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Mechanisms of spontaneous symmetry breaking in the fermionic construction of superstring models

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Mechanisms for spontaneous gauge symmetry breaking are presented in the context of the four-dimensional superstring models. Some phenomenologically interesting models are constructed to highlight the simplicity of the procedure.

I. INTRODUCTION

Recent interest in superstring theories was generated by the discovery[1] that these theories which unify gravity with other interactions may be finite and are free of gauge and gravitational anomalies.[2] This interest was further boosted by the construction of ten-dimensional heterotic superstring theories[3] which initiated the search for all the superstring solutions that may reproduce a realistic low-energy phenomenology. Since then, many different ways of constructing superstring solutions directly in four dimensions have been developed.[4–7] Among them, the one we would like to develop further is the free fermionic construction.[5–7] In this approach, the extra two-dimensional (world-sheet) degrees of freedom that are needed for the cancellations of conformal anomalies are free fermions instead of the usual bosonic coordinates interpreted as the position of the string in higher dimensions. For an interacting string theory at higher loop level, the world sheet can be represented by a Riemann surface of genus $g \geq 1$. While the bosonic field on the world sheet has to be periodic when it is translated around one of the nontrivial (noncontractible) loops, the complex fermionic field can change by a phase. The set of these phases associated with all possible loops is the boundary condition for the fermions. Besides the local conformal invariance, at the loop levels, the conformal reparametrization invariance has an extra discrete (or global) subgroup that has to be checked separately; it is called the modular transformation (MT). A particular boundary condition is typically not invariant under the modular transformation. Thus, modular invariance immediately implies that one has to use a set of boundary conditions for each fermion simultaneously. This set is called the spin structure. Therefore, to define a superstring solution, one does not only have to write down a two-dimensional conformal field theory for the fermions and bosons, it is also necessary to specify the spin structure. Each spin structure which respects conformal or superconformal invariance and modular invariance defines a superstring solution. General solutions for the constraints that these self-consistency conditions impose on the spin structure were worked out not long ago.[5,6] Many models with interesting gauge groups and chiral fermions were also demonstrated.[5]

The massless modes of these string models describe a gauge theory coupled to gravity or supergravity at the Planck scale. The hope is that the usual field-theory techniques,[8] such as the renormalization-group equation and the Higgs mechanism can be used for the symmetry breaking and the mass generation in order to obtain realistic low-energy theories. While it is certainly not an easy task to obtain the realistic low-energy phenomenology out of a supergravity theory in four dimensions, the procedure itself assumes the decoupling of the infinite number of massive particles in the string theory which may not be justified in certain cases. In any case, it is interesting to explore other means of constructing a consistent superstring solution which is more akin to the low-energy observations.

In this paper we explore new ways of constructing superstring solutions using the Thirring interactions between the fermions and the generalized Scherk-Schwarz (SS) mechanism. It was well known that the two-dimensional fermions with the Thirring interactions can define a conformal field theory. It was demonstrated[9] that a fermionic theory with properly defined Thirring interactions can reproduce the partition function (PF) of a generalized torus defined by a simply laced, self-dual, Lorentzian lattice. It also has been used as a mechanism of breaking the gauge symmetry at a scale different from the Planck mass.[10]

Scherk and Schwarz (SS) proposed a particular compactification scheme using a higher-dimensional supergravity which breaks the supersymmetry spontaneously in the dimensional-reduction process.[11] An explicitly stringy generalization of this scheme was proposed by Ferrara, Kounnas, and Porrati[12] (FKP). In the SS mechanism, the compactification scheme depends on the conserved charges of the higher-dimensional theory. As a result the symmetry is broken after the compactification. A stringy generalization of such a mechanism can be obtained by coupling the modes of the PF for the compactified bosonic coordinate to the charges relating to the other fermionic coordinates.[12,13] FKP start with a five-dimensional fermionic construction and then compactify one of the dimensions to a circle of arbitrary radius R. They realized that in a proper mode expansion (see Sec. III) of this PF in terms of the winding number and the momentum, the individual modes can have a simple modular transformation property. Therefore, it is possible to provide charge-dependent mixing of the PF's of the other fermionic coordinates with this compactified bosonic one. The modular invariance holds irrespective of the amount of mixing. These SS mixing parameters, called e^i, are arbitrary parameters and therefore provide a new scale and also introduce gauge symmetry breaking into the theory.

In a previous paper,[10] we generalized the idea of FKP even further and abandoned the geometric picture of a higher-dimensional theory altogether. The SS mechanism can be introduced into a four-dimensional free fermionic theory directly as a modular-invariant mixing between the partition functions associated with different fermions. This mechanism combined with the Thirring interactions between some of the fermions provides a powerful tool for model building.[10] In this paper we present the method in detail and illustrate the symmetry-breaking mechanism through explicit examples. Mathematical formulas referred to in Ref. 10 are worked out explicitly and proofs of many statements made previously are also included. The generalization was motivated by the observation that if two left- and two right-moving fermions have the same boundary condition, then one can expand the PF in a particular way such that the individual modes transform in a very simple way under the modular transformation. In this case the individual modes correspond to the charge sectors of the fermionic theory. These fermions will be called "decomposed" fermions. Once these mode expansions are identified for each boundary condition, one can easily follow FKP's construction to introduce the SS mixing into each PF characterized by the different boundary condition for these decomposed fermions. The mixing parameters e^i now denote the SS mixing between the PF of these decomposed fermions with other fermionic coordinates. FKP's result (for $R=1$) is obtained as a special case in this approach when the decomposed fermion is chosen such that the total PF factorizes into the PF of the decomposed fermion and other fermionic and bosonic coordinates. The PF of the decomposed fermion will be that of the torus with radius one in this special case. The parameters corresponding to the radius R are introduced in the fermionic picture through the Thirring interaction with the coupling $2g=R-1/R$ among the decomposed fermions.[9]

A short review of the free fermionic constructions is given in Sec. II, since it is also the starting point in our construction. Then the symmetry breaking can be introduced through a mixture of Thirring interaction and SS mixing. First, we concentrate on the models using only a SS mixing mechanism and the possibilities of spontaneous symmetry breaking in the free fermionic constructions are investigated. For this purpose the construction of the Hilbert space in the free fermionic models are studied in Sec. III. Two equivalent ways of constructing the states are reviewed. The states obtained by the operation of the current creation operators on the vacuum states in various charge sectors are more convenient for writing the PF of the decomposed fermions. In Sec. IV, the SS mixing is introduced in the context of the free fermionic models. This mechanism can easily be used to break the gauge symmetries without the tachyon-generation problem. In Sec. V it is found that the modular-transformation property for the individual modes of Thirring fermions remain the same as those for free fermions. However, the world-sheet supersymmetry together with modular invariance puts some restrictions on when the Thirring interaction can be introduced. In particular, Thirring interaction cannot be introduced in a model when the starting point is a free fermionic $N=1$ superstring model. The use of Thirring interaction as a gauge-symmetry-breaking tool is also sketched. In Sec. VI the ideas of the SS mixing and Thirring interactions are combined. Conclusions and some critical discussions are presented in Sec. VII. Some technical details are presented in Appendixes A–D.

We would like to point out that the Scherk-Schwarz mechanism presented here can be reformulated as a Thirring interaction[9] among world-sheet fermions. However, separate treatment of the two in this paper makes the connection with Refs. 12 and 13 more transparent.

