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Polarizations and spin correlations in the $NN \rightarrow N\Delta$ reaction

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A systematic formalism is developed for the spin observables of the $NN \rightarrow N\Delta$ reaction, according to the polarization states of the four baryons involved in the transition. This formalism allows us to express all observables by means of a compact formula. Special emphasis is put on the search for relationships between different spin observables and particularly on relations equivalent to the so-called "Bohr's rule" in $NN \rightarrow NN$. These relationships are useful when choosing specific sets of orthonormalized Δ-spin operators adapted to analysis of experimental data.

I. INTRODUCTION

In recent years a great interest has been taken in the production of the nucleon isobar $\Delta(1232)$ in proton-proton[1-5] and proton-nucleus interactions.[6,7] The role of the propagation of isobaric intermediate states in particle-nucleus collisions has also been pointed out.[8-10] In view of the importance of the isobaric resonance in intermediate-energy physics, a precise knowledge of the $NN \rightarrow N\Delta$ transition is more and more necessary. Experimentally, information can be extracted from the $NN \rightarrow NN\pi$ reactions.[11] Furthermore, recent data from Argonne[4] provide us with the first set of spin observables of the Δ production on a wide energy range.

From the theoretical point of view, we are facing the amplitude analysis of the $NN \rightarrow N\Delta$ transition as well as its interpretation in terms of various models. From the number of helicity states, and using parity conservation, 16 complex functions are needed to specify the full spin dependence of the $NN \rightarrow N\Delta$ reaction.[12] They correspond to the 16 independent operators describing this reaction in the spin space. They lead to a total number of 512 possible experimental quantities, though not independent. In Ref. 3 formulas are presented in terms of the spin amplitudes for observables in which up to two nucleon spins (such as those of beam and target or of beam and recoil nucleon) are measured. There are 19 such observables.

The purpose of the present paper is to develop a systematic formalism for the spin observables, according to the polarization states of the four baryons involved in the transition. This formalism allows us to express all observables by means of a compact formula. Special emphasis is put on the search for relationships between different spin observables. Particularly, we derive relationships due to the invariance under reflection with respect to the scattering plane, the equivalent of the so-called "Bohr's rule" applied, for instance, in the nucleon-nucleon elastic scattering.[13] These relationships are useful when choosing specific sets of orthonormalized Δ-spin operators adapted to the analysis of experimental data.

Much of the material presented here can be found in previous work, such as the earlier paper by Csonka, Moravcsik, and Scadron,[14] published some 20 years ago. However, because spin observables are not easy to handle, except in some well-known trivial cases, we found it useful to treat explicitly the $NN \rightarrow N\Delta$ reaction, which involves a spin-$\frac{3}{2}$ particle.

The outline of this paper is as follows. In Sec. II the decomposition of the $NN \rightarrow N\Delta$ transition in spin-space amplitudes is shortly recalled,[3] and we describe the construction of polarization tensors for the Δ particle. Relationships between transition and polarization matrices are also given. Section III contains the derivation of a compact formula for the spin observables, and relationships among these observables are established. Section IV deals with a particular set of Δ-spin operators well adapted to the analysis of the experimental situation.

This paper is thus devoted to the algebraic formalism, the first unavoidable step towards discussion and interpretation in terms of models. This last aspect, as well as comparison with experimental data, will be the subject of a forthcoming publication.

II. SPIN FORMALISM

A. Transition spin amplitudes

A convenient spin-space decomposition of the $NN \rightarrow \Delta N$ antisymmetrized production amplitude is given by[3]

$$M = \sum_{i=1}^{8} [f_i(\theta)Q_i + g_i(\theta)Q_i(\boldsymbol{\sigma}_2 \cdot \hat{\mathbf{n}})] , \qquad (2.1)$$

where $\boldsymbol{\sigma}_2$ stands for the usual Pauli operator acting on nucleon 2, assuming nucleon 1 to undergo the transition and to become the Δ. A right-handed orthonormal basis $(\hat{\mathbf{l}}, \hat{\mathbf{m}}, \hat{\mathbf{n}})$ is used as the reference frame. The unit vectors are defined by

$$\hat{l}=\hat{\mathbf{k}}, \quad \hat{\mathbf{n}}=\frac{\mathbf{k}\times\mathbf{k}_\Delta}{|\mathbf{k}\times\mathbf{k}_\Delta|}, \quad \hat{\mathbf{m}}=\hat{\mathbf{n}}\times\hat{l} , \tag{2.2}$$

where $\mathbf{k}$ and $\mathbf{k}_\Delta$ are, respectively, the initial-beam-nucleon and final-Δ center-of-mass three-momenta, θ being the production angle.

For purposes of implementing parity conservation, note that $\hat{\mathbf{n}}$ is a pseudovector while $\hat{l}$, and $\hat{\mathbf{m}}$ are true polar vectors. All the dynamics is contained in the 16 complex-spin amplitudes $f_i(\theta)$ and $g_i(\theta)$, analogous to the spin-nonflip and spin-flip amplitudes of pion-nucleon scattering. The eight Q_i in Eq. (2.1) are spin-space operators which transform as true scalar because of parity conservation. We recall them for the sake of completeness:

$$\begin{aligned}
&Q_1=(\mathbf{S}\cdot\hat{l})(\sigma_2\cdot\hat{l}), \quad Q_2=\frac{2i}{\sqrt{3}}(\hat{\mathbf{m}}\cdot\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}})(\sigma_2\cdot\hat{l}) ,\\
&Q_3=(\mathbf{S}\cdot\hat{\mathbf{m}})(\sigma_2\cdot\hat{\mathbf{m}}), \quad Q_4=-\frac{2i}{\sqrt{3}}(\hat{l}\cdot\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}})(\sigma_2\cdot\hat{\mathbf{m}}) ,\\
&Q_5=(\mathbf{S}\cdot\hat{\mathbf{n}})(\sigma_2\cdot\hat{\mathbf{n}}), \quad Q_6=\frac{2i}{\sqrt{3}}(\hat{l}\cdot\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{m}})(\sigma_2\cdot\hat{\mathbf{n}}) ,\\
&Q_7=(\hat{l}\cdot\overleftrightarrow{\mathbf{T}}\cdot\hat{l}), \quad Q_8=\frac{1}{\sqrt{3}}[(\hat{\mathbf{m}}\cdot\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{m}})-(\hat{\mathbf{n}}\cdot\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{n}})] .
\end{aligned} \tag{2.3}$$

They satisfy orthonormality conditions:

$$\begin{aligned}
&\tfrac{1}{4}\mathrm{Tr}(Q_i^\dagger Q_j)=\delta_{ij} ,\\
&\tfrac{1}{4}\mathrm{Tr}[Q_i^\dagger Q_j(\sigma_2\cdot\hat{\mathbf{a}})]=0 \quad \text{with } \hat{\mathbf{a}}=\hat{l},\hat{\mathbf{m}},\hat{\mathbf{n}} .
\end{aligned} \tag{2.4}$$

The $\mathbf{S}$ and $\overleftrightarrow{\mathbf{T}}$ quantities in Eq. (2.3) are the rank-1 and -2 irreducible tensorial operators which link the nucleon 1 spin space to the Δ spin space, respectively.

Anticipating further developments, it is instructive to recall how operators in the spin space can be generated by means of the Wigner-Eckart theorem. Indeed, the spherical component of a rank-J irreducible tensor operator is connected to the $j'\to j$ transition by

$$[T(j,j')]_J^M=\frac{(-)^{2J}}{\sqrt{2j+1}}\langle j\|T_J\|j'\rangle \times\sum_{mm'}\langle j'Jm'M\mid jm\rangle\mid jm\rangle\langle j'm'\mid \tag{2.5}$$

in terms of the Clebsch-Gordan coefficient and reduced matrix element taken with Racah's definition.

For spin-$\frac{1}{2}$ particles, the unit matrix and the three Pauli-spin matrices σ can be obtained from Eq. (2.5) by setting $j=j'=\frac{1}{2}$, for $J=0$, and 1, respectively. The corresponding reduced matrix elements are

$$\begin{aligned}
&\langle\tfrac{1}{2}\|T_{J=0}\|\tfrac{1}{2}\rangle\equiv\langle\tfrac{1}{2}\|\mathbb{1}\|\tfrac{1}{2}\rangle=\sqrt{2} ,\\
&\langle\tfrac{1}{2}\|T_{J=1}\|\tfrac{1}{2}\rangle\equiv\langle\tfrac{1}{2}\|\sigma\|\tfrac{1}{2}\rangle=\sqrt{6} .
\end{aligned} \tag{2.6}$$

Similarly, the $\mathbf{S}$ and $\overleftrightarrow{\mathbf{T}}$ operators of Eq. (2.3) are generated by setting $j=\frac{3}{2}$, $j'=\frac{1}{2}$, for $J=1$, and 2, respectively. The reduced matrix elements are chosen to be

$$\begin{aligned}
&\langle\tfrac{3}{2}\|T_{J=1}\|\tfrac{1}{2}\rangle\equiv\langle\tfrac{3}{2}\|\mathbf{S}\|\tfrac{1}{2}\rangle=\sqrt{6} ,\\
&\langle\tfrac{3}{2}\|T_{J=2}\|\tfrac{1}{2}\rangle\equiv\langle\tfrac{3}{2}\|\overleftrightarrow{\mathbf{T}}\|\tfrac{1}{2}\rangle=5\sqrt{3} .
\end{aligned} \tag{2.7}$$

While contracting the rank-2 tensor $\overleftrightarrow{\mathbf{T}}$, use is made in Eq. (2.3) of the dyadic notation defined by

$$(\hat{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{b}})\equiv[[T(\tfrac{3}{2},\tfrac{1}{2})]_2\otimes[\hat{\mathbf{a}}\otimes\hat{\mathbf{b}}]_2]_0^0 \tag{2.8}$$

with $\hat{\mathbf{a}},\hat{\mathbf{b}}=\hat{l},\hat{\mathbf{m}},\hat{\mathbf{n}}$. This dyadic product is related to $\mathbf{S}$ transition and σ Pauli operators acting on the same nucleon by

$$(\hat{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{b}})=\tfrac{1}{2}[(\mathbf{S}\cdot\hat{\mathbf{a}})(\sigma\cdot\hat{\mathbf{b}})+(\mathbf{S}\cdot\hat{\mathbf{b}})(\sigma\cdot\hat{\mathbf{a}})] . \tag{2.9}$$

Because of the trace condition, we have

$$\sum_{\hat{\mathbf{a}}=\hat{l},\hat{\mathbf{m}},\hat{\mathbf{n}}}(\hat{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{a}})=0 .$$

B. Δ-spin-space operators

It is quite obvious that Eq. (2.5) can be used to build a complete basis of spin-space operators for a particle of arbitrary j. Together with a convenient orthonormal vector triad like the $\{\hat{l},\hat{\mathbf{m}},\hat{\mathbf{n}}\}$ basis of Eq. (2.2), such a set of operators is particularly suitable to describe the polarization states or the density matrix.

