# Static minimum-energy path from a vacuum to a sphaleron in the Weinberg-Salam model

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In the Weinberg-Salam model with vanishing mixing angle $\theta_W = 0$, the static minimum-energy path is obtained within the general spherically symmetric ansatz, which connects a vacuum to the sphaleron of Klinkhamer and Manton. The physical significance of the path is discussed in connection with a possible baryon-number-violating effect at high temperature.

## I. INTRODUCTION

As first observed by 't Hooft,[1] baryon number is not conserved at the quantum level in the standard Weinberg-Salam model. The divergence of the baryon-number current is in fact nonvanishing because of the presence of an SU(2) anomaly. This baryon nonconservation is basically associated with the instanton physics[2] that describes tunneling transitions between topologically distinct vacua. The probability for these phenomena is exponentially suppressed as $\exp(-4\pi/\alpha_W)$ with $\alpha_W = \alpha/\sin^2\theta_W$ and the proton is in practice stable. However, it has been claimed[3,4] that such a suppression is absent at high temperature $T \sim 100$ GeV, because the thermal fluctuations induce classical transitions passing over the potential barrier between the different vacua. If this is indeed the case, there arise new possibilities[5] for generating the cosmological baryon-number excess without resorting to grand unified theories (GUT's).

The crucial observation is due to Klinkhamer and Manton.[3] They found a static, but unstable solution (called sphaleron) to classical field equations of the Weinberg-Salam model, that corresponds to a saddle point of the potential barrier between two topologically distinct vacua. They also presented a finite-energy path from a vacuum to the sphaleron. However, the detailed structure of the potential between the distinct vacua which seems very useful for calculating the baryon-number-violation rate at high temperature, is not known yet.

The purpose of this paper is to show a static minimum-energy path (SMEP) connecting a vacuum to the sphaleron by making the general spherically symmetric ansatz. The SMEP seems to be useful for understanding adiabatic production and decay of the sphaleron, though further analysis is necessary to estimate its effects on baryon-number nonconservation at high temperature.

In Sec. II we study the static field equations with the general spherically symmetric ansatz. We confirm that the only stationary solution is the sphaleron. In Sec. III we numerically determine the SMEP with use of a Lagrange multiplier that is introduced to fix the topological charge. We also show the potential energy along the SMEP. Finally, in Sec. IV a brief discussion is given.

## II. THE SU(2) SPHALERON IN THE GENERAL SPHERICALLY SYMMETRIC ANSATZ

Klinkhamer and Manton[3] obtained a static saddle-point solution (sphaleron) of the classical field equations of the Weinberg-Salam model in the limit that the weak-mixing angle $\theta_W$ vanishes. The Weinberg-Salam model reduces to an SU(2) theory with a doublet Higgs scalar, since the U(1) gauge field decouples in the field equations. They also argued that the energy functional changes smoothly with $\theta_W$ and hence the introduction of the U(1) sector does not, presumably, affect the sphaleron solution as $\theta_W$ is small. In this paper we focus on the SU(2) field equations taking the limit $\theta_W = 0$. To solve the classical field equations we shall make a general spherically symmetric ansatz for field configurations. We shall show in this section that our saddle-point solution is exactly the same as the sphaleron of Klinkhamer and Manton who used a more restricted form of field configuration.

The field equations for static, classical fields are

$$(D_i F_{ij})^a = \frac{i}{2} g [\phi^\dagger \tau^a (D_j \phi) - (D_j \phi)^\dagger \tau^a \phi] , \tag{1a}$$

$$D_i D_i \phi = 2\lambda(\phi^\dagger \phi - \tfrac{1}{2} v^2) \phi , \tag{1b}$$

with

$$F_{ij}^a = \partial_i W_j^a - \partial_j W_i^a + g \epsilon^{abc} W_i^b W_j^c , \tag{2a}$$

$$D_i \phi = \partial_i \phi - \tfrac{1}{2} i g \tau^a W_i^a \phi . \tag{2b}$$

Here $W_i^a$ and $\phi$ are the SU(2) gauge potential and Higgs scalar, respectively. The static energy functional $E_{\rm stat}$ is given by

$$E_{\rm stat} = \int d^3x [\tfrac{1}{4} F_{ij}^a F_{ij}^a + (D_i \phi)^\dagger (D_i \phi) + \lambda(\phi^\dagger \phi - \tfrac{1}{2} v^2)^2] . \tag{3}$$