II. FREE FERMIONIC CONSTRUCTIONS

In this section the free fermionic construction of four-dimensional superstring solutions without spontaneous symmetry breaking is briefly reviewed. We restrict our description to the light-cone gauge and to the heterotic string models.[3,5–7] In this gauge the four-dimensional superstring models are constructed by taking left-moving bosonic coordinates X_L^i $(i=1,2)$ and fermionic coordinates ψ_L^i $(i=1,2)$, α^I, β^I, and γ^I $(I=1,\ldots,6)$ and right-moving bosonic coordinates X_R^i $(i=1,2)$ and fermionic coordinates ϕ^a $(a=1,\ldots,44)$. The left-moving part has an underlying world-sheet supersymmetry provided the fermions α^I, β^I, and γ^I are in the adjoint representation of a Lie group. In the present case all the fermions are being taken as real and it is assumed that the fermions α^I, β^I, and γ^I (for each I) are in the adjoint representation of a SO(3) (Refs. 5 and 6). The superconformal algebra in the left-moving sector is generated by the Virasoro generator

$$T_B \equiv \frac{1}{2}\left[\sum_{i=1}^{2}\partial X_L^i \partial X_L^i+\sum_{i=1}^{2}\psi_L^i\partial\psi_L^i+\sum_{I=1}^{6}\alpha^I\partial\alpha^I+\sum_{I=1}^{6}\beta^I\partial\beta^I+\sum_{I=1}^{6}\gamma^I\partial\gamma^I\right], \tag{2.1a}$$

and the super-Virasoro generator

$$T_F=\frac{1}{2}\left[\sum_{i=1}^{2}\partial X_L^i\psi_L^i+\sum_{I=1}^{6}\alpha^I\beta^I\gamma^I\right]. \tag{2.1b}$$

The conformal algebra in the right-moving sector is generated by

$$\bar{T}_B \equiv \frac{1}{2}\left[\sum_{i=1}^{2}\bar{\partial}X_R^i\,\bar{\partial}X_R^i+\sum_{\alpha=1}^{44}\phi^\alpha\,\bar{\partial}\phi^\alpha\right], \tag{2.2}$$

where ∂ and $\bar{\partial}$ denote the partial derivatives in the light-cone coordinates. Modular invariance of the one-loop path integral puts restrictions on the way the boundary conditions on the fermions are chosen and on the relative coefficient between different terms. A boundary condition on the fermions in either the σ_1 or σ_2 direction of the torus can be specified by the sets α of fermions that are periodic in this boundary condition. The modular invariance requires that one uses a collection of boundary conditions, that is, a collection $\Xi=\{\alpha^i\}$ of sets which we shall call spin structure. The sets in Ξ must form a group under the following product of sets:[6]

$$\alpha\cdot\beta=\alpha\cup\beta-\alpha\cap\beta \; . \tag{2.3}$$

As a result, one can generate Ξ using a set of basis $\{b_0,b_1,\ldots,b_n\}$ when Ξ contains 2^{N+1} elements. Since $\alpha\cdot\alpha=\varnothing$ for every set, the empty set $\varnothing$ is in Ξ. Modular invariance also requires the set F of all fermions to be contained in Ξ and one can always adopt it as one of the basis. We shall label $b_0\equiv F$. The choice of basis are constrained first by the world-sheet supercurrent which requires the two terms in Eq. (2.1b) to have the same boundary condition for any b_i; and then by the modular invariance which requires that the numbers $n(b)$ $[=n_L(b)-n_R(b)]$ in each basis have to satisfy

$$\begin{aligned} &n(b_i)=0 \pmod 8 \; , \\ &n(b_i\cap b_j)=0 \pmod 4 \; , \\ &n(b_i\cap b_j\cap b_k\cap b_l)=0 \pmod 2 \; . \end{aligned} \tag{2.4}$$

The contribution of the world-sheet fermions to the PF can be written as

$$Z=\sum_{\alpha,\beta\in\Xi} C(\alpha\,|\,\beta)Z(\alpha\,|\,\beta) \; , \tag{2.5}$$

where α and β are the sets specifying the boundary conditions of the world-sheet fermions in the σ_1 and σ_2 directions, respectively. Modular invariance restricts the coefficients $C(\alpha\,|\,\beta)$ to be ±1. Once the coefficients $C(b^i\,|\,b^j)$ for $i>j$ and $C(F\,|\,F)$ are specified, all the other $C(\alpha\,|\,\beta)$'s for $\alpha,\beta\in\Xi$ are determined by the modular invariance of the theory. Therefore, to build a solution one first picks a basis for Ξ which is consistent with the world-sheet supercurrent and Eqs. (2.4). Then, each choice of coefficients $C(F\,|\,F)$ and $C(b_i\,|\,b_j)$, $i>j$, to be ±1 generates a solution. If Ξ is generated by $N+1$ basis elements, there will be $2^{N(N+1)/2+1}$ solutions associated with the basis. However, not all the solutions generated this way are independent. For example, it is interesting to note that in ten dimensions, the condition stated above plus the existence of a space-time gravitino actually restrict both the bosonic and the fermionic PF of the model to be unique and independent of how they are constructed.[14] The simplest four-dimensional model is generated by a single basis element F. This model has SO(44) gauge invariance and is tachyonic. A superstring model with $N=4$ supersymmetry and SO(44) gauge symmetry is obtained by taking two basis elements F and S, where S is a set of eight left-moving fermions only, including ψ_L^i. In fact, the existence of space-time supersymmetry requires, in addition to the existence of S, that $C(s\,|\,\alpha)=-1$ for every α in Ξ that is disjoint to S. We will always take $S=\{\psi^i(i=1,2),\alpha^I(I=1,\ldots,6)\}$.

The relation that the two fermions share the same boundary conditions for every set in Ξ is an equivalence relation. This relation divides the set F into subsets called the minimal intersections. For example, in the $N=4$ model specified above there are two minimal intersections: $\Omega_1=\{\psi^i,\alpha^I\}$, $\Omega_2=\{\beta^I,\gamma^I,\phi^a(a=1,\ldots,44)\}$. The minimal intersections are helpful in understanding the gauge groups and the spectrum of the models.

III. CONSTRUCTION OF HILBERT SPACE IN A FERMIONIC THEORY

A. Partition function of a fermion

This section is devoted to a general discussion about the construction of Hilbert space in a two-dimensional free fermionic theory. For the purpose of discussions, we shall consider fermions which can have complex boundary conditions in this section and restrict ourselves to the real fermions in the later sections.

Let us consider the propagation of a free left-moving Weyl (complex) fermion on a toroidal world sheet. It picks up a phase while going around a loop in σ_1 or σ_2 directions of the torus:

$$\begin{aligned} &\psi_2(\sigma_1+2\pi,\sigma_2)=e^{i\pi v_L}\psi_2(\sigma_1,\sigma_2) \; , \\ &\psi_2(\sigma_1,\sigma_2+2\pi)=e^{i\pi u_L}\psi_2(\sigma_1,\sigma_2) \; . \end{aligned} \tag{3.1}$$

The partition function of this fermion is

$$\begin{bmatrix} v_L \\ u_L \end{bmatrix} \equiv \mathrm{Tr}(e^{2\pi i\tau H_v}e^{-\pi i u_L \hat{N}_v}) \; , \tag{3.2}$$

where H_v and $\hat{N}_v$ are, respectively, the Hamiltonian and the fermion number operators. The trace can be calculated by summing over a Hilbert space created by the modes of the fermions to give

$$\begin{bmatrix} v_L \\ u_L \end{bmatrix} = p^{(v_L^2-1/3)/4}\prod_{n=1}^{\infty}(1+p^{2n-1+v_L}e^{+i\pi u_L}) \times(1+p^{2n-1-v_L}e^{-i\pi u_L}) \; , \tag{3.3}$$

where $p\equiv e^{i\pi\tau}$. Using the Jacobi triple-product identity

$$\sum_{k\in Z} p^{(k+v/2)^2}t^k=\eta(\tau)p^{(v^2-1/3)/4} \times\prod_{n=1}^{\infty}(1+t^{-1}p^{2n-1+v})(1+tp^{2n-1-v}) \; , \tag{3.4}$$