For the familiar $j=\frac{1}{2}$ case, this set involves only $\mathbb{1}$ and σ (as quoted above), and the polarization states are fully defined by the four matrices $\mathbb{1}$, $\sigma\cdot\hat{l}$, $\sigma\cdot\hat{\mathbf{m}}$, and $\sigma\cdot\hat{\mathbf{n}}$. In order to promote a compact notation, these four matrices can be recast into $\mathcal{P}(\alpha)$, with $\alpha=0$, $\hat{l}$, $\hat{\mathbf{m}}$, or $\hat{\mathbf{n}}$, respectively.

In the case of a $j=\frac{3}{2}$ particle such as the Δ, the same procedure generates 16 spin-space operators, which correspond to the spherical components of irreducible tensors $[T(\frac{3}{2},\frac{3}{2})]_J$ with rank $J=0$, 1, 2, and 3, respectively. By convention they will be denoted as the unit matrix $\mathbb{1}$, generalized Pauli spin operator σ_Δ, $\overleftrightarrow{\mathbf{T}}_\Delta$, and T_3 tensorial operators. The explicit values of their reduced matrix elements are listed below:

$$\begin{aligned}
&\langle\tfrac{3}{2}\|T_{J=0}\|\tfrac{3}{2}\rangle\equiv\langle\tfrac{3}{2}\|\mathbb{1}\|\tfrac{3}{2}\rangle=2 ,\\
&\langle\tfrac{3}{2}\|T_{J=1}\|\tfrac{3}{2}\rangle\equiv\langle\tfrac{3}{2}\|\sigma_\Delta\|\tfrac{3}{2}\rangle=2\sqrt{15} ,\\
&\langle\tfrac{3}{2}\|T_{J=2}\|\tfrac{3}{2}\rangle\equiv\langle\tfrac{3}{2}\|\overleftrightarrow{\mathbf{T}}_\Delta\|\tfrac{3}{2}\rangle=20\sqrt{6} ,\\
&\langle\tfrac{3}{2}\|T_{J=3}\|\tfrac{3}{2}\rangle\equiv\langle\tfrac{3}{2}\|T_3\|\tfrac{3}{2}\rangle=84\sqrt{2} .
\end{aligned} \tag{2.10}$$

The dyadic notation of Eq. (2.8), used for the $N\to\Delta$ transition, can be further extended. Thus, we have

$$(\hat{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{b}})=[[T(\tfrac{3}{2},\tfrac{3}{2})]_2\otimes[\hat{\mathbf{a}}\otimes\hat{\mathbf{b}}]_2]_0^0 , \tag{2.11a}$$

and similarly,

$$(T_3\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}})_0=[[T(\tfrac{3}{2},\tfrac{3}{2})]_3\otimes[\hat{\mathbf{a}}\otimes[\hat{\mathbf{b}}\otimes\hat{\mathbf{c}}]_2]_3]_0^0 . \tag{2.11b}$$

This notation has the advantage of underlining the invariance properties under the exchange of the basis vectors $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$.

The rank-2 and -3 tensorial operators defined above can be expressed in terms of σ_Δ. It leads to (see Appendix A)

$$(\hat{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{b}})=\tfrac{1}{2}[(\sigma_\Delta\cdot\hat{\mathbf{a}})(\sigma_\Delta\cdot\hat{\mathbf{b}})+(\sigma_\Delta\cdot\hat{\mathbf{b}})(\sigma_\Delta\cdot\hat{\mathbf{a}})]-5(\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}) \tag{2.12}$$

and

$$(T_3\hat{a}\hat{b}\hat{c})_0=-\tfrac{1}{6}[(\sigma_\Delta\cdot\hat{a})(\sigma_\Delta\cdot\hat{b})(\sigma_\Delta\cdot\hat{c})+(\sigma_\Delta\cdot\hat{a})(\sigma_\Delta\cdot\hat{c})(\sigma_\Delta\cdot\hat{b})+(\sigma_\Delta\cdot\hat{b})(\sigma_\Delta\cdot\hat{c})(\sigma_\Delta\cdot\hat{a})$$
$$+(\sigma_\Delta\cdot\hat{b})(\sigma_\Delta\cdot\hat{a})(\sigma_\Delta\cdot\hat{c})+(\sigma_\Delta\cdot\hat{c})(\sigma_\Delta\cdot\hat{a})(\sigma_\Delta\cdot\hat{b})+(\sigma_\Delta\cdot\hat{c})(\sigma_\Delta\cdot\hat{b})(\sigma_\Delta\cdot\hat{a})]$$
$$+\tfrac{41}{15}[(\sigma_\Delta\cdot\hat{a})(\hat{b}\cdot\hat{c})+(\sigma_\Delta\cdot\hat{b})(\hat{c}\cdot\hat{a})+(\sigma_\Delta\cdot\hat{c})(\hat{a}\cdot\hat{b})]\ . \tag{2.13}$$

Note that the expression (2.11a) refers to nine operators. Symmetry and trace conditions reduce them to five independent ones:

$$(\hat{a}\cdot\overleftrightarrow{T}_\Delta\cdot\hat{b})=(\hat{b}\cdot\overleftrightarrow{T}_\Delta\cdot\hat{a})\ ,\qquad \sum_{\hat{a}=\hat{l},\hat{m},\hat{n}}(\hat{a}\cdot\overleftrightarrow{T}_\Delta\cdot\hat{a})=0\ . \tag{2.14}$$

Similarly, out of the 27 quantities defined by $(T_3\hat{a}\hat{b}\hat{c})_0$, only seven independent operators remain, once invariance under permutation and trace conditions are taken into account:

$$\sum_{\hat{b}=\hat{l},\hat{m},\hat{n}}(T_3\hat{a}\hat{b}\hat{b})_0=0\quad \forall\hat{a}\ . \tag{2.15}$$

Relationships connecting $N\rightarrow\Delta$ transition, nucleon- and Δ-spin-space operators are displayed in Appendix A.

As for the case of the spin-$\frac{1}{2}$ particle, we have combined the Δ-spin operators $\mathbb{1}$, σ_Δ, $\overleftrightarrow{T}_\Delta$, T_3 and a convenient orthonormal triad $(\hat{l},\hat{m},\hat{n})$. In this way, we have constructed 16 independent quantities (4×4 matrices) which are Cartesian components of the Δ-spin-space operators in the chosen vector basis. They will be useful to describe polarization states of the Δ particle or to write its density matrix. In order to introduce a compact notation, the quantities $\mathbb{1}$, $(\sigma_\Delta\cdot\hat{a})$, $(\hat{a}\cdot\overleftrightarrow{T}_\Delta\cdot\hat{b})$, and $(T_3\hat{a}\hat{b}\hat{c})_0$ will be denoted $\mathcal{P}_\Delta(\beta)$, with $\beta=0$, $(\hat{a})$, $(\hat{a},\hat{b})$, and $(\hat{a},\hat{b},\hat{c})$, respectively. Note that these $\mathcal{P}_\Delta(\beta)$ are not yet orthonormal. This will be examined in Sec. IV. Explicit expressions of $\mathcal{P}_\Delta(\beta)$ in terms of projectors are given in Appendix B.

C. Relationships between transition and spin-space matrices

Various relationships can be established between the set of operators building $\mathcal{P}(\alpha),\mathcal{P}_\Delta(\beta)$ and the Q_j operators describing the $NN\rightarrow N\Delta$ transition. They are very useful to study the spin observables.

The first one concerns commutation rules of $\sigma_2\cdot\hat{n}$ with operators of $\mathcal{P}_2(\alpha)$, acting in the spin space of nucleon 2:

$$\mathcal{P}_2(\alpha)(\sigma_2\cdot\hat{n})=(\sigma_2\cdot\hat{n})\mathcal{P}_2(\alpha)Z_\alpha\ , \tag{2.16}$$

with $Z_0=Z_n=+1$, $Z_l=Z_m=-1$. Similarly, the commutation rules of $\mathcal{P}_2(\alpha)$ with the Q_j can be expressed with the help of a set of diagonal matrices $s(\alpha)$:

$$Q_j\mathcal{P}_2(\alpha)=\mathcal{P}_2(\alpha)\sum_{k=1}^{8}Q_k s_{jk}(\alpha)\ , \tag{2.17}$$

with the properties

$$s(0)=1,\quad s(\alpha_1)s(\alpha_2)=s(\,|\alpha_1\times\alpha_2|\,) \tag{2.18}$$

for α_1 and $\alpha_2=\hat{l},\hat{m},\hat{n}$.

Connecting the spin-space operators of nucleons 1 and 2 through the Q_j operators defines a set of matrices $v(\alpha)$:

$$Q_j\mathcal{P}_1(\alpha)=\mathcal{P}_2(\alpha)\sum_{k=1}^{8}Q_k v_{jk}(\alpha)\ , \tag{2.19}$$

with the properties

$$v(0)=1,\quad v^2(\alpha)=1\ ,\qquad v(\alpha_1)v(\alpha_2)=v(\alpha_2)v(\alpha_1)=v(\,|\alpha_1\times\alpha_2|\,) \tag{2.20}$$

for $\alpha_1,\alpha_2=\hat{l},\hat{m},\hat{n}$. These matrices are given explicitly in

TABLE I. Quantities defined in Eq. (2.21).

β	$\mathcal{P}_\Delta(\beta)$	q	p	$K(\beta)$
0	$\mathbb{1}$	1	0	$\mathbb{1}$
$\hat{a}$	$(\sigma_\Delta\cdot\hat{a})$	1	$\hat{a}$	$w(\hat{a})$
$(\hat{a},\hat{a})$	$(\hat{a}\cdot\overleftrightarrow{T}_\Delta\cdot\hat{a})$	1	0	$-5+w^2(\hat{a})$
$(\hat{a},\hat{b})$ with $\hat{a}\neq\hat{b}$	$(\hat{a}\cdot\overleftrightarrow{T}_\Delta\cdot\hat{b})$	i	$\hat{a}\times\hat{b}$	$\frac{1}{2}[w(\hat{a})w(\hat{b})-w(\hat{b})w(\hat{a})]$
$(\hat{a},\hat{a},\hat{a})$	$(T_3\hat{a}\hat{a}\hat{a})_0$	1	$\hat{a}$	$\frac{41}{5}w(\hat{a})-w^3(\hat{a})$
$(\hat{a},\hat{b},\hat{b})$ with $\hat{a}\neq\hat{b}$	$(T_3\hat{a}\hat{b}\hat{b})_0$	1	$\hat{a}$	$\frac{17}{5}w(\hat{a})-\frac{1}{2}[w(\hat{a})w^2(\hat{b})+w^2(\hat{b})w(\hat{a})]$
$(\hat{a},\hat{b},\hat{c})$ with $\hat{a}\neq\hat{b}\neq\hat{c}$	$(T_3\hat{l}\hat{m}\hat{n})_0$	i	0	$-\frac{1}{6}\{w(\hat{l})[w(\hat{m})w(\hat{n})-w(\hat{n})w(\hat{m})]+w(\hat{m})[w(\hat{n})w(\hat{l})-w(\hat{l})w(\hat{n})]+w(\hat{n})[w(\hat{l})w(\hat{m})-w(\hat{m})w(\hat{l})]\}$

Appendix C.