The general spherically symmetric ansatz which yields spherically symmetric distributions both of the energy and topological charge is written as

$$W_j^a(x) = \frac{1}{g} [ A(r) \epsilon_{jam} x_m + B(r)(r^2 \delta_{ja} - x_j x_a) + C(r) x_j x_a ] , \tag{4a}$$

 

$$\phi(x)=\frac{v}{\sqrt{2}}\left[H(r)+K(r)\frac{i\tau\cdot\mathbf{r}}{r}\right]\begin{bmatrix}0\\1\end{bmatrix} . \qquad (4b)$$

We can furthermore impose the radial gauge condition

$$x_j W_j^a(x)=0 , \qquad (5)$$

setting $C(r)=0$. For the ansatz (4) with $C(r)=0$, the field equations (1) reduce to a set of differential equations:

$$f_A''-\frac{1}{r^2}(f_A^2+f_B^2-1)f_A=m_W^2[(H^2+K^2)f_A+(K^2-H^2)] , \qquad (6a)$$

$$f_B''-\frac{1}{r^2}(f_A^2+f_B^2-1)f_B=m_W^2[(H^2+K^2)f_B-2HK] , \qquad (6b)$$

$$f_Af_B'-f_A'f_B-(m_Wr)^2(H'K-HK')=0 , \qquad (6c)$$

$$(rH)''-\frac{1}{2r^2}[(f_A-1)^2+f_B^2](rH)+\frac{1}{r^2}f_B(rK)=\tfrac{1}{2}m_H^2(H^2+K^2-1)(rH) , \qquad (6d)$$

$$(rK)''-\frac{1}{2r^2}[(f_A-1)^2+f_B^2](rK)-\frac{2}{r^2}f_A(rK)+\frac{1}{r^2}f_B(rH)=\tfrac{1}{2}m_H^2(H^2+K^2-1)(rK) , \qquad (6e)$$

where

$$f_A=1+r^2A,\quad f_B=r^3B , \qquad (7)$$

and $m_W$ and $m_H$ denote the gauge-boson mass $m_W=gv/2$ and the Higgs-boson mass $m_H=\sqrt{2\lambda}v$, and $f'\equiv df/dr$. The static energy functional (3) now becomes

$$E_{\text{stat}}=\frac{4\pi}{g^2}\int_0^\infty dr\left[f_A'^2+f_B'^2+\frac{1}{2r^2}(f_A^2+f_B^2-1)^2+2m_W^2[\,r^2(H'^2+K'^2)+\tfrac{1}{2}(H^2+K^2)(f_A^2+f_B^2+1)\right.$$
$$\left.-(H^2-K^2)f_A-2HKf_b+\tfrac{1}{4}m_H^2r^2(H^2+K^2-1)^2]\right] . \qquad (8)$$

Notice that all the equations in (6) are not independent. In fact, the left-hand side of Eq. (6c) is an integration constant of the other equations and it turns out to vanish by the following boundary condition at $r=\infty$:

$$\tau^aW_j^a(x)\to-\frac{2i}{g}\partial_jUU^{-1} ,$$
$$\phi(x)\to\frac{v}{\sqrt{2}}U\begin{bmatrix}0\\1\end{bmatrix} , \qquad (9)$$

where

$$U=\exp\left[\frac{i}{2}q\frac{\tau\cdot\mathbf{r}}{r}\right] . \qquad (10)$$

Here, $q$ is a free parameter varying from 0 to $2\pi$. This boundary condition (9) yields the asymptotic behavior

$$f_A\approx\cos q+(d_A\cos q-d_B\sin q)\exp(-m_Wr) ,$$
$$f_B\approx\sin q+(d_A\sin q+d_B\cos q)\exp(-m_Wr) ,$$
$$H\approx\cos\frac{q}{2}\left[1+\frac{d_H}{m_Wr}\exp(-m_Hr)\right]+\sin\frac{q}{2}\frac{d_B}{m_W^2r^2}\exp(-m_Wr) , \qquad (11)$$
$$K\approx\sin\frac{q}{2}\left[1+\frac{d_H}{m_Wr}\exp(-m_Hr)\right]-\cos\frac{q}{2}\frac{d_B}{m_W^2r^2}\exp(-m_Wr) .$$

On the other hand, the regularity condition on the functions $A$, $B$, $H$, and $K$ at the origin ($r=0$) requires, near $r=0$,

$$f_A\approx1+c_A(m_Wr)^2 ,$$
$$f_B\approx-\tfrac{1}{3}c_Hc_K(m_Wr)^3 , \qquad (12)$$
$$H\approx c_H+\tfrac{1}{12}c_H(c_H^2-1)(m_Hr)^2 ,$$
$$K\approx c_K(m_Wr) .$$

$c_A$, $c_H$, $c_K$, $d_A$, $d_B$, and $d_H$ are constants of order unity that can be numerically determined by solving Eqs. (6).