Eq. (3.3) can be rewritten as

$$\begin{bmatrix} v_L \\ u_L \end{bmatrix} = \frac{1}{\eta(\tau)}\sum_{n_L=-\infty}^{+\infty} p^{(n_L+v_L/2)^2}e^{-i\pi n_L u_L} \; , \tag{3.5}$$

where the Dedekin eta function $\eta(\tau)$ is given by

$$\eta(\tau)=p^{1/12}\prod_{n=1}^{\infty}(1-p^{2n}) . \quad (3.6)$$

There is a simple physical interpretation for Eq. (3.5) given by an alternate formulation of this free fermionic theory. Consider the Sugawara-type energy-momentum tensor

$$T_L \equiv \tfrac{1}{2}:J^L J^L: \quad (3.7)$$

and current

$$J^L \equiv :\psi_L^* \psi_L: , \quad (3.8)$$

where the normal ordering in Eq. (3.7) is defined with respect to the modes of the current,

$$J^L(z)\equiv \sum_m J_m^L z^{-m-1} , \quad (3.9)$$

where $z\equiv e^{\sigma_2+i\sigma_1}$. These modes satisfy the algebra $[J_m^L,J_n^L]=m\delta_{m+n}$. The transformation of the fermion field under the U(1)-symmetry group can be written as a commutation relation:

$$[J^L(z_1),\psi_L(z_2)]=\psi_L(z_2)\delta(z_1-z_2) . \quad (3.10)$$

The vacuum state is defined with respect to the modes of the current:

$$J_m^L |0\rangle=0,\quad m>0 , \qquad Q^L|0\rangle=q^L|0\rangle , \quad (3.11)$$

where $Q^L\equiv J_0^L$ will be called the charge operator and q^L is the vacuum charge. Higher excitations are created by the modes of the current. A generic state can be written as

$$|\{\lambda\}\rangle \equiv \prod_m (J_{-m}^L)^{\lambda_m}|0\rangle . \quad (3.12)$$

The state in Eq. (3.12) constitutes only one of the "charge sectors" of the theory. Equation (3.10) implies that by operating ψ_L or ψ_L^* n_L times on the vacuum defined in Eq. (3.11), we create a different charge sector with charge eigenvalues q^L+n_L or q^L-n_L, respectively. The Hamiltonian for the system is the zero mode of the stress-energy tensor in Eq. (3.7), and can be written as

$$H_L=\tfrac{1}{2}Q^2+N_L , \quad (3.13)$$

where $N_L=\sum_{n>0}J_{-n}J_n$ is a number operator. It can be shown that the vacuum charge q^L of a fermion with boundary condition (v_L,u_L) in (σ_1,σ_2) directions is given as

$$q^L=\frac{v_L}{2} . \quad (3.14)$$

Therefore, the Hamiltonian (3.13) has eigenvalues

$$E_L=\frac{1}{2}\left[\frac{v_L}{2}+n_L\right]^2+N_L^0-\tfrac{1}{24} , \quad (3.15)$$

where N_L^0 is the eigenvalue of the number operator N_L and $-\frac{1}{24}$ is the vacuum energy. The PF in Eq. (3.5) can now be rewritten as

$$\begin{bmatrix} v_L \\ u_L \end{bmatrix} = \mathrm{Tr}(e^{2\pi i\tau H_L-\pi i u_L Q_L})e^{+(\pi i/2)u_L v_L} = (\mathrm{Tr}e^{2\pi i\tau(N_L-1/24)}) \times\left[\sum_{n_L=-\infty}^{+\infty} e^{\pi i\tau(n_L+v_L/2)^2}e^{-\pi i u_L n_L}\right] . \quad (3.16)$$

However, in this language, the states are labeled by the quantum numbers n_L and N_L^0 and the vacuum charge. Similarly the PF of a right-moving fermion can be written as

$$\begin{bmatrix} v_R \\ u_R \end{bmatrix} = \frac{1}{\bar\eta(\bar\tau)}\sum_{n_R=-\infty}^{+\infty}\bar p^{(n_R+v_R/2)^2}e^{\pi i n_R u_R} . \quad (3.17)$$

B. Mode expansion of the partition function

If the boundary conditions of the left- and the right-moving fermions are identical, i.e., $v_L=v_R=v$ and $u_L=u_R=u$, then these Weyl fermions can be combined to form a Dirac fermion. In this section some properties of the PF of such a fermion are studied. The PF of a Dirac fermion is a product of left- and right-moving Weyl fermions:

$$Z^T\equiv\left|\begin{matrix} v \\ u\end{matrix}\right|^2\equiv\begin{bmatrix} v \\ u\end{bmatrix}\begin{bmatrix} \bar v \\ u\end{bmatrix} . \quad (3.18)$$

Using Eqs. (3.16) and (3.17) this PF can be written as

$$\left|\begin{matrix} 0 \\ 0\end{matrix}\right|^2=\frac{1}{|\eta|^2}\sum_{n_L,n_R}p^{n_L^2}\bar p^{n_R^2} , \quad (3.19a)$$

$$\left|\begin{matrix} 0 \\ 1\end{matrix}\right|^2=\frac{1}{|\eta|^2}\sum_{n_L,n_R}p^{n_L^2}\bar p^{n_R^2}(-)^{n_L+n_R} , \quad (3.19b)$$

$$\left|\begin{matrix} 1 \\ 0\end{matrix}\right|^2=\frac{1}{|\eta|^2}\sum_{n_L,n_R}p^{(n_L+1/2)}\bar p^{(n_L+1/2)^2} , \quad (3.19c)$$

$$\left|\begin{matrix} 1 \\ 1\end{matrix}\right|^2=\frac{1}{|\eta|^2}\sum_{n_L,n_R}p^{(n_L+1/2)}\bar p^{(n_L+1/2)^2}(-)^{n_L+n_R} = 0 . \quad (3.19d)$$

The left-hand sides (LHS's) of Eqs. (3.19) have the usual transformation property under modular transformations. But the individual terms in the summation do not transform in any simple way under $\tau\to-1/\tau$. However, as shown in Appendix A, writing $\tau=\tau_r+i\tau_i$ and using Poisson resummation formula, Eqs. (3.19) can be rewritten as

$$\left|\begin{matrix} 0 \\ 0\end{matrix}\right|^2=\frac{1}{|\eta|^2}\frac{1}{\sqrt{2\tau_i}}\sum_{m,n}(-)^{mn}\exp\left[-\frac{\pi}{2\tau_i}(n^2+|\tau|^2m^2-2mn\tau_r)\right] \equiv\sum_{m,n}Z_{n,m}^{(0,0)} , \quad (3.20a)$$

$$\left|\begin{matrix}0\\1\end{matrix}\right|^2=\sum_{n,m}(-)^m Z^{(0,0)}_{n,m}\equiv\sum_{n,m}Z^{(0,1)}_{n,m}\ , \tag{3.20b}$$

$$\left|\begin{matrix}1\\0\end{matrix}\right|^2=\sum_{n,m}(-)^n Z^{(0,0)}_{n,m}\equiv\sum_{n,m}Z^{(1,0)}_{n,m}\ , \tag{3.20c}$$

$$\left|\begin{matrix}1\\1\end{matrix}\right|^2=\sum_{n,m}(-)^{n+m} Z^{(0,0)}_{n,m}\equiv\sum_{n,m}Z^{(1,1)}_{n,m}=0\ . \tag{3.20d}$$

With this particular type of mode expansion, each individual term in the sum in Eqs. (3.20) has very simple modular-transformation properties:

(a) $\tau\to\tau+1$,

$$\begin{aligned}&Z^{(0,0)}_{n,m}\to Z^{(0,1)}_{n-m,m}\ ,\\&Z^{(0,1)}_{n,m}\to Z^{(0,0)}_{n-m,m}\ ,\\&Z^{(1,0)}_{n,m}\to Z^{(1,0)}_{n-m,m}\ .\end{aligned} \tag{3.21}$$

(b) $\tau\to-1/\tau$,

$$\begin{aligned}&Z^{(0,0)}_{n,m}\to Z^{(0,0)}_{m,-n}\ ,\\&Z^{(0,1)}_{n,m}\to Z^{(1,0)}_{m,-n}\ ,\\&Z^{(1,0)}_{n,m}\to Z^{(0,1)}_{m,-n}\ .\end{aligned} \tag{3.22}$$

Equations (3.20)–(3.22) are the tools for constructing new modular-invariant string theories. In most of the previously constructed string solutions, the PF's $\binom{v}{u}$ are taken as the basic building blocks. FKP are the ones to use the modes as the building blocks of the new solutions. However, they use the modes of the torus and therefore are restricted by the torus construction. The simple transformation properties of the individual modes in Eqs. (3.21) and (3.22) allows us to be liberated from this severe constraint and apply the generalized SS mixing mechanism to almost any fermionic formulations.