The rules are completed by the relationships, which link the spin-space operators of nucleon 2 with those of the Δ:

$$\mathcal{P}_\Delta(\beta)Q_j = q\mathcal{P}_2(p)\sum_{k=1}^{8} Q_k K_{jk}(\beta) , \tag{2.21}$$

where $q=1$ or i and $p=0$, $\hat{l}$, $\hat{\mathbf{m}}$, or $\hat{\mathbf{n}}$. The K matrices q and p depend on the β value. Seven cases have to be considered and can be found in Table I. The matrices $K(\beta)$ are real. They are given in terms of matrices w which are displayed in Appendix C, and are defined by

$$(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{a}})Q_j = \mathcal{P}_2(\hat{\mathbf{a}})\sum_{k=1}^{8} Q_k w_{jk}(\hat{\mathbf{a}}) , \tag{2.22}$$

with $\hat{\mathbf{a}}=\hat{l},\hat{\mathbf{m}},\hat{\mathbf{n}}$.

III. OBSERVABLES OF THE $NN\to N\Delta$ TRANSITION

A. Definition and general expression

It is very convenient to denote the spin observables of the $NN\to N\Delta$ transition according to the four indices introduced by Bystricky, Lehar, and Winternitz[13] in the case of nucleon-nucleon elastic scattering. This is done by using the symbol $X_{(\beta)\alpha_2'\alpha_1\alpha_2}$ where α_1 refers to the spin orientation of the beam particle (nucleon 1), α_2 to the target particle (nucleon 2), (β) to the outgoing Δ particle, and α_2' to the recoil nucleon. Here each index β, α_2', α_1, and α_2 stands for its associated particle having a vector or tensor polarization along one of the directions $\{\hat{l},\hat{\mathbf{m}},\hat{\mathbf{n}}\}$. For the Δ particle the possible expressions of β are listed in Table I. In the case of an unpolarized initial particle, or if the polarization of a final particle is not detected, the corresponding index is set equal to 0.

Bystricky, Lehar, and Winternitz[13] propose to call the observables defined as above "pure" center-of-mass experiments. This is somewhat unfortunate and we prefer to call them IFE, which stands for "initial-frame experiments." When helicity frames have to be used, linear combinations of these quantities are performed with appropriate coefficients.[15]

In terms of the M amplitude of Eq. (2.1), the spin observables for $NN\to N\Delta$ take the form

$$\sigma X_{(\beta)\alpha_2'\alpha_1\alpha_2} = \tfrac{1}{4}\mathrm{Tr}[M^\dagger\mathcal{P}_\Delta(\beta)\mathcal{P}_2(\alpha_2')M\mathcal{P}_1(\alpha_1)\mathcal{P}_2(\alpha_2)] , \tag{3.1}$$

where σ is the differential cross section for unpolarized particles up to phase-space factors, so that $X_{(0)000}=1$. Substituting Eqs. (2.1) and (2.16), Eq. (3.1) becomes

$$\sigma X_{(\beta)\alpha_2'\alpha_1\alpha_2} = \sum_{i,j=1}^{8}(f_i^*f_j+g_i^*g_jZ_{\alpha_2})\tfrac{1}{4}\mathrm{Tr}[Q_i^\dagger\mathcal{P}_\Delta(\beta)\mathcal{P}_2(\alpha_2')Q_j\mathcal{P}_1(\alpha_1)\mathcal{P}_2(\alpha_2)]$$
$$+\sum_{i,j=1}^{8}(f_i^*g_j+g_i^*f_jZ_{\alpha_2})\tfrac{1}{4}\mathrm{Tr}[Q_i^\dagger\mathcal{P}_\Delta(\beta)\mathcal{P}_2(\alpha_2')Q_j(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{n}})\mathcal{P}_1(\alpha_1)\mathcal{P}_2(\alpha_2)] . \tag{3.2}$$

Making use of the relations between operators given in Eqs. (2.17), (2.19), and (2.21), we can write

$$\mathcal{P}_\Delta(\beta)\mathcal{P}_2(\alpha_2')Q_j\mathcal{P}_1(\alpha_1)\mathcal{P}_2(\alpha_2) = q\mathcal{P}_2(\alpha_2')\mathcal{P}_2(p)\mathcal{P}_2(\alpha_1)\mathcal{P}_2(\alpha_2)\sum_{i=1}^{8}Q_i[K(\beta)v(\alpha_1)s(\alpha_2)]_{ji} . \tag{3.3}$$

If we restrict ourselves to physical polarizations which conserve parity, we obtain

$$\mathcal{P}_2(\alpha_2')\mathcal{P}_2(p)\mathcal{P}_2(\alpha_1)\mathcal{P}_2(\alpha_2) = C_0\mathbb{1}+C_1(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{n}}) , \tag{3.4}$$

and we get

$$\sigma X_{(\beta)\alpha_2'\alpha_1\alpha_2} = \sum_{i,j=1}^{8}(f_i^*f_j+g_i^*g_jZ_{\alpha_2})qC_0[K(\beta)v(\alpha_1)s(\alpha_2)]_{ji} + \sum_{i,j=1}^{8}(f_i^*g_jZ_{\alpha_2}+g_i^*f_j)qC_1[K(\beta)v(\alpha_1)s(\alpha_2)s(\hat{\mathbf{n}})]_{ji} . \tag{3.5}$$

The quantities C_0 and C_1, which depend on α_1, α_2, α_2', and p, are displayed in Table II.

By symmetrizing or antisymmetrizing the expressions appearing in Eq. (3.5), it is possible to exhibit the real and imaginary part of the amplitudes. The spin observables become

$$\sigma X_{(\beta)\alpha_2'\alpha_1\alpha_2} = \mathrm{Re}q\left\{\mathrm{Re}C_0\sum_{i,j=1}^{8}\mathrm{Re}(f_i^*f_j+g_i^*g_jZ_{\alpha_2})F_{ij}+\mathrm{Im}C_0\sum_{i,j=1}^{8}\mathrm{Im}(f_i^*f_j+g_i^*g_jZ_{\alpha_2})F_{ij}\right.$$
$$\left.+\mathrm{Re}C_1\sum_{i,j=1}^{8}2\,\mathrm{Re}(f_i^*g_j)G_{ij}+\mathrm{Im}C_1\sum_{i,j=1}^{8}2\,\mathrm{Im}(f_i^*g_j)G_{ij}\right\}$$
$$+\mathrm{Im}q\left\{\mathrm{Re}C_0\sum_{i,j=1}^{8}\mathrm{Im}(f_i^*f_j+g_i^*g_jZ_{\alpha_2})F_{ij}-\mathrm{Im}C_0\sum_{i,j=1}^{8}\mathrm{Re}(f_i^*f_j+g_i^*g_jZ_{\alpha_2})F_{ij}\right.$$
$$\left.+\mathrm{Re}C_1\sum_{i,j=1}^{8}2\,\mathrm{Im}(f_i^*g_j)G_{ij}-\mathrm{Im}C_1\sum_{i,j=1}^{8}2\,\mathrm{Re}(f_i^*g_j)G_{ij}\right\} , \tag{3.6}$$

with

$$F_{ij}=[K(\beta)v(\alpha_1)s(\alpha_2)]_{ij}\ ,$$
$$G_{ij}=[K(\beta)v(\alpha_1)s(\alpha_2)s(\hat{\mathbf{n}})]_{ij}\ . \tag{3.7}$$

We recall that the C_0 and C_1 coefficients are displayed in Table II. The Z_{α_2} and q quantities are defined by Eqs. (2.16) and (2.21) and listed in Table I. The $s(\alpha)$ and $v(\alpha)$ matrices are defined by Eqs. (2.17) and (2.19) and displayed in Appendix C. The $K(\beta)$ matrices are defined in Eq. (2.21) and in Table I in terms of $w(\hat{\mathbf{a}})$ matrices which are given by Eq. (2.22) and displayed in Appendix C.

In the X notation, the target-polarization asymmetry A_{0n} becomes $X_{(0)00n}$, the depolarization of the target D_{ab} becomes $X_{(0)a0b}$, and the polarization transfer from beam to recoil particle K_{ab} becomes $X_{(0)ab0}$.

Although the expression (3.6) looks rather extended, it may end up in a compact formula for some observables. Two particular examples are given below.

For the case $\alpha_1=\alpha_2=\hat{l}$, $\alpha_2'=\beta=\hat{\mathbf{n}}$, Table I gives $q=1$, $p=\hat{\mathbf{n}}$, and $K(\hat{\mathbf{n}})=w(\hat{\mathbf{n}})$. Table II gives $C_0=1$ and one gets

$$\sigma X_{(n)nll}=\sum_{i,j=1}^{8}\mathrm{Re}(f_i^*f_j-g_i^*g_j)[w(\hat{\mathbf{n}})v(\hat{l})s(\hat{l})]_{ij}\ . \tag{3.8}$$

For the case $\alpha_1=\alpha_2'=\hat{\mathbf{n}}$, $\alpha_2=0$, $\beta=(\hat{l},\hat{\mathbf{m}})$, Table I gives $q=i$, $p=\hat{\mathbf{n}}$, and $K(\hat{l},\hat{\mathbf{m}})=\frac{1}{2}[w(\hat{l})w(\hat{\mathbf{m}})-w(\hat{\mathbf{m}})w(\hat{l})]$. Table II gives $C_1=1$ and we obtain

$$\sigma X_{(lm)nno}=\sum_{i,j=1}^{8}2\ \mathrm{Im}(f_i^*g_j)[K(\hat{l},\hat{\mathbf{m}})v(\hat{\mathbf{n}})s(\hat{\mathbf{n}})]_{ij}\ . \tag{3.9}$$

For the $NN\to N\Delta$ reaction, 1024 IFE can be defined. Each spin-space operator of $\mathcal{P}_\Delta$ carries 64 observables; each of the three nucleons can be unpolarized or polarized along $\hat{l},\hat{\mathbf{m}},\hat{\mathbf{n}}$. Accounting for the parity conservation, the number of IFE is reduced by a factor of 2. In practice, the nonvanishing experiments correspond to an even number of l and m indices among the four subscripts β, α_2', α_1, and α_2.

