New low-energy theorems in nucleon Compton scattering

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By using a second lemma in Compton scattering, we derive four new low-energy theorems of second order in the frequency of the incoming photon.

I. INTRODUCTION

Some time ago we proved a second lemma[1] in hadron Compton scattering, according to which the time-space component of the excited-state hadron Compton amplitude can be written as

$$E_{0n}^{\alpha\beta}=k'_m\,\Gamma_{mn}^{\alpha\beta}(\mathbf{k},\omega,\mathbf{k}',\omega')\ , \tag{1}$$

where α and β are isospin indices, $k_\mu=(k,i\omega)$ and k'_μ are incident and outgoing photon momenta, respectively, and $\Gamma_{mn}^{\alpha\beta}$ is odd under crossing in the Breit frame:

$$\Gamma_{mn}^{\alpha\beta}(\mathbf{k},\omega,\mathbf{k}',\omega')=-\Gamma_{nm}^{\beta\alpha}(-\mathbf{k}',-\omega',-\mathbf{k},-\omega)\ . \tag{2}$$

Here we shall use this result to obtain low-energy theorems in nucleon Compton scattering to second order in the photon energy.

II. THE BASIC EQUATIONS

When one considers the consequence of gauge invariance for each photon separately the nucleon scattering amplitude tensor $T_{\mu\nu}^{\alpha\beta}$ is seen to satisfy the relation[2]

$$k'_\mu T_{\mu\nu}^{\alpha\beta}=T_{\nu\lambda}^{\alpha\beta}k_\lambda=i(2\pi)^3(EE'/M^2)^{1/2}\epsilon^{\alpha\beta\gamma}\langle \mathbf{p}'\,|\,J_\nu^\gamma(0)\,|\,\mathbf{p}\rangle\ , \tag{3}$$

where $\mathbf{p}$ and $\mathbf{p}'$ are the momenta of the initial and final nucleons, respectively, E and E' the corresponding energies, and J_ν^γ is the electromagnetic current. Next, we make the usual decomposition $T=U+E$ where U is the unexcited part of the amplitude and E is the excited part. Using Eq. (1) we obtain from (3) the two independent relations

$$k'_m(E_{mn}^{\alpha\beta}-\omega'\Gamma_{mn}^{\alpha\beta})=-k'_m U_{mn}^{\alpha\beta}+\omega' U_{0n}^{\alpha\beta}+i(2\pi)^3(EE'/M^2)^{1/2}\epsilon^{\alpha\beta\gamma}\langle \mathbf{p}'\,|\,J_n^\gamma\,|\,\mathbf{p}\rangle \tag{4}$$

and

$$k'_m k_n(\Gamma_{mn}^{\alpha\beta}-\omega\Lambda_{mn}^{\alpha\beta})=-U_{0n}^{\alpha\beta}k_n+\omega U_{00}^{\alpha\beta}+i(2\pi)^3(EE'/M^2)^{1/2}\epsilon^{\alpha\beta\gamma}\langle \mathbf{p}'\,|\,J_0^\gamma\,|\,\mathbf{p}\rangle\ , \tag{5}$$

where the second term on the left-hand side of Eq. (5) is a consequence of Singh's lemma,[3] according to which

$$E_{00}^{\alpha\beta}=k'_m k_n\Lambda_{mn}^{\alpha\beta}\ , \tag{6}$$

where $\Lambda_{mn}^{\alpha\beta}$ is even under crossing.

The fact that the property (2) is satisfied only in the Breit frame is particularly gratifying since it is in this frame that the requirement of T invariance achieves its simplest form, as used by Pais[4] to find the minimal basis $B_{mn}^{(i)}$ in which $E_{mn}^{\alpha\beta}$ is to be expanded. To order ω^2 we have, for the isospin-antisymmetric part of the amplitude,