As far as the "decomposed" fermion is concerned, the individual modes of the PF are the basic building blocks. The summation over m and n in the total PF allows us to introduce modular transformations on these indices so as to simplify the transformation properties of the individual modes even further. Under the modular transformation

$$\tau\to\frac{a\tau+b}{c\tau+d},\quad ad-bc=1,\quad a,b,c,d\in Z,\quad c\geq 0\ , \tag{3.23}$$

if we let

$$\begin{bmatrix}n\\m\end{bmatrix}\to\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}n\\m\end{bmatrix}\ , \tag{3.24}$$

then under the combined transformation of τ, m and n individual modes of the PF transform exactly as the PF itself:

(a) Under $\tau\to\tau+1$, ~ $\binom{n}{m}\to\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\binom{n}{m}$,

$$\begin{aligned}&Z^{(0,0)}_{n,m}\to Z^{(0,1)}_{n,m}\ ,\\&Z^{(0,1)}_{n,m}\to Z^{(0,0)}_{n,m}\ ,\\&Z^{(1,0)}_{n,m}\to Z^{(1,0)}_{n,m}\ .\end{aligned} \tag{3.25}$$

(b) Under $\tau\to-1/\tau$, $\binom{n}{m}\to\left(\begin{smallmatrix}0&0\\1&-1\end{smallmatrix}\right)\binom{n}{m}$,

$$\begin{aligned}&Z^{(0,0)}_{n,m}\to Z^{(0,0)}_{n,m}\ ,\\&Z^{(0,1)}_{n,}\to Z^{(1,0)}_{n,m}\ ,\\&Z^{(1,0)}_{n,m}\to Z^{(0,1)}_{n,m}\ .\end{aligned} \tag{3.26}$$

These simple transformation properties will allow us to introduce n,m dependence into the partition function of the other fermions. We call this SS mixing.

IV. SS MIXING, AND SPONTANEOUS SYMMETRY BREAKING

A. Spectrum of a superstring model with decomposed fermion

As an example we shall work out the spectrum of the $N=4$ superstring model defined earlier, in the language of Eq. (3.15). For the fermions that are decomposed, the states are labeled by the quantum numbers m, n, and N_L^0. Of course, the spectrum is independent of how the states are labeled. This model is constructed by taking two basis sets F and S. The minimal intersections are $\Omega_1=\{\psi^i,\alpha^I\}$ and $\Omega_2=\{\beta^I,\gamma^I,\phi^a\ (a=1,\ldots,44)\}$. This structure of minimal intersections allows one to use the set $T=\{\beta^1,\gamma^1;\phi^1,\phi^2\}$ for decomposition. The generators of the superconformal algebra are [see Eqs. (2.1) and (2.2)]

$$T_B=\frac{1}{2}\left[\sum_{i=1}^{2}:\partial X_L^i\partial X_L^i+\sum_{i=1}^{2}\psi_L^i\partial\psi_L^i:+\sum_{I=1}^{6}:\alpha^I\partial\alpha^I:\right.$$
$$\left.+\sum_{I=2}^{6}:\beta^I\partial\beta^I:+\sum_{I=2}^{6}:\gamma^I\partial\gamma^I:+:J^LJ^L:\right]\ , \tag{4.1}$$

$$T_F=\frac{1}{2}\left[\sum_{i=1}^{2}:\partial x_L^i\psi_L^i+\sum_{I=2}^{6}:\alpha^I\beta^I\gamma^I:+:\alpha^1J^L:\right]\ , \tag{4.2}$$

and

$$\bar{T}_B=\frac{1}{2}\left[\sum_{i=1}^{2}:\bar{\partial}X_R^i\,\bar{\partial}X_R^i:+\sum_{a=3}^{44}:\phi^a\bar{\partial}\phi^a:+:J^RJ^R:\right]\ , \tag{4.3}$$

where $J^L=:\beta^1\gamma^1:$ and $J^R=:\phi^1\phi^2:$. The partition function of this model can be written as

$$Z=\frac{1}{|\eta|^2}\left[\begin{bmatrix}0\\0\end{bmatrix}^4-\begin{bmatrix}0\\0\end{bmatrix}^4-\begin{bmatrix}1\\0\end{bmatrix}^4+\delta_1\begin{bmatrix}1\\1\end{bmatrix}^4\right]$$
$$\times\left[\begin{bmatrix}0\\0\end{bmatrix}^6\begin{bmatrix}\bar{0}\\0\end{bmatrix}^{22}-\begin{bmatrix}0\\1\end{bmatrix}^6\begin{bmatrix}\bar{0}\\1\end{bmatrix}^{22}\right.$$
$$\left.+\begin{bmatrix}1\\0\end{bmatrix}^6\begin{bmatrix}\bar{1}\\0\end{bmatrix}^{22}+\delta\delta_1\begin{bmatrix}1\\1\end{bmatrix}^6\begin{bmatrix}\bar{1}\\1\end{bmatrix}^{22}\right]\ , \tag{4.4}$$

where δ and δ_1 can take values ±1 independently. Every term on the right-hand side of Eq. (4.4) has argument $(0,0,\tau)$ (see Appendix B). Also, $\binom{1}{1}(0,0,\tau)=0$, but it is kept for later reference. When the contribution of the fermions in T are decomposed into modes, the same PF can be written as

$$Z=\frac{1}{|\eta|^2}\sum_{n,m}Z_{n,m}^{(0,0)}\left[\begin{bmatrix}0\\0\end{bmatrix}^4-\begin{bmatrix}0\\1\end{bmatrix}^4-\begin{bmatrix}1\\0\end{bmatrix}^4+\delta_1\begin{bmatrix}1\\1\end{bmatrix}^4\right]$$
$$\times\left[\begin{bmatrix}0\\0\end{bmatrix}^5\begin{bmatrix}\bar{0}\\0\end{bmatrix}^{21}+(-)^m\begin{bmatrix}0\\1\end{bmatrix}^5\begin{bmatrix}\bar{0}\\1\end{bmatrix}^{21}+(-)^n\begin{bmatrix}1\\0\end{bmatrix}^5\begin{bmatrix}\bar{1}\\0\end{bmatrix}^{21}+\delta\delta_1(-)^{m+n}\begin{bmatrix}1\\1\end{bmatrix}^5\begin{bmatrix}\bar{1}\\1\end{bmatrix}^{21}\right] \tag{4.5}$$

$$=\frac{1}{|\eta|^4}\frac{1}{\sqrt{2\tau_i}}\sum_{n,m}(-)^{mn}\exp\left[-\frac{\pi}{2\tau_i}(n^2+|\tau|^2m^2-2mn\tau_r)\right]$$
$$\times\left\{\left[\begin{bmatrix}0\\0\end{bmatrix}^4-\begin{bmatrix}0\\1\end{bmatrix}^4-\begin{bmatrix}1\\0\end{bmatrix}^4+\delta_1\begin{bmatrix}1\\1\end{bmatrix}^4\right]\left[\begin{bmatrix}0\\0\end{bmatrix}^5\begin{bmatrix}\bar{0}\\0\end{bmatrix}^{21}+(-)^m\begin{bmatrix}0\\1\end{bmatrix}^5\begin{bmatrix}\bar{0}\\1\end{bmatrix}^{21}+(-)^n\begin{bmatrix}1\\0\end{bmatrix}^5\begin{bmatrix}\bar{1}\\0\end{bmatrix}^{21}\right.\right.$$
$$\left.\left.+\delta\delta_1(-)^{m+n}\begin{bmatrix}1\\1\end{bmatrix}^5\begin{bmatrix}\bar{1}\\1\end{bmatrix}^{21}\right]\right\}. \tag{4.6}$$