B. Invariance relations between observables

The 512 observables defined above are not independent. A very useful method for checking relations among them relies on the search for operators satisfying

$$\mathcal{P}_\Delta(\beta)\mathcal{P}_2(\alpha_2')M\mathcal{P}_1(\alpha_1)\mathcal{P}_2(\alpha_2)=M\ . \tag{3.10}$$

Note that this relation is equivalent to the so-called Bohr's rule used in Ref. 13 in $NN\to NN$ and obtained by invariance under reflection in the scattering plane.

Applying the relations given in Eqs. (2.17), (2.19), and (2.21), the Q_j part of M leads to

$$\mathcal{P}_\Delta(\beta)\mathcal{P}_2(\alpha_2')Q_j\mathcal{P}_1(\alpha_1)\mathcal{P}_2(\alpha_2)$$
$$=q\mathcal{P}_2(\alpha_2')\mathcal{P}_2(p)\mathcal{P}_2(\alpha_1)\mathcal{P}_2(\alpha_2)$$
$$\times\sum_{i=1}^{8}Q_i[K(\beta)v(\alpha_1)\times s(\alpha_2)]_{ji}\ . \tag{3.11}$$

TABLE II. C_0 and C_1 coefficients of Eqs. (3.6), (4.13), and (4.14).

	$\alpha_1=0,\ \alpha_2=0$	$\alpha_1=0,\ \alpha_2\neq0$	$\alpha_1\neq0,\ \alpha_2=0$	$\alpha_1\neq0,\ \alpha_2\neq0$
$p=0,\ \alpha_2'=0$	$C_0=1,\ C_1=0$	$C_0=0,\ C_1=(\hat{\alpha}_2\cdot n)$	$C_0=0,\ C_1=(\hat{\alpha}_1\cdot\hat{\mathbf{n}})$	$C_0=(\hat{\alpha}_1\cdot\hat{\alpha}_2)$, $C_1=i(\hat{\alpha}_1\times\hat{\alpha}_2\cdot\hat{\mathbf{n}})$
$p=0,\ \alpha_2'\neq0$	$C_0=0,\ C_1=(\hat{\alpha}_2'\cdot\hat{\mathbf{n}})$	$C_0=(\hat{\alpha}_2\cdot\hat{\alpha}_2')$, $C_1=i(\hat{\alpha}_2'\times\hat{\alpha}_2\cdot\hat{\mathbf{n}})$	$C_0=(\hat{\alpha}_1\cdot\hat{\alpha}_2')$, $C_1=i(\hat{\alpha}_2'\times\hat{\alpha}_1\cdot\hat{\mathbf{n}})$	$C_1=(\hat{\alpha}_1\cdot\hat{\alpha}_2)(\hat{\alpha}_2'\cdot\hat{\mathbf{n}})$ $+(\hat{\alpha}_1\cdot\hat{\alpha}_2')(\hat{\alpha}_2\cdot\hat{\mathbf{n}})$ $-(\hat{\alpha}_2\cdot\hat{\alpha}_2')(\hat{\alpha}_1\cdot\hat{\mathbf{n}})$, $C_0=i(\hat{\alpha}_1\times\hat{\alpha}_2\cdot\hat{\alpha}_2')$
$p\neq0,\ \alpha_2'=0$	$C_0=0,\ C_1=(\hat{\mathbf{p}}\cdot\hat{\mathbf{n}})$	$C_0=(\hat{\mathbf{p}}\cdot\hat{\alpha}_2)$, $C_1=i(\hat{\mathbf{p}}\times\hat{\alpha}_2\cdot\hat{\mathbf{n}})$	$C_0=(\hat{\mathbf{p}}\cdot\hat{\alpha}_1)$, $C_1=i(\hat{\mathbf{p}}\hat{\alpha}\times{}_1\cdot\hat{\mathbf{n}})$	$C_1=(\hat{\alpha}_1\cdot\hat{\alpha}_2)(\hat{\mathbf{p}}\cdot\hat{\mathbf{n}})$ $+(\hat{\mathbf{p}}\cdot\hat{\alpha}_1)(\hat{\alpha}_2\cdot\hat{\mathbf{n}})$ $-(\hat{\mathbf{p}}\cdot\hat{\alpha}_2)(\hat{\alpha}_1\cdot\hat{\mathbf{n}})$, $C_0=i(\hat{\alpha}_1\times\hat{\alpha}_2\cdot\hat{\mathbf{p}})$
$p\neq0,\ \alpha_2'\neq0$	$C_0=(\hat{\mathbf{p}}\cdot\hat{\alpha}_2')$, $C_1=i(\hat{\alpha}_2'\times\hat{\mathbf{p}}\cdot\hat{\mathbf{n}})$	$C_1=(\hat{\mathbf{p}}\cdot\hat{\alpha}_2)(\hat{\alpha}_2'\cdot\hat{\mathbf{n}})$ $+(\hat{\mathbf{p}}\cdot\hat{\alpha}_2')(\hat{\alpha}_2\cdot\hat{\mathbf{n}})$, $-(\hat{\alpha}_2\cdot\hat{\alpha}_2')(\hat{\mathbf{p}}\cdot\hat{\mathbf{n}})$, $C_0=i(\hat{\mathbf{p}}\times\hat{\alpha}_2\cdot\hat{\alpha}_2')$	$C_1=(\hat{\mathbf{p}}\cdot\hat{\alpha}_1)(\hat{\alpha}_2'\cdot\hat{\mathbf{n}})$ $+(\hat{\mathbf{p}}\cdot\hat{\alpha}_2')(\hat{\alpha}_1\cdot\hat{\mathbf{n}})$ $-(\hat{\alpha}_1\cdot\hat{\alpha}_2')(\hat{\mathbf{p}}\cdot\hat{\mathbf{n}})$, $C_0=i(\hat{\mathbf{p}}\times\hat{\alpha}_1\cdot\hat{\alpha}_2')$	$C_0=(\hat{\alpha}_1\cdot\hat{\alpha}_2)(\hat{\mathbf{p}}\cdot\hat{\alpha}_2')$ $+(\hat{\alpha}_1\times\hat{\alpha}_2\cdot\hat{\mathbf{p}}\times\hat{\alpha}_2')$, $C_1=i[(\hat{\mathbf{p}}\cdot\hat{\alpha}_2')(\hat{\alpha}_1\times\hat{\alpha}_2\cdot\hat{\mathbf{n}})$ $-(\hat{\alpha}_1\cdot\hat{\alpha}_2)(\hat{\mathbf{p}}\times\hat{\alpha}_2'\cdot\hat{\mathbf{n}})$ $+(\hat{\alpha}_1\cdot\hat{\mathbf{n}})(\hat{\mathbf{p}}\times\hat{\alpha}_2'\cdot\hat{\alpha}_2)$ $-(\hat{\alpha}_2\cdot\hat{\mathbf{n}})(\hat{\mathbf{p}}\times\hat{\alpha}_2'\cdot\hat{\alpha}_1)]$

The operators $Q_j(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{n}})$ obey the same kind of relation with an extra Z_{α_2} coefficient [see Eq. (2.16)].

Finally we find that Eq. (3.10) is verified for

$$[\tfrac{1}{6}(T_3\hat{\mathbf{n}}\hat{\mathbf{n}}\hat{\mathbf{n}})_0-\tfrac{1}{5}(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{n}})](\boldsymbol{\sigma}_2\cdot\hat{\mathbf{n}})M(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{n}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{n}})=M\ . \tag{3.12}$$

At this stage, it is helpful to define a new Δ-spin-space operator

$$\Sigma(\hat{\mathbf{a}})=\tfrac{1}{6}(T_3\hat{\mathbf{a}}\hat{\mathbf{a}}\hat{\mathbf{a}})_0-\tfrac{1}{5}(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{a}}) \tag{3.13}$$

which satisfies the relation

$$\Sigma(\hat{\mathbf{a}})\Sigma(\hat{\mathbf{b}})=\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}+i\Sigma(\hat{\mathbf{a}}\times\hat{\mathbf{b}})\ , \tag{3.14}$$

analogous to the well-known relation with Pauli operators

$$(\boldsymbol{\sigma}\cdot\hat{\mathbf{a}})(\boldsymbol{\sigma}\cdot\hat{\mathbf{b}})=\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}+i\boldsymbol{\sigma}\cdot(\hat{\mathbf{a}}\times\hat{\mathbf{b}})\ . \tag{3.15}$$

Multiplying now what we call the first Bohr's relation Eq. (3.12) by $\Sigma(\hat{\mathbf{m}})$ and using Eq. (3.14), we obtain the second Bohr's relation

$$\Sigma(\hat{\mathbf{l}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{l}})M(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{l}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{l}})=\Sigma(\hat{\mathbf{m}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{m}})\times M(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{m}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{m}})\ . \tag{3.16}$$

Using properties and relations of this new Σ operator given in Appendix A, we can write six other Bohr's relations:

$$-(1+\tfrac{1}{2}\hat{\mathbf{n}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{n}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{n}})M(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{n}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{n}})=(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{n}})M\ , \tag{3.17}$$

$$[\tfrac{1}{2}(T_3\hat{\mathbf{m}}\hat{\mathbf{n}}\hat{\mathbf{n}})_0-\tfrac{1}{5}(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{m}})](\boldsymbol{\sigma}_2\cdot\hat{\mathbf{m}})M(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{m}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{m}})=(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{l}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{l}})M(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{l}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{l}})\ , \tag{3.18}$$

$$[\tfrac{1}{2}(T_3\hat{\mathbf{l}}\hat{\mathbf{n}}\hat{\mathbf{n}})_0-\tfrac{1}{5}(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{l}})](\boldsymbol{\sigma}_2\cdot\hat{\mathbf{l}})M(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{l}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{l}})=(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{m}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{m}})M(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{m}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{m}})\ , \tag{3.19}$$

$$[\tfrac{1}{2}(T_3\hat{\mathbf{n}}\hat{\mathbf{m}}\hat{\mathbf{m}})_0-\tfrac{1}{5}(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{n}})](\boldsymbol{\sigma}_2\cdot\hat{\mathbf{n}})M(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{n}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{n}})=-(1+\tfrac{1}{2}\hat{\mathbf{l}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{l}})M\ , \tag{3.20}$$

$$(\hat{\mathbf{m}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{n}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{l}})M(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{l}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{l}})=(\hat{\mathbf{n}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{l}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{m}})M(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{m}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{m}})\ , \tag{3.21}$$

$$(T_3\hat{\mathbf{l}}\hat{\mathbf{m}}\hat{\mathbf{n}})_0(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{n}})M(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{n}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{n}})=(\hat{\mathbf{l}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{m}})M\ . \tag{3.22}$$

It is easy to see that the 16 components of $\mathcal{P}_\Delta$ (or their combinations) are all contained in the eight Bohr's relations. Taking into account these relations divides the number of independent observables by a factor of 2, so we are left with 256 of them.

