121\pm0.008\ .\end{aligned}\tag{22}$$

From Table I, together with the unitarity constraints of Fig. 1, it is clear that a lower limit for m_t of about 40 GeV arises from the consideration of B_d-$\bar{B}_d$ mixing alone. For lower values of m_t, one obtains a value of q too high to be consistent with unitarity. However, it should be emphasized that this lower limit on m_t strongly depends on the error bars and theoretical uncertainties one allows in Eq. (22). Perhaps more significant is the fact that for the "central" values of the various parameters, Table I tells us that one should expect $m_t \approx 100$ GeV.

In conclusion, we have suggested two Wolfenstein-type parametrizations of the CKM matrix, which offer some advantages over the traditional parametrizations through generalized Euler angles and one phase.

In the first parametrization, the CKM matrix coincides with Wolfenstein's parametrization to order λ^3, but differs from it in higher orders. It has the advantage of having parameters simply related to measurable quantities, and allowing for an easy calculation of higher-order corrections in the expansion parameter λ. We have carried out this expansion up to order λ^8.

In the second parametrization, the parameters are simply related to four independent moduli squared of the CKM matrix. We have illustrated the usefulness of this parametrization through the discussion of B_d-$\bar{B}_d$ mixing.

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