Using a Poisson resummation over m, Eq. (4.6) can be rewritten as

$$Z=\frac{1}{|\eta|^4}\left[\begin{bmatrix}0\\0\end{bmatrix}^4-\begin{bmatrix}0\\1\end{bmatrix}^4-\begin{bmatrix}1\\0\end{bmatrix}^4+\delta_1\begin{bmatrix}1\\1\end{bmatrix}^4\right]$$
$$\times\sum_{n,m}\left\{\exp[\pi i\tau(m+n)^2-\pi i\bar{\tau}n^2]\left[\begin{bmatrix}0\\0\end{bmatrix}^5\begin{bmatrix}\bar{0}\\0\end{bmatrix}^{21}+(-)^m\begin{bmatrix}0\\1\end{bmatrix}^5\begin{bmatrix}\bar{0}\\1\end{bmatrix}^{21}\right]\right.$$
$$\left.+\exp[\pi i\tau(m+n+\tfrac{1}{2})^2-\pi i\bar{\tau}(n+\tfrac{1}{2})^2]\left[\begin{bmatrix}1\\0\end{bmatrix}^5\begin{bmatrix}\bar{1}\\0\end{bmatrix}^{21}+\delta\delta_1(-)^m\begin{bmatrix}1\\1\end{bmatrix}^5\begin{bmatrix}\bar{1}\\1\end{bmatrix}^{21}\right]\right\}. \tag{4.7}$$

Therefore the Virasoro operators can be written as

$$L_0=\tilde{L}_0+\frac{1}{2}\left[m+n+\frac{v}{2}\right]^2,\qquad \bar{L}_0=\bar{\tilde{L}}_0+\frac{1}{2}\left[n+\frac{v}{2}\right]^2, \tag{4.8}$$

and $v\sim 1$ is the σ-boundary condition for the decomposed fermions T. $\tilde{L}_0$ and $\bar{\tilde{L}}_0$ are the zero modes of the conformal generators T_B and $\bar{T}_B$ in Eqs. (4.1) and (4.3) excluding the terms depending on J_0^L and J_0^R. In this model there are only two sectors containing the massless particles, i.e., ϕ and S.

To write down the Gliozzi-Scherk-Olive (GSO) projections, the number operators for the sets Ω_1 and Ω_2-T are denoted as N_{Ω_1} and N_{Ω_2T}.

1. Sector ϕ

The GSO projections are

$$(-)^{N_{\Omega_1}+N_{\Omega_2T}+m}=-1,\qquad (-)^{N_{\Omega_1}}=-1. \tag{4.9}$$

The spectrum in this sector consists of the following states.

(i) Graviton (including antisymmetric tensor field and dilaton):

$$\psi^i_{-1/2}|0\rangle_L\otimes X^j_{-1}|0\rangle_R,\quad i,j\in\{1,2\}. \tag{4.10}$$

(ii) Six gauge bosons:

$$\alpha^I_{-1/2}|0\rangle_L\otimes X^i_{-1}|0\rangle_R,\quad I\in\{1,\ldots,6\}. \tag{4.11}$$

(iii) Gauge bosons in the adjoint representation of SO(44):

$$\psi^i_{-1/2}|0\rangle_L\otimes\phi^a_{-1/2}\phi^b_{-1/2}|0\rangle_R,\quad a,b\in\{3,\ldots,44\}, \tag{4.12a}$$

$$\psi^i_{-1/2}|0\rangle_L\otimes\phi^a_{-1/2}|m=1,n=-1\rangle,\quad a\in\{3,\ldots,44\} \tag{4.12b}$$

and

$$\psi^i_{-1/2}|0\rangle_L\otimes\phi^a_{-1/2}|m=-1,n=1\rangle,\qquad \psi^i_{-1/2}|0\rangle_L\otimes J^R_{-1}|0\rangle_R. \tag{4.12c}$$

(iv) Six scalar fields in the adjoint representation of SO(44):

$$\alpha^I_{-1/2}|0\rangle_L\otimes\phi^a_{-1/2}\phi^b_{-1/2}|0\rangle_R, \tag{4.13a}$$

$$\alpha^I_{-1/2}|0\rangle_L\otimes\phi^a_{-1/2}|m=1,n=-1\rangle, \tag{4.13b}$$

and

$$\alpha^{I}_{-1/2}|0\rangle_L \otimes \phi^{a}_{-1/2}|m=-1,n=1\rangle ,$$
$$\alpha^{I}_{-1/2}|0\rangle_L \otimes J^{R}_{-1}|0\rangle_R . \qquad (4.13c)$$

2. Sector S

The GSO projections in this sector, with the choice $\delta_1=-1$, are

$$(-)^{N_{\Omega_1}+N_{\Omega_2 T}+m}=-\delta_1=1$$

and (4.14)

$$(-)^{N_{\Omega_1}}=-\delta_1=1 .$$

In this sector, $(-)^{N_{\Omega_1}}$ also contains a chirality projection operator. The states in this sector are as follows.

(i) Gravitino ⊕ dilatino:

$$|S\rangle_s \otimes X^{i}_{-1}|0\rangle_R , \qquad (4.15)$$

where $|S\rangle_s$ is an eight-dimensional spinor in the space of the zero modes of the fermions ψ^i and α^I. These states can be classified more precisely by specifying the chirality projections for the pairs $(\psi_1\psi_2)$, $(\alpha_1\alpha_2)$, $(\alpha_3\alpha_4)$, and $\alpha_5\alpha_6$). Thus the four states with positive helicity are

$$|S\rangle_s^{+}\equiv \begin{bmatrix}+\\+\\+\\+\end{bmatrix}\begin{bmatrix}+\\-\\-\\+\end{bmatrix}\begin{bmatrix}+\\-\\+\\-\end{bmatrix}\begin{bmatrix}+\\+\\-\\-\end{bmatrix} . \qquad (4.16a)$$

The other four states are the CPT conjugates of these:

$$|S\rangle_s^{-}\equiv \begin{bmatrix}-\\-\\-\\-\end{bmatrix}\begin{bmatrix}-\\+\\+\\-\end{bmatrix}\begin{bmatrix}-\\+\\-\\+\end{bmatrix}\begin{bmatrix}-\\-\\+\\+\end{bmatrix} . \qquad (4.16b)$$

(ii) Gauginos in the adjoint representation of SO(44):

$$|S\rangle_s \otimes \phi^{a}_{-1/2}\phi^{b}_{-1/2}|0\rangle_R,\quad a,b\in\{3,\ldots,44\} , \qquad (4.17a)$$

$$|S\rangle_s \otimes \phi^{a}_{-1/2}|m=1,n=-1\rangle,\quad a\in\{3,\ldots,44\} , \qquad (4.17b)$$

and

$$|S\rangle_s \otimes \phi^{a}_{-1/2}|m=-1,n=1\rangle ,\quad |S\rangle_s \otimes J^{R}_{-1}|0\rangle_R . \qquad (4.17c)$$

The treatment of this section can be applied to any superstring model where some fermions can be decomposed. The boundary conditions for the left-moving fermions in σ_1 (σ_2) directions are represented as vectors $\mathbf{A}_L$ ($\mathbf{B}_L$) and for the right-moving fermions as $\mathbf{A}_R$ ($\mathbf{B}_R$). The boundary condition for the fermions in the set $F-T$ is represented by $\mathbf{a}_L$ ($\mathbf{b}_L$), $\mathbf{a}_R$ ($\mathbf{b}_R$). Then the PF for the four-dimensional superstring solution is given by

$$Z(\tau,\bar{\tau})=\tau_i^{-1}|\eta(\tau)|^{-2}\sum_{\substack{\text{spin}\\ \text{structure}}} C\begin{bmatrix}\mathbf{A}_L & \mathbf{A}_R\\ \mathbf{B}_L & \mathbf{B}_R\end{bmatrix} Z_T\begin{bmatrix}v\\u\end{bmatrix}(\tau,\bar{\tau}) Z_L\begin{bmatrix}\mathbf{a}_L\\ \mathbf{b}_L\end{bmatrix}(0,0,\tau) Z_R\begin{bmatrix}\mathbf{a}_R\\ \mathbf{b}_R\end{bmatrix}(0,0,\bar{\tau}) . \qquad (4.18)$$

So far, we have merely used Eqs. (4.8) to relabel the spectrum of a superstring model in the language of decomposed fermions. The language turns out to be very useful for the introduction of Thirring interaction or SS mixing.