We searched for solutions of Eq. (6) numerically with each fixed value of $q$. We found indeed no solution in the full region of $q=0$ to $2\pi$ except for the sphaleron and vacua. Our sphaleron solution was seen at $q=\pi$. As shown in Fig. 1 our solution is exactly the same as that of Klinkhamer and Manton.[3] Their solution $f(\xi)$ and $h(\xi)$ in Ref. 3 corresponds to

$$f_A(r)=1-2f(\xi) ,$$
$$f_B(r)=H(r)=0 , \qquad (13)$$
$$K(r)=h(\xi) ,$$

where $\xi\equiv2m_Wr$. It should be stressed that we recovered the sphaleron solution in the SU(2) theory by making a more general ansatz (4).

The topological baryon number $Q_B$ in the unitary gauge is defined as

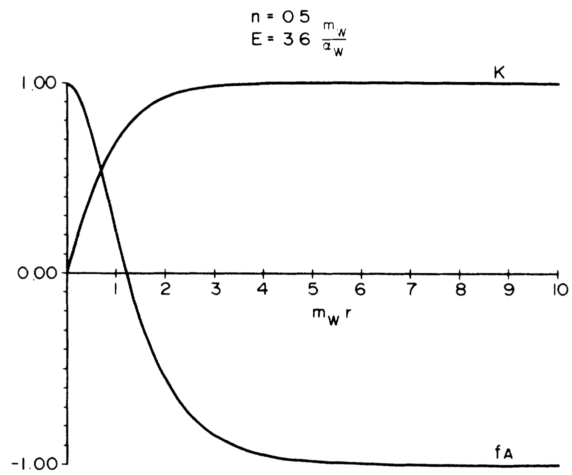


FIG. 1. The functions $f_A(r)$ and $K(r)$ for the sphaleron. The other functions $f_B(r)$ and $H(r)$ vanish identically.

$$Q_B = \frac{g^2}{32\pi^2}\int d^3x\, K^0 = \frac{g^2}{16\pi^2}\int d^3x\, \epsilon^{ijk}\left[\partial_i W_j^a W_k^a + \frac{g}{3}\epsilon^{abc} W_i^a W_j^b W_k^c\right] . \tag{14}$$

The topological current

$$K^\mu = -\frac{g^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}(F_{\nu\rho}^a W_\sigma^a - \tfrac{1}{3} g \epsilon^{abc} W_\nu^a W_\rho^b W_\sigma^c) \tag{15}$$

is the quantity whose divergence is $\partial_\mu K^\mu = -(g^2/32\pi^2)F_{\mu\nu}^a \tilde{F}^{a\mu\nu}$. In the radial gauge the topological baryon number is not simply given by $Q_B$ because of the presence of a nonvanishing surface term $\int d\mathbf{S}\cdot\mathbf{K}$, but rather by the quantity

$$\tilde{Q}_B = Q_B + \frac{q - \sin q}{2\pi} . \tag{16}$$

Here we have defined the topological baryon number $\tilde{Q}_B$ as a difference from that of a reference vacuum, say, with $q = 0$. This $\tilde{Q}_B$ is a gauge-invariant quantity and the actual baryon-number violation is related to $n_f \tilde{Q}_B$, where $n_f$ denotes the family number.

For the sphaleron configuration ($q = \pi$) we easily find $Q_B = 0$ and hence its topological baryon number is

$$\tilde{Q}_B(\text{sphaleron}) = \tfrac{1}{2} , \tag{17}$$

while $\tilde{Q}_B(\text{vacua}) = \text{integers}$.

## III. A STATIC MINIMUM-ENERGY PATH FROM A VACUUM TO THE SPHALERON

In the previous section, we presented the static unstable solution, i.e., the sphaleron. However, the energy functional $E_{\text{stat}}$ between a vacuum and the sphaleron is not yet clear to us. We shall find, in this section, the field configurations which minimize the energy functional $E_{\text{stat}}$ under the constraint $\tilde{Q}_B = n$, and show the $n$ dependence of the minimum energy $E(n)$ from the vacuum ($n = 0$) to the sphaleron ($n = \frac{1}{2}$).