In order to exemplify the way the Bohr's rules are translated into observables, we look at the first and the last ones, which give

$$\tfrac{1}{6}X_{(nnn)bcd}-\tfrac{1}{5}X_{(n)bcd}=\pm X_{(0)b'c'd'}\ , \tag{3.23}$$

$$X_{(lmn)bcd}=\pm X_{(lm)b'c'd'}\ . \tag{3.24}$$

The indices b', c', and d' are related to the indices b, c, and d, respectively, by the transformations

$$0\leftrightarrow n \quad\text{and}\quad l\leftrightarrow m\ .$$

As can be verified, because of the Z_α coefficient, the Bohr's rule holds only between operators with $p=0$ and $\hat{\mathbf{n}}$, on the one hand, and $p=\hat{\mathbf{l}}$ or $\hat{\mathbf{m}}$, on the other hand. The $\pm$ sign appearing in (3.23) and (3.24) is due to relations between Pauli-spin operators [see Eq. (3.15)]. For instance, we have

$$\begin{aligned}
&\tfrac{1}{6}X_{(nnn)000}-\tfrac{1}{5}X_{(n)000}=X_{(0)nnn}\ ,\\
&\tfrac{1}{6}X_{(lll)nmn}-\tfrac{1}{5}X_{(l)nmn}=-(\tfrac{1}{6}X_{(mmm)0l0}-\tfrac{1}{5}X_{(m)0l0})\ ,\\
&(X_{(0)l0m}+\tfrac{1}{2}X_{(nn)l0m})=X_{(n)mnl}\ ,\\
&\tfrac{1}{2}X_{(mnn)00l}-\tfrac{1}{5}X_{(m)00l}=-X_{(l)nnm}\ ,\\
&X_{(mn)0l0}=X_{(nl)nmn},\quad X_{(lmn)mln}=-X_{(lm)lm0}\ .
\end{aligned} \tag{3.25}$$

Taking account of the antisymmetrization of the two identical initial nucleons leads to a relation between the observables at θ and $\pi-\theta$. We can show that

$$X_{(\beta)\alpha_2'\alpha_1\alpha_2}(\theta)=\pm X_{(\beta)\alpha_2'\alpha_2\alpha_1}(\pi-\theta)\ , \tag{3.26}$$

where a + sign is assigned to observables written with $f_i^*f_j$ and $g_i^*g_j$ products, and a − sign to those written with $f_i^*g_j$ products. When the α_1 and α_2 indices are identical, this equation simply defines the symmetry character of the observables. When α_1 and α_2 are different, it defines a relation between two observables. For each Δ-spin-space operator with $p=0$, we have

$$\begin{aligned}
&X_{(\beta)000}(\theta)=+X_{(\beta)000}(\pi-\theta)\quad 0\text{ relation}\ ,\\
&X_{(\beta)n00}(\theta)=-X_{(\beta)n00}(\pi-\theta)\quad 0\text{ relation}\ ,\\
&X_{(\beta)0n0}(\theta)=-X_{(\beta)00n}(\pi-\theta)\quad 1\text{ relation}\ ,\\
&X_{(\beta)aa0}(\theta)=X_{(\beta)a0a}(\pi-\theta)\quad 3\text{ relations}\ ,\\
&X_{(\beta)bc0}(\theta)=-X_{(\beta)b0c}(\pi-\theta)\quad 2\text{ relations}\ ,\\
&X_{(\beta)0aa}(\theta)=+X_{(\beta)0aa}(\pi-\theta)\quad 0\text{ relation}\ ,\\
&X_{(\beta)0bc}(\theta)=-X_{(\beta)0cb}(\pi-\theta)\quad 1\text{ relation}\ ,\\
&X_{(\beta)bcd}(\theta)=+X_{(\beta)bdc}(\pi-\theta)\quad 3\text{ relations}\ ,\\
&X_{(\beta)naa}(\theta)=-X_{(\beta)naa}(\pi-\theta)\quad 0\text{ relation}\ ,\\
&X_{(\beta)bbn}(\theta)=-X_{(\beta)bnb}(\pi-\theta)\quad 2\text{ relations}\ ,
\end{aligned} \tag{3.27}$$

where $b\neq c\neq d$. These 12 relations reduce the number of independent observables from 32 to 20 for each polarization state of the Δ. For the Δ-spin-space operators with $p\neq 0$, the number of such relations is the same, although the relations are different.

To sum up, using the parity conservation, the so-called Bohr's relations, and the Pauli principle, the number of IFE describing the $NN\to N\Delta$ reaction reduces from 1024 to 160. It is still more than enough to overdetermine the 16 complex spin amplitudes f_i and g_i, only 31 independent experiments being necessary (the overall phase is irrelevant). Nonlinear relations between observables as in the $NN\to NN$ case[13,16,17] can certainly further reduce this number, but it is not the subject of this paper to look for them.

IV. ORTHONORMALIZATION OF THE Δ-SPIN-SPACE OPERATORS

As was stated previously, the 16 components of $\mathcal{P}_\Delta(\beta)$ given in Sec. II B are not orthonormal. For practical purposes, it is more convenient to deal with an orthonormal basis. Let us denote by $\Omega_I(\mathbf{p})$, combinations of the 16 components of $\mathcal{P}_\Delta(\beta)$, where $p=0,\hat{l},\hat{\mathbf{m}},\hat{\mathbf{n}}$ and I runs from 1 to 4. These operators satisfy

$$[\Omega_I(p)]^2=1 \tag{4.1}$$

and

$$\tfrac{1}{4}\mathrm{Tr}[\Omega_I^\dagger(p)\Omega_I'(p')]=\delta_{II'}\delta_{pp'} . \tag{4.2}$$

One of the aims is to search for a complete basis simplifying the use of the Bohr's rules. As noted in Sec. III, it implies connections between $p=0$ and $\hat{\mathbf{n}}$, and $p=\hat{l}$ and $\hat{\mathbf{m}}$, only. Thus we can write

$$\Omega_I(0)1_2M1_11_2=\Omega_I(\hat{\mathbf{n}})(\sigma_2\cdot\hat{\mathbf{n}})M(\sigma_1\cdot\hat{\mathbf{n}})(\sigma_2\cdot\hat{\mathbf{n}}) , \tag{4.3}$$

$$\Omega_I(\hat{l})(\sigma_2\cdot\hat{l})M(\sigma_1\cdot\hat{l})(\sigma_2\cdot\hat{l}) =\Omega_I(\hat{\mathbf{m}})(\sigma_2\cdot\hat{\mathbf{m}})M(\sigma_1\cdot\hat{\mathbf{m}})(\sigma_2\hat{\mathbf{m}}) . \tag{4.4}$$

The two equations are verified if

$$\Omega_I(\hat{\mathbf{n}})=\Omega_I(0)\Sigma(\hat{\mathbf{n}}) , \tag{4.5}$$

$$\Omega_I(\hat{l})\Sigma(\hat{l})=\Omega_I(\hat{\mathbf{m}})\Sigma(\hat{\mathbf{m}}) . \tag{4.6}$$

The operators $\Sigma(\hat{\mathbf{a}})$ have been defined in the preceding section. The last equation suggests the simple choice

$$\Omega_I(\hat{l})=\Omega_I(0)\Sigma(\hat{l}) , \tag{4.7}$$

$$\Omega_I(\hat{\mathbf{m}})=\Omega_I(0)\Sigma(\hat{\mathbf{m}}) . \tag{4.8}$$

By virtue of (4.5), (4.7), and (4.8), it is sufficient to specify the four $\Omega_I(0)$ to reach a complete knowledge of the 16 operators. Among many possibilities, our choice is dictated by the fact that $\hat{\mathbf{n}}$ is invariant in the outgoing-Δ and incident-nucleon helicity frames. Therefore, we take

$$\begin{aligned}
&\Omega_1(0)=1,\quad \Omega_2(0)=\tfrac{1}{4}\hat{\mathbf{n}}\cdot\overleftrightarrow{\mathrm{T}}_\Delta\cdot\hat{\mathbf{n}} ,\\
&\Omega_3(0)=\frac{1}{4\sqrt{3}}(\hat{l}\cdot\overleftrightarrow{\mathrm{T}}_\Delta\cdot\hat{l}-\hat{\mathbf{m}}\cdot\overleftrightarrow{\mathrm{T}}_\Delta\cdot\hat{\mathbf{m}}) ,\\
&\Omega_4(0)=\frac{1}{2\sqrt{3}}(T_3\hat{l}\hat{\mathbf{m}}\hat{\mathbf{n}})_0 .
\end{aligned} \tag{4.9}$$

After multiplication by $\Sigma(\hat{\mathbf{n}})$, $\Sigma(\hat{l})$, and $\Sigma(\hat{\mathbf{m}})$, we are left with

$$\begin{aligned}
&\Omega_1(\hat{\mathbf{n}})=\Sigma(\hat{\mathbf{n}}),\quad \Omega_2(\hat{\mathbf{n}})=-\tfrac{1}{2}[\Sigma(\hat{\mathbf{n}})+(\sigma_\Delta\cdot\hat{\mathbf{n}})] ,\\
&\Omega_3(\hat{\mathbf{n}})=\frac{1}{4\sqrt{3}}[(T_3\hat{\mathbf{n}}\hat{l}\hat{l})_0-(T_3\hat{\mathbf{n}}\hat{\mathbf{m}}\hat{\mathbf{m}})_0] ,\\
&\Omega_4(\hat{\mathbf{n}})=\frac{1}{2\sqrt{3}}\hat{l}\cdot\overleftrightarrow{\mathrm{T}}_\Delta\cdot\hat{\mathbf{m}},\quad \Omega_1(\hat{l})=\Sigma(\hat{l}) ,\\
&\Omega_2(\hat{l})=\frac{1}{2}\left[\frac{\Sigma(\hat{l})+\sigma_\Delta\cdot\hat{l}}{2}\right]+\frac{\sqrt{3}}{2}\left[\frac{(T_3\hat{l}\hat{\mathbf{n}}\hat{\mathbf{n}})_0-(T_3\hat{l}\hat{\mathbf{m}}\hat{\mathbf{m}})_0}{4\sqrt{3}}\right] ,\\
&\Omega_3(\hat{l})=-\frac{\sqrt{3}}{2}\left[\frac{\Sigma(\hat{l})+\sigma_\Delta\cdot\hat{l}}{2}\right]+\frac{1}{2}\left[\frac{(T_3\hat{l}\hat{\mathbf{n}}\hat{\mathbf{n}})_0-(T_3\hat{l}\hat{\mathbf{m}}\hat{\mathbf{m}})_0}{4\sqrt{3}}\right] ,\\
&\Omega_4(\hat{l})=\frac{1}{2\sqrt{3}}\hat{\mathbf{m}}\cdot\overleftrightarrow{\mathrm{T}}_\Delta\cdot\hat{\mathbf{n}},\quad \Omega_1(\hat{\mathbf{m}})=\Sigma(\hat{\mathbf{m}}) ,\\
&\Omega_2(\hat{\mathbf{m}})=\frac{1}{2}\left[\frac{\Sigma(\hat{\mathbf{m}})+\sigma_\Delta\cdot\hat{\mathbf{m}}}{2}\right]+\frac{\sqrt{3}}{2}\left[\frac{(T_3\hat{\mathbf{m}}\hat{\mathbf{n}}\hat{\mathbf{n}})_0-(T_3\hat{\mathbf{m}}\hat{l}\hat{l})_0}{4\sqrt{3}}\right] ,\\
&\Omega_3(\hat{\mathbf{m}})=\frac{\sqrt{3}}{2}\left[\frac{\Sigma(\hat{\mathbf{m}})+\sigma_\Delta\cdot\hat{\mathbf{m}}}{2}\right]-\frac{1}{2}\left[\frac{(T_3\hat{\mathbf{m}}\hat{\mathbf{n}}\hat{\mathbf{n}})_0-(T_3\hat{\mathbf{m}}\hat{l}\hat{l})_0}{4\sqrt{3}}\right] ,\\
&\Omega_4(\hat{\mathbf{m}})=\frac{1}{2\sqrt{3}}\hat{\mathbf{n}}\cdot\overleftrightarrow{\mathrm{T}}_\Delta\cdot\hat{l} .
\end{aligned} \tag{4.10}$$