$$\begin{aligned}E_{mn}^{[\alpha\beta]}&=[I^\alpha,I^\beta]\sum_i a_i B_{mn}^{(i)}\\&=[I^\alpha,I^\beta]\{\,\omega a_{11}\delta_{mn}+[a_2(0)+a_{21}\mathbf{k}\cdot\mathbf{k}'+a_{22}\omega^2]\epsilon_{mnj}\sigma_j+a_3(0)[\delta_{mn}\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{k})-\mathbf{k}\cdot\mathbf{k}'\epsilon_{mnj}\sigma_j]\\&\quad+a_4(0)[k_m(\boldsymbol{\sigma}\times\mathbf{k})_n-k'_n(\boldsymbol{\sigma}\times\mathbf{k}')_m]+a_5(0)[k'_m(\boldsymbol{\sigma}\times\mathbf{k}')_n-k_n(\boldsymbol{\sigma}\times\mathbf{k})_m-2\omega^2\epsilon_{mnj}\sigma_j]\\&\quad+a_6(0)[k_m(\boldsymbol{\sigma}\times\mathbf{k}')_n-k'_n(\boldsymbol{\sigma}\times\mathbf{k})_m-2\mathbf{k}\cdot\mathbf{k}'\epsilon_{mnj}\sigma_j]+a_7(0)[k'_m(\boldsymbol{\sigma}\times\mathbf{k})_n-k_n(\boldsymbol{\sigma}\times\mathbf{k}')_m]\}\ ,\end{aligned} \tag{7}$$

where I^α is the αth component of isospin and the expansion of the coefficients, to order ω^2, are in accordance with the even crossing symmetry property of $E_{mn}^{\alpha\beta}$. For convenience we have used parity- and time-reversal-invariant basis elements $B_{mn}^{(i)}$ for $i=4-7$ of which the corresponding basis elements $E_{mn}^{(i)}$ of Pais,[4] there numbered $i=8-11$, are linear combinations, as we discuss in the Appendix.

The usual procedure to obtain low-energy theorems uses a relation involving $k'_m k_n E_{mn}^{\alpha\beta}$ and then it can give no information on the partial amplitudes for which $k'_m k_n B_{mn}^{(i)}=0$. This is the case for $i=3-7$ in Eq. (7), which will now be accessible to us.

As $\Gamma_{mn}^{\alpha\beta}$ and $\Lambda_{mn}^{\alpha\beta}$ are three-tensors, as is $E_{mn}^{\alpha\beta}$, they can be expanded in the same minimal basis. To get information on $E_{mn}^{[\alpha\beta]}$ to order ω^2 we see from Eq. (4) that we need to consider $\Gamma_{mn}^{[\alpha\beta]}$ only to order ω and consequently Eq. (5) indicates that it is sufficient to consider $\Lambda_{mn}^{[\alpha\beta]}$ only to order one. As $\Gamma_{mn}^{\alpha\beta}$ is odd under crossing we have, to order ω,

$$\Gamma_{mn}^{[\alpha\beta]}=[I^\alpha,I^\beta][b_1(0)\delta_{mn}+\omega b_{21}\epsilon_{mnj}\sigma_j] \; . \tag{8}$$

On the other hand, as $\Lambda_{mn}^{\alpha\beta}$ is even under crossing we have, to order one,

$$\Lambda_{mn}^{[\alpha\beta]}=[I^\alpha,I^\beta]c_2(0)\epsilon_{mnj}\sigma_j \; . \tag{9}$$

Using Eqs. (7)–(9) the isospin-antisymmetric parts of the left-hand sides of Eqs. (4) and (5) become

$$\begin{aligned} k'_m(E_{mn}^{[\alpha\beta]}-\omega\Gamma_{mn}^{[\alpha\beta]})=[I^\alpha,I^\beta](\omega(a_{11}-b_1(0)k'_n+(\boldsymbol{\sigma}\times\mathbf{k}')_n\{a_2(0)+\mathbf{k}\cdot\mathbf{k}'[a_{21}-a_3(0)-a_6(0)]+\omega^2(a_{22}-a_5(0)-b_{21})\} \\ +(\boldsymbol{\sigma}\times\mathbf{k})_n[\mathbf{k}\cdot\mathbf{k}'a_4(0)+\omega^2a_7(0)]+k_n\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{k})a_5(0)+k'_n\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{k})[a_3(0)+a_6(0)]) \end{aligned} \tag{10}$$

and

$$k'_m k_n(\Gamma_{mn}^{[\alpha\beta]}-\omega\Lambda_{mn}^{[\alpha\beta]})=[I^\alpha,I^\beta]\{\mathbf{k}\cdot\mathbf{k}'b_1(0)+\omega[b_{21}-c_2(0)]\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{k})\} \; . \tag{11}$$