We are able to derive $E(n)$ by finding a stationary configuration of the following functional:[6]

$$W[\psi] = E_{\text{stat}}[\psi] + \eta \tilde{Q}_B[\psi] , \tag{18}$$

where $\psi$ denotes a set of functions $f_A(r)$, $f_B(r)$, $H(r)$, and $K(r)$, and $\eta$ is a Lagrange multiplier. We restrict field configurations to the same ones as given in Eq. (4).

The variational procedure for $W[\psi]$ in Eq. (18) leads to a set of equations for $f_A(r)$, $f_B(r)$, $H(r)$, and $K(r)$. Equations (6a) and (6b) are modified as

$$f_A'' - \frac{1}{r^2}(f_A^2 + f_B^2 - 1)f_A = m_W^2[(H^2+K^2)f_A + (K^2 - H^2)] - \zeta m_W f_B' , \tag{19a}$$

$$f_B'' - \frac{1}{r^2}(f_A^2 + f_B^2 - 1)f_B = m_W^2[(H^2+K^2)f_B - 2HK] + \zeta m_W f_A' , \tag{19b}$$

where $\zeta = \alpha_W \eta / 2\pi m_W$. The boundary conditions at $r = 0$ are given by Eq. (12) with a modification

$$f_B = \tfrac{1}{3}(-c_H c_K + \zeta c_A)(m_W r)^3 . \tag{20}$$

On the other hand, the asymptotic conditions at $r = \infty$ are as follows:

$$\begin{aligned} f_A &= \cos q + \text{Re}[(d_A \cos q - d_B \sin q)\exp(-\alpha r)] , \\ f_B &= \sin q + \text{Re}[(d_A \sin q + d_B \cos q)\exp(-\alpha r)] , \\ H &= \cos\frac{q}{2}\left[1 + \frac{d_H}{m_W r}\exp(-m_H r)\right] + \sin\frac{q}{2}\text{Re}\left[\frac{d_B}{\alpha^2 r^2}\exp(-\alpha r)\right] , \\ K &= \sin\frac{q}{2}\left[1 + \frac{d_H}{m_W r}\exp(-m_H r)\right] - \cos\frac{q}{2}\text{Re}\left[\frac{d_B}{\alpha^2 r^2}\exp(-\alpha r)\right] , \end{aligned} \tag{21}$$

where $d_H$ is real, while $d_A$, $d_B$, and $\alpha$ are complex numbers with the condition $d_B = m_W \alpha \zeta d_A / (m_W^2 - \alpha^2)$. The parameter $\alpha$ is determined as a solution having a positive real part from the equation

$$(\alpha^2 - m_W^2)^2 + m_W^2 \zeta^2 \alpha^2 = 0 . \tag{22}$$

We can see that if $|\eta|$ is larger than a critical value $\eta_c$, where $\eta_c = 4\pi m_W / \alpha_W$, the parameter $\alpha$ from Eq. (22) is pure imaginary. In this case there is no stationary solution $\psi$ which makes $W[\psi]$ convergent. Thus we restrict $\eta$ to the region $|\eta| < \eta_c$.

For a given $\eta$ (or equivalently a given topological baryon number $\tilde{Q}_B = n$) these equations can be solved nu-