With this $\Omega_I(p)$ orthonormal basis, the Δ polarization states are no longer labeled by β as before but specified by the values of I and p. An equation similar to Eq. (2.21) can be written

$$\Omega_I(p)Q_j=q\mathcal{P}_2(p)\sum_{k=1}^{8}Q_k[K_Is(p)v(p)]_{jk} , \tag{4.11}$$

with $q=1$ for $I=1,2,3$ and $q=i$ for $I=4$. The four (8×8) matrices K_I are defined by

$$\begin{aligned}
&K_1=\mathbb{1},\quad K_2=\tfrac{1}{4}[w^2(\hat{\mathbf{n}})-5] ,\\
&K_3=\frac{1}{4\sqrt{3}}[w^2(\hat{l})-w^2(\hat{\mathbf{m}})] ,\\
&K_4=-\frac{1}{12\sqrt{3}}\{w(\hat{l})[w(\hat{\mathbf{m}})w(\hat{\mathbf{n}})-w(\hat{\mathbf{n}})w(\hat{\mathbf{m}})]\\
&\qquad+w(\hat{\mathbf{m}})[w(\hat{\mathbf{n}})w(\hat{l})-w(\hat{l})w(\hat{\mathbf{n}})]\\
&\qquad+w(\hat{\mathbf{n}})[w(\hat{l})w(\hat{\mathbf{m}})-w(\hat{\mathbf{m}})w(\hat{l})]\} .
\end{aligned} \tag{4.12}$$

They are explicitly given in Appendix C.

For a Δ polarization state defined by (I,p) indices, spin observables can be written by means of expressions similar to Eq. (3.6):

$$\sigma X_{(I,p)\alpha_2'\alpha_1\alpha_2} = \sum_{i,j=1}^{8} [\mathrm{Re}C_0\mathrm{Re}(f_i^* f_j + g_i^* g_j Z_{\alpha_2}) + \mathrm{Im}C_0\mathrm{Im}(f_i^* f_j + g_i^* g_j Z_{\alpha_2})][K_I s(p)v(p)v(\alpha_1)s(\alpha_2)]_{ij}$$
$$+[2\,\mathrm{Re}C_1\mathrm{Re}(f_i^* g_j) + 2\,\mathrm{Im}C_1\mathrm{Im}f_i^* g_j][K_I s(p)v(p)v(\alpha_1)s(\alpha_2)s(\hat{\mathbf{n}})]_{ij} \tag{4.13}$$

for $I=1,2,3$ and

$$\sigma X_{(4,p)\alpha_2'\alpha_1\alpha_2} = \sum_{i,j=1}^{8} [\mathrm{Re}C_0\mathrm{Im}(f_i^* f_j + g_i^* g_j Z_{\alpha_2}) - \mathrm{Im}C_0\mathrm{Re}(f_i^* f_j + g_i^* g_j Z_{\alpha_2})][K_4 s(p)v(p)v(\alpha_1)s(\alpha_2)]_{ij}$$
$$+[2\,\mathrm{Re}C_1\mathrm{Im}(f_i^* g_j) - 2\,\mathrm{Im}C_1\mathrm{Re}(f_i^* g_j)][K_4 s(p)v(p)v(\alpha_1)s(\alpha_2)s(\hat{\mathbf{n}})]_{ij} \ . \tag{4.14}$$

We recall that the C_0 and C_1 coefficients are displayed in Table II as functions of p, α_2', α_1, and α_2. Products of s and v matrices can be simplified by taking into account properties of these products as given by Eqs. (2.18) and (2.20). The Z_{α_2} coefficient is given in Eq. (2.16). As an example, for a Δ polarization state described by $\Omega_4(\hat{\mathbf{n}})$ and a polarized beam along $\hat{\mathbf{n}}$, i.e., $p=\alpha_1=\hat{\mathbf{n}}$ and $\alpha_2=\alpha_2'=0$, we get $C_0=1$ and $C_1=0$ from Table II. Thus Eqs. (4.14) reduces to

$$\sigma X_{(4,n)0n0} = \sum_{i,j=1}^{8} \mathrm{Im}(f_i^* f_j + g_i^* g_j)[K_4 s(\hat{\mathbf{n}})]_{ij} \ . \tag{4.15}$$

Finally, Eqs. (4.13) and (4.14) describe in a rather compact and simple way the 512 observables of the $NN\to N\Delta$ transition, once the polarization states of Δ are expanded on the $\Omega_I(p)$ orthonormal basis. The density matrix method can also be used through a decomposition on this basis.

V. CONCLUSIONS

The present work has been devoted to vector- and tensor-spin observables in the $NN\to N\Delta$ transition. Within the relations between transition and spin operators, we have established a compact formula for spin observables (IFE). These observables are connected to the 16 spin amplitudes describing the $NN\to N\Delta$ transition by only nine 8×8 matrices. With the parity conservation, the so-called Bohr's relations, and the Pauli principle, the number of IFE is reduced from 1024 to 160. A particular set of Δ-spin operators well adapted to the analysis of the experimental situation is given.

The dynamics of the Δ production is contained in the 16 spin amplitudes f_i and g_i. They can be reached either by model calculations or by phenomenological analysis. Studies of experimental data from this point of view will be the subject of a forthcoming paper.

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APPENDIX A: PROPERTIES OF TRANSITION AND SPIN OPERATORS

The $N\Delta$-spin transition operators $\mathbf{S}$ and $\overleftrightarrow{\mathbf{T}}$ defined by Eq. (2.6) are identical to those of Refs. 2, 3, and 7. For the Δ-spin-space operators, the Pauli-spin operator $\boldsymbol{\sigma}_\Delta$ for spin-$\frac{3}{2}$ particles as well as $\overleftrightarrow{\mathbf{T}}_\Delta$ and T_3 tensors of rank 2 and 3 are defined in Sec. II B. Note that $\boldsymbol{\sigma}_\Delta$ is the same as in Ref. 7, where the 4×4 matrices are explicitly given. The reader can refer to Ref. 18, where some of these operators are given with various notations and reduced matrix elements.

Below we list some relations among operators which are needed in the derivation of Eqs. (2.19) and (2.21) for the calculation of v and w matrices (see Appendix C):

$$(\mathbf{S}\cdot\hat{\mathbf{a}})(\boldsymbol{\sigma}\cdot\hat{\mathbf{b}}) = -\frac{i}{2}(\mathbf{S}\cdot\hat{\mathbf{a}}\times\hat{\mathbf{b}}) + (\hat{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{b}}) \ , \tag{A1}$$

$$(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{b}})(\mathbf{S}\cdot\hat{\mathbf{a}}) = -\frac{5i}{2}(\mathbf{S}\cdot\hat{\mathbf{a}}\times\hat{\mathbf{b}}) + (\hat{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{b}}) \ , \tag{A2}$$

$$(\hat{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{b}})(\boldsymbol{\sigma}\cdot\hat{\mathbf{c}}) = \tfrac{1}{4}[3(\mathbf{S}\cdot\hat{\mathbf{a}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{c}}) + 3(\mathbf{S}\cdot\hat{\mathbf{b}})(\hat{\mathbf{c}}\cdot\hat{\mathbf{a}}) - 2(\mathbf{S}\cdot\hat{\mathbf{c}})(\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})] + \frac{i}{2}(\hat{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{b}}\times\hat{\mathbf{c}} + \hat{\mathbf{b}}\cdot\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{a}}\times\hat{\mathbf{c}}) \ , \tag{A3}$$

$$(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{c}})(\hat{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{b}}) = \tfrac{1}{4}(3(\mathbf{S}\cdot\hat{\mathbf{a}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{c}}) + 3(\mathbf{S}\cdot\hat{\mathbf{b}})(\hat{\mathbf{c}}\cdot\hat{\mathbf{a}}) - 2(\mathbf{S}\cdot\hat{\mathbf{c}})(\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})) - \frac{3i}{2}(\hat{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{b}}\times\hat{\mathbf{c}} + \hat{\mathbf{b}}\cdot\overleftrightarrow{\mathbf{T}}\cdot\hat{\mathbf{a}}\times\hat{\mathbf{c}}) \ . \tag{A4}$$

In particular, we note that $(\boldsymbol{\sigma}_\Delta\cdot\mathbf{S})=(\mathbf{S}\cdot\boldsymbol{\sigma})=0$. Equation (2.9) is directly obtained from Eq. (A1). Writing expressions of $\overleftrightarrow{\mathbf{T}}_\Delta$ and T_3 in terms of the $\boldsymbol{\sigma}_\Delta$ Pauli operator in Eqs. (2.12) and (2.13) and, more generally, demonstrating the invariance rules requires tensorial products of different $\mathcal{P}_\Delta(\beta)$ operators. We first recall the well-known formula of a tensorial product:

$$[T(j,j')]_{j_3}^{m_3}[T(j',j'')]_{j_4}^{m_4}=\langle j\|T_{j_3}\|j'\rangle\langle j'\|T_{j_4}\|j''\rangle\sum_{JM}(2J+1)\begin{Bmatrix} j_3 & j_4 & J\\ j'' & j & j'\end{Bmatrix}\frac{1}{\langle j\|T_J\|j''\rangle}\times[T(j,j'')]_J^M\begin{pmatrix} j_3 & j_4 & J\\ -m_3 & -m_4 & M\end{pmatrix}(-)^{j''+2j'-j-M}, \quad \text{(A5)}$$

with the usual notation for 3-j and 6-j symbols. For $j=j'=j''=\frac{3}{2}$, $j_3=j_4=1$ it gives

$$(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{a}})(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{b}})=5(\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})+i(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{a}}\times\hat{\mathbf{b}})+(\hat{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{b}}) . \quad \text{(A6)}$$