Equations (10) and (11) are now to be substituted in Eqs. (4) and (5), respectively. The terms $a_{11}-b_1(0)$, $a_{22}-a_5(0)-b_{21}$, and all the other terms of Eq. (10) will be determined by the right-hand side of Eq. (4) and $b_1(0)$ and $b_{21}-c_2(0)$ will be determined by the right-hand side of Eq. (5). Knowing $b_1(0)$ we shall then have the value of a_{11} but as b_{21} receives an unknown contribution, $c_2(0)$, it cannot be determined. Therefore, a_{22} will remain unknown. From this discussion we conclude that we shall have seven low-energy theorems for the scattering amplitude, expressed in the values of a_{11}, $a_2(0)$, a_{21}, $a_4(0)$, $a_5(0)$, $a_3(0)+a_6(0)$, and $a_7(0)$. The first two of these coefficients have been determined before by Bég[2] and the third one by Singh,[5] by considering a relation for $k'_m k_n E_{mn}^{[\alpha\beta]}$. The rest constitutes the four new second-order low-energy theorems and we shall concentrate only on them.

III. THE LOW-ENERGY THEOREMS

To establish the expression for the new low-energy theorems we have to calculate the right-hand side of Eq. (4) in the Breit frame. We have

$$(2\pi)^{-6}\left[\frac{M^2}{EE'}\right]^{1/2}U_{\mu\nu}^{\alpha\beta}=\frac{\langle\mathbf{p}'\,|\,J_\mu^\alpha\,|\,\mathbf{p}+\mathbf{k}\rangle\langle\mathbf{p}+\mathbf{k}\,|\,J_\nu^\beta\,|\,\mathbf{p}\rangle}{E(\mathbf{p}+\mathbf{k})-E-\omega}+\frac{\langle\mathbf{p}'\,|\,J_\nu^\beta\,|\,\mathbf{p}-\mathbf{k}'\rangle\langle\mathbf{p}-\mathbf{k}'\,|\,J_\mu^\alpha\,|\,\mathbf{p}\rangle}{E(\mathbf{p}-\mathbf{k}')-E'+\omega'} \; , \tag{12}$$

where a summation over the intermediate nucleon spin states is implied and

$$\mathbf{p}'=-\mathbf{p}=\frac{\mathbf{k}-\mathbf{k}'}{2}, \quad \omega'=\omega, \quad E'=E \; . \tag{13}$$

Notice that as $\mathbf{p}+\mathbf{k}=-(\mathbf{p}-\mathbf{k}')$ it follows that $E(\mathbf{p}+\mathbf{k})=E(\mathbf{p}-\mathbf{k}')$.

For the current matrix element we shall use the convenient form

$$\langle\mathbf{p}'\,|\,J_\mu^\alpha\,|\,\mathbf{p}\rangle=\frac{I^\alpha}{(2\pi)^3}\left[\frac{M^2}{EE'}\right]^{1/2}\bar{u}(\mathbf{p}')\left[i[F_1^V(q^2)+F_2^V(q^2)]\gamma_\mu-\frac{F_2^V}{2M}(p'_\mu+p_\mu)\right]u(\mathbf{p}) \; , \tag{14}$$

where $q_\mu=p'_\mu-p_\mu$ and

$$F_1^V(0)=1, \quad F_2^V(0)=\mu^V-1 \; , \tag{15}$$

μ^V being the isovector magnetic dipole moment in units of $1/2M$. Equation (14) is a consequence of the use of the identity

$$i\langle\mathbf{p}'\,|\,\sigma_{\mu\nu}q_\nu\,|\,\mathbf{p}\rangle=\langle\mathbf{p}'\,|\,(p'_\mu+p_\mu)-2Mi\gamma_\mu\,|\,\mathbf{p}\rangle$$

in the usual expression of the current matrix element.