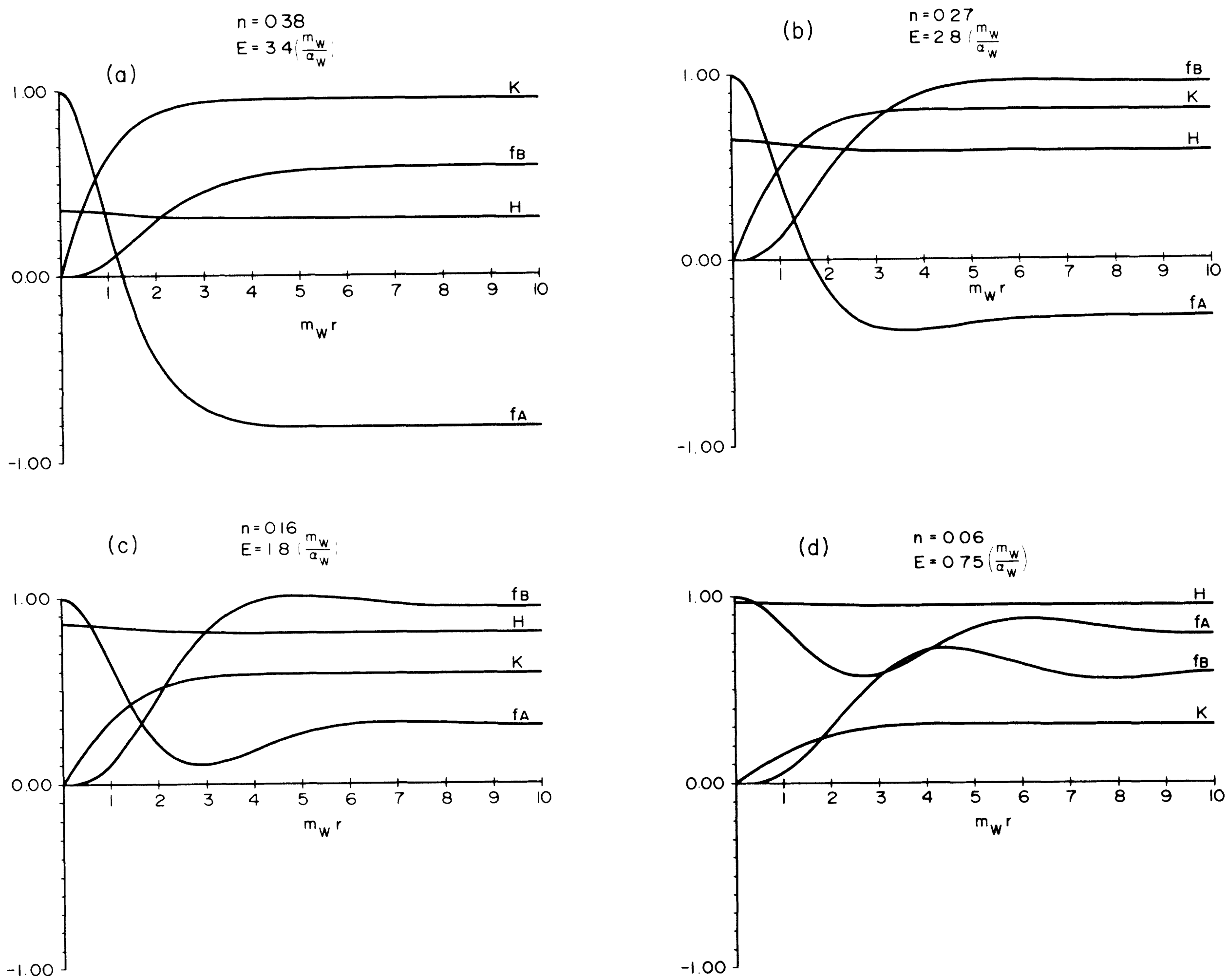


FIG. 2. The functions $f_A(r)$, $f_B(r)$, $H(r)$, and $K(r)$ of stationary solutions for several values of $n$.

merically by adjusting the boundary parameter $q$. In Fig. 2 we show the solutions for several values of $n$, where the Higgs-boson mass $m_H$ is tentatively chosen as $m_H = m_W$. The energy distributions for $n = 0.5$, 0.27, and 0.06 are shown in Fig. 3.

In Fig. 4 we show $E(n)$ vs $n$. It should be noted that $E(n)$ can be regarded as an effective potential and any stationary solution of $E_{\text{stat}}[\psi]$ corresponds to a zero point of $\partial E(n)/\partial n$. In order to show this, let us substitute the stationary solutions of $W[\psi]$ into Eq. (18). We have then

$$W(\eta) = E_{\text{stat}}(\eta) + \eta \tilde{Q}_B(\eta) , \tag{23}$$

which yields

$$\frac{dW}{d\eta} = \tilde{Q}_B(\eta) \equiv n . \tag{24}$$

Using $n$ as a variable instead of $\eta$, we can define $E(n)$ by the Legendre transform

$$E(n) = W(\eta) - \eta n . \tag{25}$$

From Eqs. (24) and (25) we obtain

$$\frac{dE}{dn} = -\eta , \tag{26}$$

which is the desired result.