B. Introducing the Scherk-Schwarz mixing

In Appendix B the generalized PF for a fermion pair has been introduced. It is observed that for any complex fermion a parameter e^i_L or e^i_R can be introduced in such a way that the generalized PF for this pair,

$$\begin{bmatrix}v^i_L\\ u^i_L\end{bmatrix}(Y^i_L,Z^i_L,\tau) \text{ or } \begin{bmatrix}v^i_R\\ u^i_R\end{bmatrix}(Y^i_R,Z^i_R,\bar{\tau}) ,$$

has the same transformation rule as the original PF

$$\begin{bmatrix}v_L\\ u_L\end{bmatrix}(0,0,\tau) \text{ or } \begin{bmatrix}v_R\\ u_R\end{bmatrix}(0,0,\bar{\tau}) .$$

The parameters

$$\mathbf{Z}_L=\mathbf{e}_L(n-\tau m),\quad \mathbf{Z}_R=\mathbf{e}_R(n-\tau m) \qquad (4.19)$$

and

$$Y_L=\mathbf{e}_L^2(n-m\tau),\quad Y_R=\mathbf{e}_R^2(n-m\bar{\tau}) \qquad (4.20)$$

have explicit m,n dependence where m,n are the mode-quantum numbers of the decomposed fermions. Therefore nonzero e^i_L or e^i_R mix the PF of the ith fermion with the decomposed fermion. The modular-transformation properties of the generalized PF (Ref. 15) make it straightforward to preserve the modular invariance while the symmetry-breaking parameters are introduced. With these mixing the PF (4.18) becomes

$$Z(\mathbf{e}_L,\mathbf{e}_R\mid\tau,\bar\tau)=\tau_i^{-1}\mid\eta\mid^{-2}\sum_{m,n}\sum_{\substack{\text{spin}\\ \text{structure}}} C\begin{bmatrix}\mathbf{A}_L & \mathbf{A}_R\\ \mathbf{B}_L & \mathbf{B}_R\end{bmatrix} Z_{n,m}^{(v,u)}(\tau,\bar\tau)Z_L\begin{bmatrix}\mathbf{a}_L\\ \mathbf{b}_L\end{bmatrix}(Y_L,\mathbf{Z}_L,\tau)Z_R\begin{bmatrix}\mathbf{a}_R\\ \mathbf{b}_R\end{bmatrix}(Y_R,\mathbf{Z}_R,\bar\tau)\ . \tag{4.21}$$

More explicitly,

$$\begin{aligned}Z(\mathbf{e}_L,\mathbf{e}_R\mid\tau,\bar\tau)=2^{-1/2}\tau_i^{-3/2}\mid\eta(\tau)\mid^{-4}\sum_{m,n}\sum_{\text{spinstructure}} C\begin{bmatrix}\mathbf{A}_L & \mathbf{A}_R\\ \mathbf{B}_L & \mathbf{B}_R\end{bmatrix}(-)^{mn+um+vn}\\ \times\exp\Bigg[\left\{-\frac{\pi}{2\tau_i}(n^2+\mid\tau\mid^2m^2-2mn\tau_r)\right\}\\ -i\pi\mathbf{e}_L^2m(n-\tau m)+i\pi\mathbf{e}_R^2m(n-\bar\tau m)\\ +2\pi i\mathbf{e}_L\cdot\mathbf{Q}_L(\mathbf{a}_L)(n-\tau m)\\ -2\pi i\mathbf{e}_R\cdot\mathbf{Q}_R(\mathbf{a}_R)(n-\bar\tau m)\Bigg]\\ \times Z_L\begin{bmatrix}\mathbf{a}_L\\ \mathbf{b}_L\end{bmatrix}(0,0,\tau)Z_R\begin{bmatrix}\mathbf{a}_R\\ \mathbf{b}_R\end{bmatrix}(0,0,\bar\tau)\ .\end{aligned} \tag{4.22}$$

In Eq. (4.22) $\mathbf{Q}_L,\mathbf{Q}_R$ are the charge operators. Since they depend only on $\mathbf{a}_L$ and $\mathbf{a}_R$, then in a given sector their eigenvalues are fixed. Equation (4.22) can be given a simple physical interpretation by rewriting it in a slightly different from. Using Poisson resummation,

$$\begin{aligned}Z(\mathbf{e}_L,\mathbf{e}_R\mid\tau,\bar\tau)=\tau_i^{-1}\mid\eta(\tau)\mid^{-4}\sum_{m,n}\sum_{\substack{\text{spin}\\ \text{structure}}} C\begin{bmatrix}\mathbf{A}_L & \mathbf{A}_R\\ \mathbf{B}_L & \mathbf{B}_R\end{bmatrix}\\ \times\exp\Bigg\{-2\pi\tau_i\Bigg[\tfrac{1}{4}m^2+\tfrac{1}{2}(\mathbf{e}_L^2+\mathbf{e}_R^2)m^2-m[\mathbf{e}_L\mathbf{Q}_L(\mathbf{a}_L)+\mathbf{e}_R\mathbf{Q}_R(\mathbf{a}_R)]\\ +\left[n-\tfrac{1}{2}(\mathbf{e}_L^2-\mathbf{e}_R^2)m+(\mathbf{e}_L\mathbf{Q}_L-\mathbf{e}_R\cdot\mathbf{Q}_R)+\frac{m+v}{2}\right]\Bigg]\\ +2\pi i\tau_r\left[m\left[n+\frac{m+v}{2}\right]\right]\Bigg\}Z_L\begin{bmatrix}\mathbf{a}_L\\ \mathbf{b}_L\end{bmatrix}(0,0,\tau)Z_R\begin{bmatrix}\mathbf{a}_R\\ \mathbf{b}_R\end{bmatrix}(0,0,\bar\tau)\ .\end{aligned} \tag{4.23}$$

Therefore when $\mathbf{e}_L,\mathbf{e}_R\neq 0$ the Virasoro operators take the form [see Eq. (4.8)]

$$\begin{aligned}L_0+\bar L_0=&\tfrac{1}{4}m^2+\tfrac{1}{2}(\mathbf{e}_L^2+\mathbf{e}_R^2)m^2\\ &-m[\mathbf{e}_L\mathbf{Q}_L(\mathbf{a}_L)+\mathbf{e}_R\mathbf{Q}_R(\mathbf{a}_R)]\\ &+\tfrac{1}{4}\{2n+m+v-(\mathbf{e}_L^2-\mathbf{e}_R^2)m\\ &\quad+2[\mathbf{e}_L\mathbf{Q}_L(\mathbf{a}_L)-\mathbf{e}_R\mathbf{Q}_R(\mathbf{a}_R)]\}^2+\tilde L_0+\bar{\tilde L}_0\end{aligned} \tag{4.24}$$

and

$$L_0-\bar L_0=mn+\frac{m(m+v)}{2}+\tilde L_0+\bar{\tilde L}_0\ , \tag{4.25}$$

where $\tilde L_0+\bar{\tilde L}_0$ are defined as in Eq. (4.8).