Using the fact that $(\hat{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{b}})$ is invariant under $\hat{\mathbf{a}},\hat{\mathbf{b}}$ exchange, this last equation leads to Eq. (2.12). To obtain Eq. (2.13), use is made of the additional expression

$$(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{a}})(\hat{\mathbf{b}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{c}})=\tfrac{8}{5}[\tfrac{3}{2}(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{b}})(\hat{\mathbf{c}}\cdot\hat{\mathbf{a}})+\tfrac{3}{2}(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{c}})(\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})-(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{a}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{c}})]-i(\hat{\mathbf{c}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{b}}\times\hat{\mathbf{a}}+\hat{\mathbf{b}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{c}}\times\hat{\mathbf{a}})-(T_3\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}})_0 . \quad \text{(A7)}$$

Other relations, which are needed in setting up the so-called "Bohr's rules" (Sec. III), are listed below:

$$\begin{aligned}(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{d}})(T_3\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}})_0=&\tfrac{2}{5}[(\hat{\mathbf{d}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{a}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{c}})+(\hat{\mathbf{d}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{b}})(\hat{\mathbf{c}}\cdot\hat{\mathbf{a}})+(\hat{\mathbf{d}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{c}})(\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})]\\ &-[(\hat{\mathbf{d}}\cdot\hat{\mathbf{a}})(\hat{\mathbf{b}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{c}})+(\hat{\mathbf{d}}\cdot\hat{\mathbf{b}})(\hat{\mathbf{c}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{a}})+(\hat{\mathbf{d}}\cdot\hat{\mathbf{c}})(\hat{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{b}})]\\ &+i[(T_3(\hat{\mathbf{d}}\times\hat{\mathbf{a}})\hat{\mathbf{b}}\hat{\mathbf{c}})_0+(T_3(\hat{\mathbf{d}}\times\hat{\mathbf{b}})\hat{\mathbf{c}}\hat{\mathbf{a}})_0+(T_3(\hat{\mathbf{d}}\times\hat{\mathbf{c}})\hat{\mathbf{a}}\hat{\mathbf{b}})_0]\end{aligned} \quad \text{(A8)}$$

and also

$$\begin{aligned}(\hat{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{b}})(\hat{\mathbf{c}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{d}})=&12(\hat{\mathbf{a}}\cdot\hat{\mathbf{c}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{d}})+12(\hat{\mathbf{a}}\cdot\hat{\mathbf{d}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{c}})-8(\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})(\hat{\mathbf{c}}\cdot\hat{\mathbf{d}})\\ &-\tfrac{12}{5}i\{[\boldsymbol{\sigma}_\Delta\cdot(\hat{\mathbf{a}}\times\hat{\mathbf{b}})\times(\hat{\mathbf{c}}\times\hat{\mathbf{d}})]-2(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{a}}\times\hat{\mathbf{c}})(\hat{\mathbf{b}}\cdot\hat{\mathbf{d}})-2(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{b}}\times\hat{\mathbf{d}})(\hat{\mathbf{a}}\cdot\hat{\mathbf{c}})\}\\ &-2i[(T_3(\hat{\mathbf{a}}\times\hat{\mathbf{c}})\hat{\mathbf{b}}\hat{\mathbf{d}})_0+(T_3(\hat{\mathbf{b}}\times\hat{\mathbf{d}})\hat{\mathbf{a}}\hat{\mathbf{c}})_0].\end{aligned} \quad \text{(A9)}$$

In particular, we get

$$(\hat{\mathbf{l}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{n}})(\hat{\mathbf{m}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{n}})=-12i[\tfrac{1}{6}(T_3\hat{\mathbf{n}}\hat{\mathbf{n}}\hat{\mathbf{n}})_0-\tfrac{1}{5}(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{n}})] , \quad \text{(A10)}$$

and two similar expressions for circular permutations of $(\hat{\mathbf{l}},\hat{\mathbf{m}},\hat{\mathbf{n}})$. These combinations of T_3 and $\boldsymbol{\sigma}_\Delta$ are found to be very powerful operators. They are denoted by

$$\boldsymbol{\Sigma}(\hat{\mathbf{a}})\equiv\tfrac{1}{6}(T_3\hat{\mathbf{a}}\hat{\mathbf{a}}\hat{\mathbf{a}})_0-\tfrac{1}{5}(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{a}}) , \quad \text{(A11)}$$

and obey the relation

$$\boldsymbol{\Sigma}(\hat{\mathbf{a}})\boldsymbol{\Sigma}(\hat{\mathbf{b}})=(\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})+i\boldsymbol{\Sigma}(\hat{\mathbf{a}}\times\hat{\mathbf{b}}) . \quad \text{(A12)}$$

This is to be compared with the analogous expression for the nucleon Pauli-spin operator

$$(\boldsymbol{\sigma}\cdot\hat{\mathbf{a}})(\boldsymbol{\sigma}\cdot\hat{\mathbf{b}})=(\hat{\mathbf{a}}\cdot\hat{\mathbf{b}})+i(\boldsymbol{\sigma}\cdot\hat{\mathbf{a}}\times\hat{\mathbf{b}}) . \quad \text{(A13)}$$

Using the $\boldsymbol{\Sigma}$ operator, we can write

$$\begin{aligned}&(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{n}})\boldsymbol{\Sigma}(\hat{\mathbf{n}})=-[1+\tfrac{1}{2}(\hat{\mathbf{n}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{n}})] ,\\ &(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{l}})\boldsymbol{\Sigma}(\hat{\mathbf{n}})=-i[\tfrac{1}{2}(T_3\hat{\mathbf{m}}\hat{\mathbf{n}}\hat{\mathbf{n}})_0-\tfrac{1}{5}(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{m}})] ,\\ &(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{m}})\boldsymbol{\Sigma}(\hat{\mathbf{n}})=i[\tfrac{1}{2}(T_3\hat{\mathbf{l}}\hat{\mathbf{n}}\hat{\mathbf{n}})_0-\tfrac{1}{5}(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{l}})] ,\\ &[1+\tfrac{1}{2}(\hat{\mathbf{l}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{l}})]\boldsymbol{\Sigma}(\hat{\mathbf{n}})=-[\tfrac{1}{2}(T_3\hat{\mathbf{n}}\hat{\mathbf{m}}\hat{\mathbf{m}})_0-\tfrac{1}{5}(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{n}})] ,\\ &(\hat{\mathbf{n}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{l}})\boldsymbol{\Sigma}(\hat{\mathbf{n}})=i(\hat{\mathbf{n}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{m}}) ,\\ &(\hat{\mathbf{l}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{m}})\boldsymbol{\Sigma}(\hat{\mathbf{n}})=(T_3\hat{\mathbf{l}}\hat{\mathbf{m}}\hat{\mathbf{n}})_0 ,\end{aligned} \quad \text{(A14)}$$

which lead to Eqs. (3.17)–(3.22).

Finally, we give the traces needed for orthonormalizing Δ-spin-space operators in Sec. IV:

$$\begin{aligned}&\tfrac{1}{4}\mathrm{Tr}1=1,\quad \tfrac{1}{4}\mathrm{Tr}(\boldsymbol{\sigma}_\Delta\cdot\hat{\mathbf{a}})^2=5 ,\\ &\tfrac{1}{4}\mathrm{Tr}(\hat{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{a}})^2=16 ,\\ &\tfrac{1}{4}\mathrm{Tr}(\hat{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{a}})(\hat{\mathbf{b}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{b}})=-8 ,\\ &\tfrac{1}{4}\mathrm{Tr}(\hat{\mathbf{a}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{b}})^2=12 ,\\ &\tfrac{1}{4}\mathrm{Tr}(T_3\hat{\mathbf{l}}\hat{\mathbf{m}}\hat{\mathbf{n}})_0^2=12 ,\\ &\tfrac{1}{4}\mathrm{Tr}(T_3\hat{\mathbf{a}}\hat{\mathbf{a}}\hat{\mathbf{a}})_0^2=\frac{(12)^2}{5} ,\\ &\tfrac{1}{4}\mathrm{Tr}(T_3\hat{\mathbf{a}}\hat{\mathbf{a}}\hat{\mathbf{a}})_0(T_3\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{b}})_0=-\frac{(12)^2}{10} ,\\ &\tfrac{1}{4}\mathrm{Tr}(T_3\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{b}})_0^2=\tfrac{96}{5} ,\\ &\tfrac{1}{4}\mathrm{Tr}(T_3\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{b}})_0(T_3\hat{\mathbf{a}}\hat{\mathbf{c}}\hat{\mathbf{c}})_0=-\tfrac{24}{5} ,\end{aligned} \quad \text{(A15)}$$

for $\hat{\mathbf{a}}\neq\hat{\mathbf{b}}\neq\hat{\mathbf{c}}$ chosen in set $(\hat{\mathbf{l}},\hat{\mathbf{m}},\hat{\mathbf{n}})$.

APPENDIX B: $\mathcal{P}_\Delta$ OPERATORS IN TERMS OF PROJECTORS

In Sec. II B construction is made of 16 $\mathcal{P}_\Delta$ operators in terms of irreducible tensors of Δ-spin-space operators and an orthonormal set of vectors $(\hat{\mathbf{l}},\hat{\mathbf{m}},\hat{\mathbf{n}})$. Relations between spherical and Cartesian components are given for the choice $\hat{\mathbf{l}}=\hat{\mathbf{u}}_z$, $\hat{\mathbf{m}}=\hat{\mathbf{u}}_x$, $\hat{\mathbf{n}}=\hat{\mathbf{u}}_y$, and the spin projectors $|\frac{3}{2},\xi\rangle\langle\frac{3}{2},\xi'|$ are denoted by $|2\xi\rangle\langle 2\xi'|$ for the sake of simplicity:

$$1=T_0^0=(|3\rangle\langle 3|+|-3\rangle\langle -3|+|1\rangle\langle 1|+|-1\rangle\langle -1|)\,,$$

$$\sigma_\Delta\cdot\hat{l}=T_1^0=3(|3\rangle\langle 3|-|-3\rangle\langle -3|)+(|1\rangle\langle 1|-|-1\rangle\langle -1|)\,,$$

$$\sigma_\Delta\cdot\hat{\mathbf{m}}=\frac{1}{\sqrt{2}}(T_1^{-1}-T_1^1)=2(|1\rangle\langle -1|+|-1\rangle\langle 1|)+\sqrt{3}(|1\rangle\langle 3|+|-1\rangle\langle -3|+|3\rangle\langle 1|+|-3\rangle\langle -1|)\,,$$