The calculation of the first two terms on the right-hand side of Eq. (4) can be simplified if one makes use of the equation of continuity that permits one to write

$$(p'_m-p_m)\langle\mathbf{p}'\,|\,J_m^\alpha\,|\,\mathbf{p}\rangle=[E(p')-E(p)]\langle\mathbf{p}'\,|\,J_0^\alpha\,|\,\mathbf{p}\rangle \; . \tag{16}$$

Using this result, one has, for the first two terms on the right-hand side of Eq. (4),

$$k'_m U^{\alpha\beta}_{mn} - \omega' U^{\alpha\beta}_{0n} = (2\pi)^6 \left[\frac{EE'}{M^2} \right]^{1/2} (\langle \mathbf{p}' | J^\alpha_0 | \mathbf{p}+\mathbf{k} \rangle \langle \mathbf{p}+\mathbf{k} | J^\beta_n | \mathbf{p} \rangle - \langle \mathbf{p}' | J^\beta_n | \mathbf{p}-\mathbf{k}' \rangle \langle \mathbf{p}-\mathbf{k}' | J^\alpha_0 | \mathbf{p} \rangle) . \quad (17)$$

By straightforward calculations, using also the identities (A5) and (A6) of the Appendix, one obtains as new low-energy theorems

$$a_4(0) = \frac{1}{i} \left[\frac{(2\mu^V - 3)u^V}{16m^3} + \frac{F_1^{V'} + F_2^{V'}}{M} \right] , \quad (18)$$

$$a_5(0) = \frac{1}{16iM^3} (2\mu^V - 1)\mu^V , \quad (19)$$

$$a_3(0) + a_6(0) = \frac{i}{16M^3} (3\mu^V - 2) , \quad (20)$$

$$a_7(0) = i \left[\frac{(2\mu^V - 1)\mu^V}{16M^3} - \frac{(\mu^V - 1)F_1^{V'} - F_2^{V'}}{2M} \right] . \quad (21)$$

There are no corresponding new low-energy theorems for the symmetric part of the amplitude. The reason is that for this part the coefficients of the basis elements $B^{(i)}_{mn}$ for $i=3-7$ in $E^{\{\alpha\beta\}}_{mn}$ are of order ω at least, that is, they start with ωs_{i1}. Therefore, $\Gamma^{\{\alpha\beta\}}_{mn}$ will have to be considered to second order. That means that it will have the same expansion for $E^{[\alpha\beta]}_{mn}$ in Eq. (7) with coefficients r_i instead of a_i. Consequently, in the isospin-symmetric equivalent of Eq. (10), $\omega[s_{i1} - r_i(0)]$ will appear in the place of $a_i(0)$ for $i=3-7$. However, these $r_i(0)$ will not be present on the left-hand side of Eq. (5) due to the fact that the corresponding basis elements are orthogonal to $k'_m k_n$, that is, $k'_m k_n B^{(i)}_{mn} = 0$ for $i=3-7$. Therefore, these $r_i(0)$ will remain unknown and consequently s_{i1} for $i=3-7$ will not be determined by the right-hand side of Eq. (4). No new results show up for $i=1,2$ either.

APPENDIX

By contracting $E^{(8)}_{mn}$ of Pais[4] with $a_m b_n$ for convenience and writing $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{k} = \boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b})\boldsymbol{\sigma}\cdot\mathbf{k} - i\boldsymbol{\sigma}\cdot[(\mathbf{a}\times\mathbf{b})\times\mathbf{k}]$ with a similar relation for $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{k}'$, it is easy to see that

$$\begin{aligned} E^{(8)}_{mn} &= \epsilon_{mnj}(k_j \boldsymbol{\sigma}\cdot\mathbf{k} + k'_j \boldsymbol{\sigma}\cdot\mathbf{k}') \\ &= (\omega^2 + \omega'^2)\epsilon_{mnj}\sigma_j - [k_m(\boldsymbol{\sigma}\times\mathbf{k})_n - k'_n(\boldsymbol{\sigma}\times\mathbf{k}')_m + k'_m(\boldsymbol{\sigma}\times\mathbf{k}')_n - k_n(\boldsymbol{\sigma}\times\mathbf{k})_m] = -B^{(4)}_{mn} - B^{(5)}_{mn} , \end{aligned} \quad (\text{A1})$$