Since in a given sector of the fermionic construction, v, $\mathbf{a}_L$ and $\mathbf{a}_R$ are fixed, therefore Eq. (4.23) implies that the GSO projections are not affected by the introduction of the SS parameters $\mathbf{e}_L$ and $\mathbf{e}_R$. Equation (4.25) implies that the level matching conditions are also unchanged. Therefore all the states already in the spectrum before the SS mixing remain even afterwards. The mass shift of the states is computed by using Eq. (4.24). It is proved in Appendix C that in any theory with space-time supersymmetry before the SS mixing, the conditions $m+n=0$ and $v=0$ are always satisfied for the massless states. After the mixing, using these conditions the mass shift is given by

$$\begin{aligned}\Delta(M^2)=&-2m\mathbf{e}_L\cdot\mathbf{Q}_L+(\mathbf{e}_L\cdot\mathbf{Q}_L-\mathbf{e}_R\cdot\mathbf{Q}_R)^2+\mathbf{e}_L^2m^2\\ &-m(\mathbf{e}_L^2-\mathbf{e}_R^2)(\mathbf{e}_L\cdot\mathbf{Q}_L-\mathbf{e}_R\cdot\mathbf{Q}_R)+\tfrac{1}{4}m^2(\mathbf{e}_L^2-\mathbf{e}_R^2)^2.\end{aligned} \tag{4.26}$$

Equation (4.26) can be generalized to the case when there are more than one set of fermions that can be decomposed. For supersymmetric theories, again $m_i + n_i = 0$ in the massless sector for each decomposed fermions and

$$\Delta(M^2) = -2\sum_i m_i \mathbf{e}_L\cdot\mathbf{Q}_L + (\mathbf{e}_L\cdot\mathbf{Q}_L - \mathbf{e}_R\cdot\mathbf{Q}_R)^2 + \mathbf{e}_L^2\left[\sum_i m_i\right]^2 - \sum_i m_i(\mathbf{e}_L^2 - \mathbf{e}_R^2)(\mathbf{e}_L\cdot\mathbf{Q}_L - \mathbf{e}_R\cdot\mathbf{Q}_R) + \frac{1}{4}\left[\sum_i m_i\right]^2(\mathbf{e}_L^2 - \mathbf{e}_R^2)^2 . \tag{4.27}$$

There is, however, one potential problem. The mass shifts in Eqs. (4.26) and (4.27) are not positive semidefinite. The states for which $\sum m_i \mathbf{e}_L\cdot\mathbf{Q}_L > 0$ become tachyonic as a result of the SS mixing. The tachyon-free condition in general puts a very strong condition on the kind of theories that can be obtained in models using free world-sheet fermions and a SS mixing mechanism. However this mechanism can be easily used for gauge symmetry breaking.

In the example of the $N = 4$ model considered in Sec. IV A, the group SO(44) can be broken down to the group $SO(44-2N)\otimes[SO(2)]^N$ by choosing $e^i{}_R$, $i = 1, \ldots, N$, to be all different. If all the $e^i{}_R$, $i = 1, \ldots, N$, are equal, then the surviving symmetry is $SO(44-2N)\otimes U(N)$.

This symmetry-breaking mechanism can be easily applied to any free fermionic solutions in which, by construction, one of the minimal intersections contains even numbers of both left-handed and right-handed real fermions. As far as the gauge boson is concerned, this breaking effect is very similar to the Wilson loop breaking[16] in the Calabi-Yau compactification scheme. (As a result, it can never reduce the rank of the gauge group.) However, the resulting mass shifts for scalar bosons and fermions are quite different in the two cases. Note also that $e^i{}_L, e^i{}_R$ are free parameters in this first-quantized string solution. As far as we know, there is nothing unnatural to assume that they can be very small. In that case the gauge-symmetry-breaking scale is $e^i M_{\rm Pl}$ which can be much lower than the Planck scale $M_{\rm Pl}$ and may give rise to some stringy prediction at an intermediate scale directly. It should be interesting to find out phenomenologically how low this scale can get.

V. THIRRING INTERACTION AND SYMMETRY BREAKING IN SUPERSTRING MODELS

The generation of tachyons due to SS mixing can be avoided by introducing the Thirring interaction among the decomposed fermions. This is due to a fundamental difference in spectrum between the models of free and interacting fermions. In general, the symmetry group enlarges when the interaction is turned off and there are more massless particles in the spectrum. It is some of these massless particles which become tachyonic due to the SS mixing. In this paper, we shall use only the simplest type of Thirring interaction to which all four fermions, i.e., two real left movers and two right movers, share the same boundary condition. A more general Thirring interaction in which the two left movers and the two right movers have different boundary conditions is possible and will be the subject of a future publication.[17] We restrict ourselves to the simplest case here because we know how to introduce SS mixing in this situation. We start with a review of its general properties.

A. Thirring model

The simplest Thirring model is described by the action

$$I_0 = \int d^2x(\bar\psi\gamma^\mu\partial_\mu\psi + 2g\bar\psi\gamma^\mu\psi\bar\psi\gamma_\mu\psi) . \tag{5.1}$$

A more general Thirring model can be described by taking more than one Dirac fermion. The Thirring action is invariant under local reparametrization as well as Weyl transformations. In terms of the two left-moving components χ_i ($i = 1,2$) and two right-moving components ϕ_i ($i = 1,2$) of the Dirac field ψ, the action (5.1) can be written as

$$I_0 = \int d^2x\left[\sum_{i=1}^{2}\chi_i\partial\chi_i + \sum_{i=1}^{2}\phi_i\bar\partial\phi_i + 2g\chi_1\chi_2\phi_1\phi_2\right] . \tag{5.2}$$

If a free left-moving Majorana-Weyl fermion χ_3 is added to this Thirring model, then the enlarged action

$$I = I_0 + \int d^2x\, \chi_3\partial\chi_3 \tag{5.3}$$

is also invariant under a supersymmetry transformation:

$$\delta_\epsilon\chi^i = \epsilon_{ijk}\chi^j\chi^k\epsilon . \tag{5.4}$$

The generators of the superconformal algebra in the left-moving part are

$$T_B = \tfrac{1}{2}{:}j^L j^L{:} + {:}\chi_3\partial\chi_3{:} , \tag{5.5a}$$

$$T_F = {:}\chi_3 j^L{:} , \tag{5.5b}$$

where $j^L \equiv {:}\chi_1\chi_2{:}$. Conformal algebra in the right-moving part is realized by

$$T_B = \tfrac{1}{2}{:}j^R j^R{:} \tag{5.6}$$

and $j^R \equiv {:}\phi_1\phi_2$. These conformal field theories can be used to construct new superstring solutions. Here we shall concentrate on one particular application. To construct modular-invariant theories the properties of the PF under modular transformation has to be studied. The PF of a Dirac fermion was obtained earlier by computing the trace of $e^{2\pi i\tau H}$ over the states created by the modes of the currents and summing over all the charge sectors. This approach is very useful in obtaining the PF of a Thirring fermion. Following the results of a recent paper by Bagger, Nemeschansky, Seiberg, and Yankielowicz, the PF of a Thirring fermion (with interaction being parametrized as $2g = R - 1/R$) can be written as[9]

$$\left|\begin{matrix} v \\ u \end{matrix}\right|_R^2 = \frac{1}{|\eta|^2} \sum_{n_L, n_R} \exp\left[\pi i \tau \left(\frac{v}{2R} + n_L \frac{a}{2} + n_R \frac{b}{2}\right)^2\right] \exp\left[-\pi i \bar{\tau} \left(\frac{v}{2R} + n_R \frac{a}{2} + n_L \frac{b}{2}\right)^2\right] \exp[-\pi i u (n_L + n_R)] , \qquad (5.7)$$

where $a = R + 1/R$ and $b = R - 1/R$.

As in the case of free fermions, the PF can be rewritten in a form in which each individual mode of the PF transforms in a simple way under modular transformations. The equation corresponding to (3.20a) of the free case can be written as

$$\left|\begin{matrix} 0 \\ 0 \end{matrix}\right|_R^2 = \frac{1}{|\eta|^2} \frac{R}{\sqrt{2\tau_i}} \sum_{m,n} (-)^{mn} \exp\left[-\frac{\pi R^2}{2\tau_i}(n^2 + |\tau|^2 m^2 - 2mn\tau_r)\right] . \qquad (5.8)$$

Other equations in (3.20) remain unchanged except that

$$\left|\begin{matrix} 1 \\ 1 \end{matrix}\right|_R^2$$

is now nonzero [cf. Eq. (3.20d)]. The modular transformation properties are exactly the same as the free case [Eqs. (3.21) and (3.22)].