$$\sigma_\Delta\cdot\hat{\mathbf{n}}=\frac{i}{\sqrt{2}}(T_1^{-1}+T_1^1)=-2i(|1\rangle\langle -1|-|-1\rangle\langle 1|)+i\sqrt{3}(|1\rangle\langle 3|-|-1\rangle\langle -3|-|3\rangle\langle 1|+|-3\rangle\langle -1|)\,,$$

$$\hat{l}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{l}=\frac{2}{\sqrt{30}}T_2^0=4(|3\rangle\langle 3|+|-3\rangle\langle -3|-|1\rangle\langle 1|-|-1\rangle\langle -1|)\,,$$

$$\hat{\mathbf{m}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{m}}-\hat{\mathbf{n}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{n}}=\frac{1}{\sqrt{5}}(T_2^{-2}+T_2^2)$$
$$=4\sqrt{3}(|1\rangle\langle -3|+|-1\rangle\langle 3|+|-3\rangle\langle 1|+|3\rangle\langle -1|)\,,$$

$$\hat{l}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{m}}=\frac{1}{2\sqrt{5}}(T_2^{-1}-T_2^1)=2\sqrt{3}(|1\rangle\langle 3|-|-1\rangle\langle -3|+|3\rangle\langle 1|-|-3\rangle\langle -1|)\,,$$

$$\hat{l}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{n}}=\frac{i}{2\sqrt{5}}(T_2^{-1}+T_2^1)=i2\sqrt{3}(|1\rangle\langle 3|+|-1\rangle\langle -3|-|3\rangle\langle 1|-|-3\rangle\langle -1|)\,,$$

$$\hat{\mathbf{m}}\cdot\overleftrightarrow{\mathbf{T}}_\Delta\cdot\hat{\mathbf{n}}=\frac{i}{2\sqrt{5}}(T_2^{-2}-T_2^2)=i2\sqrt{3}(-|1\rangle\langle -3|+|-1\rangle\langle 3|+|-3\rangle\langle 1|-|3\rangle\langle -1|)\,,$$

$$(T_3\hat{l}\hat{l}\hat{l})_0=-\frac{2}{\sqrt{70}}T_3^0=-\tfrac{12}{5}[|3\rangle\langle 3|-|-3\rangle\langle -3|-3(|1\rangle\langle 1|-|-1\rangle\langle -1|)]\,,$$

$$(T_3\hat{l}\hat{\mathbf{m}}\hat{\mathbf{m}})_0-(T_3\hat{l}\hat{\mathbf{n}}\hat{\mathbf{n}})_0=-\frac{1}{\sqrt{21}}(T_3^{-2}+T_3^2)=4\sqrt{3}(|1\rangle\langle -3|-|-1\rangle\langle 3|+|-3\rangle\langle 1|-|3\rangle\langle -1|)\,,$$

$$(T_3\hat{l}\hat{\mathbf{m}}\hat{\mathbf{n}})_0=\frac{-i}{2\sqrt{21}}(T_3^{-2}-T_3^2)=-i2\sqrt{3}(|1\rangle\langle -3|+|-1\rangle\langle 3|-|-3\rangle\langle 1|-|3\rangle\langle -1|)\,,$$

$$(T_3\hat{\mathbf{m}}\hat{l}\hat{l})_0=\frac{-2}{\sqrt{210}}(T_3^{-1}-T_3^1)$$
$$=-\frac{8\sqrt{3}}{5}[-\sqrt{3}(|1\rangle\langle -1|+|-1\rangle\langle 1|)+|1\rangle\langle 3|+|-1\rangle\langle -3|+|3\rangle\langle 1|+|-3\rangle\langle -1|]\,,$$

$$(T_3\hat{\mathbf{n}}\hat{l}\hat{l})_0=-\frac{2i}{\sqrt{210}}(T_3^{-1}+T_3^1)=-\frac{8i\sqrt{3}}{5}[\sqrt{3}(|1\rangle\langle -1|-|-1\rangle\langle 1|)+|1\rangle\langle 3|-|-1\rangle\langle -3|$$
$$-|3\rangle\langle 1|+|-3\rangle\langle -1|]\,,$$

$$(T_3\hat{\mathbf{m}}\hat{\mathbf{m}}\hat{\mathbf{m}})_0-(T_3\hat{\mathbf{m}}\hat{\mathbf{n}}\hat{\mathbf{n}})_0+\tfrac{1}{2}(T_3\hat{\mathbf{m}}\hat{l}\hat{l})_0=-\frac{1}{\sqrt{14}}(T_3^{-3}-T_3^3)=-12(|3\rangle\langle -3|+|-3\rangle\langle 3|)\,,$$

$$(T_3\hat{\mathbf{n}}\hat{\mathbf{n}}\hat{\mathbf{n}})_0-(T_3\hat{\mathbf{n}}\hat{\mathbf{m}}\hat{\mathbf{m}})_0+\tfrac{1}{2}(T_3\hat{\mathbf{n}}\hat{l}\hat{l})_0=\frac{i}{\sqrt{14}}(T_3^{-3}+T_3^3)=-12i(|3\rangle\langle -3|-|-3\rangle\langle 3|)\,.$$

With the choice of the $(\hat{l},\hat{\mathbf{m}},\hat{\mathbf{n}})$ fixed basis, the beam nucleon only is in its helicity frame. For outgoing particles, helicity projections can be formed as linear combinations of $\mathcal{P}_\Delta$ with appropriate rotation coefficients.

APPENDIX C

In this appendix we give the explicit expressions of matrices v, w, and s of Eqs. (2.19), (2.22), and (2.17):

$$v(\hat{\mathbf{n}})=\begin{pmatrix} 0 & 0 & -\frac{1}{2} & -\sqrt{3}/2 & & & & \\ 0 & 0 & -\sqrt{3}/2 & \frac{1}{2} & & & 0 & \\ -\frac{1}{2} & -\sqrt{3}/2 & 0 & 0 & & & & \\ -\sqrt{3}/2 & \frac{1}{2} & 0 & 0 & & & & \\ & & & & 0 & 0 & -\frac{1}{2} & -\sqrt{3}/2 \\ & & 0 & & 0 & 0 & -\sqrt{3}/2 & \frac{1}{2} \\ & & & & -\frac{1}{2} & -\sqrt{3}/2 & 0 & 0 \\ & & & & -\sqrt{3}/2 & \frac{1}{2} & 0 & 0 \end{pmatrix},$$

$$v(\hat{\boldsymbol{l}})=\left(\begin{array}{cc} 0 & \begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -\frac{1}{2} & \sqrt{3}/2 & 0 & 0 \\ \sqrt{3}/2 & \frac{1}{2} & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & -\frac{1}{2} & \sqrt{3}/2 \\ 0 & 0 & \sqrt{3}/2 & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{matrix} & 0 \end{array}\right),$$

$$v(\hat{\mathbf{m}})=\left(\begin{array}{cc} 0 & \begin{matrix} -\frac{1}{2} & \sqrt{3}/2 & 0 & 0 \\ \sqrt{3}/2 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \sqrt{3}/2 \\ 0 & 0 & -\sqrt{3}/2 & -\frac{1}{2} \end{matrix} \\ \begin{matrix} -\frac{1}{2} & \sqrt{3}/2 & 0 & 0 \\ -\sqrt{3}/2 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \sqrt{3}/2 \\ 0 & 0 & \sqrt{3}/2 & -\frac{1}{2} \end{matrix} & 0 \end{array}\right),$$

$$w(\hat{\mathbf{n}})=\left(\begin{array}{cc} \begin{matrix} 0 & 0 & -\frac{5}{2} & -\sqrt{3}/2 \\ 0 & 0 & -\sqrt{3}/2 & -\frac{3}{2} \\ -\frac{5}{2} & -\sqrt{3}/2 & 0 & 0 \\ -\sqrt{3}/2 & -\frac{3}{2} & 0 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & 0 & -\frac{1}{2} & -\sqrt{3}/2 \\ 0 & 0 & 3\sqrt{3}/2 & -\frac{3}{2} \\ -\frac{1}{2} & 3\sqrt{3}/2 & 0 & 0 \\ -\sqrt{3}/2 & -\frac{3}{2} & 0 & 0 \end{matrix} \end{array}\right),$$

$$w(\hat{\boldsymbol{l}})=\left(\begin{array}{cc} 0 & \begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ -\frac{5}{2} & \sqrt{3}/2 & 0 & 0 \\ \dfrac{\sqrt{3}}{2} & -\frac{3}{2} & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & -\frac{5}{2} & \sqrt{3}/2 \\ 0 & 0 & \sqrt{3}/2 & -\frac{3}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{matrix} & 0 \end{array}\right),$$

$$w(\hat{\mathbf{m}})=\left(\begin{array}{cc} 0 & \begin{matrix} -\frac{5}{2} & -\sqrt{3}/2 & 0 & 0 \\ \sqrt{3}/2 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \sqrt{3}/2 \\ 0 & 0 & 3\sqrt{3}/2 & \frac{3}{2} \end{matrix} \\ \begin{matrix} -\frac{5}{2} & \sqrt{3}/2 & 0 & 0 \\ -\sqrt{3}/2 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 3\sqrt{3}/2 \\ 0 & 0 & \sqrt{3}/2 & \frac{3}{2} \end{matrix} & 0 \end{array}\right),$$

$$s(\hat{\mathbf{n}})=\begin{bmatrix} \begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{matrix} & 0 \\ 0 & \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \end{bmatrix}, \quad s(\hat{\boldsymbol{l}})=\begin{bmatrix} \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{matrix} & 0 \\ 0 & \begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \end{bmatrix},$$

$$s(\hat{\mathbf{m}})=\begin{bmatrix} \begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} & 0 \\ 0 & \begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \end{bmatrix}.$$

We recall that $v(0)$ and $s(0)$ are the unit (8×8) matrices. The matrices defined in Eq. (4.12) are also displayed below:

$$K_2=\begin{bmatrix} \begin{matrix} \frac{1}{2} & \sqrt{3}/2 & 0 & 0 \\ \sqrt{3}/2 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \sqrt{3}/2 \\ 0 & 0 & \sqrt{3}/2 & -\frac{1}{2} \end{matrix} & 0 \\ 0 & \begin{matrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\sqrt{3}/2 \\ 0 & 0 & -\sqrt{3}/2 & -\frac{1}{2} \end{matrix} \end{bmatrix},$$

$$K_3=\begin{bmatrix} \begin{matrix} -\sqrt{3}/2 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \sqrt{3}/2 & 0 & 0 \\ 0 & 0 & \sqrt{3}/2 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & -\sqrt{3}/2 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{3}/2 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \sqrt{3}/2 \end{matrix} \end{bmatrix}, \quad K_4=\begin{bmatrix} \begin{matrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{matrix} \end{bmatrix}.$$

We recall that K_1 is the unit (8×8) matrix.

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