where the B's are the parity- and time-reversal-invariant basis elements used in Eq. (7). Also

$$\begin{aligned} E^{(9)}_{mn} &= \epsilon_{mnj}(k_j \boldsymbol{\sigma}\cdot\mathbf{k}' + k'_j \boldsymbol{\sigma}\cdot\mathbf{k}) \\ &= 2\mathbf{k}\cdot\mathbf{k}'\epsilon_{mnj}\sigma_j - [k_m(\boldsymbol{\sigma}\times\mathbf{k}')_n - k'_n(\boldsymbol{\sigma}\times\mathbf{k})_m + k'_m(\boldsymbol{\sigma}\times\mathbf{k})_n - k_n(\boldsymbol{\sigma}\times\mathbf{k}')_m] = -B^{(6)}_{mn} - B^{(7)}_{mn} . \end{aligned} \quad (\text{A2})$$

From these relations one can detect some misprints in the basis elements $i=10,11$ of Pais. As they are written they are equal to the expressions in square brackets on the right-hand side of Eqs. (A1) and (A2) and, therefore, would not be independent basis elements. Instead, the plus and minus signs in both equations should be interchanged, that is, one should read

$$\begin{aligned} E^{(10)}_{mn} &= k_m(\boldsymbol{\sigma}\times\mathbf{k})_n - k'_m(\boldsymbol{\sigma}\times\mathbf{k}')_n + (m \leftrightarrow n) \\ &= B^{(4)}_{mn} - B^{(5)}_{mn} - (\omega^2 + \omega'^2)B^{(2)}_{mn} \end{aligned} \quad (\text{A3})$$

and

$$\begin{aligned} E^{(11)}_{mn} &= k_m(\boldsymbol{\sigma}\times\mathbf{k}')_n - k'_m(\boldsymbol{\sigma}\times\mathbf{k})_n + (m \leftrightarrow n) \\ &= B^{(6)}_{mn} - B^{(7)}_{mn} + 2\mathbf{k}\cdot\mathbf{k}' B^{(2)}_{mn} . \end{aligned} \quad (\text{A4})$$

Also the T-invariant statement in Eq. (3.14) of Pais should be read

$$E_{mn}(\mathbf{k},\mathbf{k}',\mathbf{S}) = E_{nm}(-\mathbf{k}', -\mathbf{k}, -\mathbf{S})$$

with $-\mathbf{k}', -\mathbf{k}$ instead of $-\mathbf{k}, -\mathbf{k}'$ on the right-hand side.

One more remark. It one chooses to work with the basis element $i=8-11$ of Pais, one must be careful with the fact that after contraction with k'_m the resulting terms such as $(\mathbf{k}\times\mathbf{k}')\boldsymbol{\sigma}\cdot\mathbf{k}$ that comes from $E^{(8)}_{mn}$ and $k_n\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{k})$ that comes from $E^{(10)}_{mn}$ are not linear independent. In fact, from (A1) one has

$$\begin{aligned} (\mathbf{k}\times\mathbf{k}')_n \boldsymbol{\sigma}\cdot\mathbf{k} &= \omega^2(\boldsymbol{\sigma}\times\mathbf{k}')_n - \mathbf{k}'\cdot\mathbf{k}(\boldsymbol{\sigma}\times\mathbf{k})_n \\ &\quad - k_n \boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{k}) . \end{aligned} \quad (\text{A5})$$

Also from (A2) or by interchanging $\mathbf{k}$ and $\mathbf{k}'$ in (A5) one has

$$\begin{aligned} (\mathbf{k}'\times\mathbf{k})_n \boldsymbol{\sigma}\cdot\mathbf{k}' &= \omega'^2(\boldsymbol{\sigma}\times\mathbf{k})_n - \mathbf{k}\cdot\mathbf{k}'(\boldsymbol{\sigma}\times\mathbf{k}')_n \\ &\quad + k'_n \boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{k}) . \end{aligned} \quad (\text{A6})$$

With the basis element that we have chosen the contraction with k'_m automatically gives independent terms, as they are written down in Eq. (10).

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