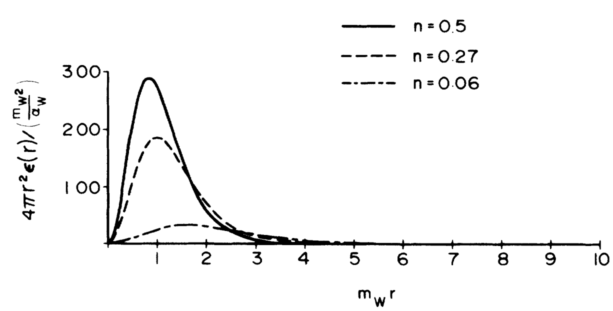


FIG. 3. The energy density $4\pi r^2 \epsilon(r)$ which is related to the total energy as $E = \int dr[4\pi r^2 \epsilon(r)]$ is shown for different values of $n$. Solid, dashed, and dashed-dotted lines correspond to $n = 0.5$, 0.27, and 0.06, respectively.

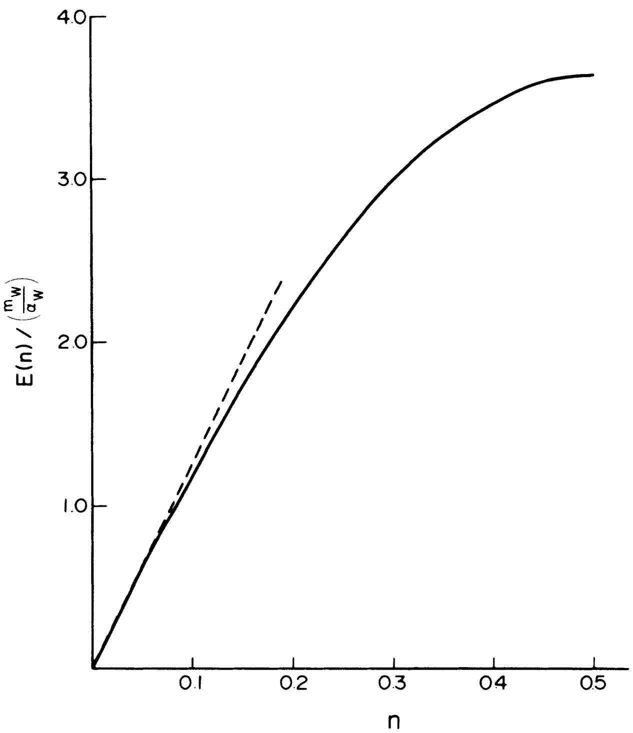


FIG. 4. The minimum static energy $E(n)$ for fixed $n$ is plotted as a solid line. The dashed line represents the critical slope $dE(n)/dn = 4\pi m_W/\alpha_W$. Note that $E(n)$ seems to approach to zero with this slope.

Figure 4 clearly shows that no stationary solution of $E_{\text{stat}}[\psi]$ exists besides vacua and the sphaleron within the general spherically symmetric ansatz. The sphaleron is smoothly connected to a vacuum and indeed there is no singularity like $dE/dn = \infty$ on the path. We remark that numerically $dE/dn$ tends to $\eta_c$ as $n$ approaches 0 from a positive value.

## IV. DISCUSSION

A description of the baryon-nonconservation process at high temperature may be given with classical statistical mechanics.[4,7] Transitions with $\tilde{Q}_B$ changing proceed along various paths in field configuration space, which start from a neighborhood of a vacuum, say, of $\tilde{Q}_B = 0$, concentrate at a sphaleron, and finally arrive at a neighborhood of another vacuum, say, of $\tilde{Q}_B = \pm 1$. It is argued that near the sphaleron most of transitions take place along a particular direction characterized by a negative eigenvalue of the second derivative $\delta^2 E_{\text{stat}}/\delta\psi\delta\psi$. Unfortunately, the eigenmodes, especially, the unstable mode has not been obtained yet.

On the other hand, in our description in terms of the SMEP the unstable mode near the sphaleron is approximately determined. The characteristic decay frequency $\omega_-$ of the sphaleron in the overdamped case[7] is almost given by $(\partial^2 E/\partial n^2)_{n=1/2}$ with some proportionality constant which depends on detailed dynamics. The frequency $\omega_-$ is an important factor to estimate the rate of the baryon nonconservation process. From Fig. 3 we obtain $(\partial^2 E/\partial n^2)_{n=1/2} = -34 m_W/\alpha_W$. This should be compared to the corresponding value estimated along the Manton path for the case of $m_H = 0$:

$$(\partial^2 E/\partial n^2)_{n=1/2} = (\pi/2)^2 (\partial^2 E/\partial^2 \mu)_{\mu=\pi/2}$$
$$= -17 m_W/\alpha_W \, .$$

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