B. Model building with the Thirring fermions

There is a restriction on when the Thirring interaction can be introduced for the superstring-model building. This restriction comes from the superconformal symmetry together with the modular invariance. For the choice of supercurrent and S as in Sec. III, the left-moving parts of the Thirring fermions can only be any of the pairs $\{\beta^I, \gamma^I\}$ $(I = 1, \ldots, 6)$ because $\alpha^I \in S$ can never belong to the same minimal intersection with any of the right movers. It is proved in Appendix C that in an $N = 1$ (space-time) supersymmetric theory, no $\{\beta^I, \alpha^I\}$ can belong to the same minimal intersection with the right movers. As a result the Thirring interaction consistent with the superconformal symmetry cannot be introduced in $N = 1$ superstring theories. The Virasoro operators analogous to Eq. (4.8) are not

$$L_0 = \tilde{L}_0 + \frac{1}{2}\left[\frac{1}{2}mR + \frac{1}{R}\left(n + \frac{m+v}{2}\right)\right]^2 ,$$
$$\bar{L}_0 = \bar{\tilde{L}}_0 + \frac{1}{2}\left[\frac{1}{2}mR - \frac{1}{R}\left(n + \frac{m+v}{2}\right)\right]^2 , \qquad (5.9)$$

or

$$L_0 + \bar{L}_0 = \tilde{L}_0 + \bar{\tilde{L}}_0 + \tfrac{1}{4}m^2R^2 + \frac{1}{4R^2}(2n + m + v)^2 , \qquad (5.10a)$$

$$L_0 - \bar{L}_0 = \tilde{L}_0 - \bar{\tilde{L}}_0 + mn + \frac{m(m+v)}{2} . \qquad (5.10b)$$

Since L_0 and $\bar{L}_0$ only on v and not on u, the Thirring interaction does not change the GSO projections. The level matching condition (5.10b) is also independent of the parameter R. The only effect of the Thirring interaction is a change in the mass formula as given in (5.10a). This formula implies that there is a mass shift for all the states with nonzero (m,n) dependence. This mass shift can be both positive and negative. However, for the $N = 2$ and 4 superstring theories that we are interested in it is proved in Appendix C that the mass-shift is positive-semidefinite.

The Thirring interaction introduced in this section can also be used for gauge symmetry breaking. However for the particularly simple type of Thirring interactions that we used here the only surviving symmetry is $[\mathrm{U}(1)]^d$ for d Thirring fermions. Models with heterotic as well as non-Abelian Thirring interactions have been investigated by the authors and will be reported elsewhere.[17]

VI. SUPERSTRING MODELS WITH THIRRING FERMIONS AND SCHERK-SCHWARZ MIXING

In this section the Scherk-Schwarz mixing and the Thirring interactions are applied together for superstring-model building. The interaction will be introduced among the decomposed fermions of Secs. III and IV. Of course, the world-sheet supersymmetry puts restrictions on when this interaction can be introduced among the decomposed fermions. As in the previous sections, the modular invariance is trivial to preserve. The GSO projections and the level matching conditions are unchanged. The mass shift can be computed by

$$\begin{aligned} L_0 + \bar{L}_0 = \tilde{L}_0 + \bar{\tilde{L}}_0 &+ \tfrac{1}{4}m^2R^2 + \tfrac{1}{2}(\mathbf{e}_L^2 + \mathbf{e}_R^2)m^2 \\ &- m(\mathbf{e}_L \cdot \mathbf{Q}_L + \mathbf{e}_R \cdot \mathbf{Q}_R) \\ &+ \frac{1}{4R^2}[2n + m + v - (\mathbf{e}_L^2 - \mathbf{e}_R^2)m \\ &+ 2(\mathbf{e}_L \cdot \mathbf{Q}_L - \mathbf{e}_R \cdot \mathbf{Q}_R)]^2 . \end{aligned} \qquad (6.1)$$

Now the mass formula depends upon two new parameters: e and R. It is shown in Appendix C that even when all e's are zero, by choosing $R \neq 1$ (for $N = 2,4$ superstring theories), all the states with nonzero (m,n) quantum numbers become massive. This mass shift can be made to be of the order M_{Pl}. Therefore the massless spectrum contains only those states with $m = 0, n = 0$. For these states the mass shift is given by

$$\Delta(M^2) = \frac{1}{R^2}(\mathbf{e}_L \cdot \mathbf{Q}_L - \mathbf{e}_R \cdot \mathbf{Q}_R)^2 , \qquad (6.2)$$

and is positive-semidefinite.

Combination of Thirring interaction and SS mixing is used to break the gauge symmetry in two steps. In the example of Sec. IV Thirring interaction can be used to break the gauge group SO(44) into U(1)×SO(42). Then by appropriate choice of $\mathbf{e}_R$, the residual gauge symmetry can be broken further.

VII. DISCUSSIONS

In the symmetry-breaking mechanism presented in this paper the rank of the gauge group is not reduced. A generalization of Riemann theta functions may be required to obtain the symmetry breaking which reduces rank. As far as the gauge-boson masses are concerned formula (6.2) is similar to the one in the Wilson loop mechanism.[16] Recently, Antoniadis, Bachas, and Kounnas[18] have proposed a method of gauge symmetry breaking which can be alternatively formulated as a general Thirring interaction.[17]

It has been proved in Appendix D that the fermionic constructions presented in the paper reduce to the five-dimensional formulation of FKP as a special case. Also the gauge symmetry breaking considered here can be applied to nonchiral models only. A twisted world-sheet Thirring interaction can also be used for the gauge symmetry breaking in chiral four-dimensional models.[19]

Mass formula (6.1) can also be obtained by the Lorentz-boosting mechanism[18] or by a Thirring interaction. Naively one may be tempted to apply SS mixing to break space-time supersymmetry. However, the SS mixing needed for supersymmetry breaking, when translated into Thirring language, implies superconformal noninvariant interactions. Therefore it is most likely inconsistent in the SS mixing picture as well.

APPENDIX A

In this appendix (3.20a) is obtained from (3.19a). Consider the expression

$$A\begin{bmatrix}0\\0\end{bmatrix}=\sum_{n_L,n_R} p^{n_L^2}\bar{p}^{\,n_R^2} \qquad \text{(A1)}$$

$$=\sum_{n_L,n_R}\exp(\pi i\tau n_L^2-\pi i\bar{\tau}n_R^2)\;. \qquad \text{(A2)}$$

Now define $m=(n_L+n_R)/2$, $n=(n_L-n_R)/2$. Then,

$$A\begin{bmatrix}0\\0\end{bmatrix}=\left[\sum_{(m,n)\in Z}+\sum_{(m,n)\in Z/2}\right]\times\exp[\pi i\tau(m+n)^2-\pi i\bar{\tau}(m-n)^2] \qquad \text{(A3)}$$

$$\equiv A^1\begin{bmatrix}0\\0\end{bmatrix}+A^2\begin{bmatrix}0\\0\end{bmatrix}. \qquad \text{(A4)}$$

Now

$$A^1\begin{bmatrix}0\\0\end{bmatrix}=\sum_{m\in Z}\exp(-2\pi\tau_i m^2)\times\sum_{n\in Z}\exp(-2\pi\tau_i n^2-4\pi i\tau_r mn)\;. \qquad \text{(A5)}$$

By using the Poisson resummation formula,

$$A^1\begin{bmatrix}0\\0\end{bmatrix}=\frac{1}{\sqrt{2\tau_i}}\sum_{m,n\in Z}\exp\left[-2\pi\tau_i m^2-\frac{\pi}{2\tau_i}(n-2\tau_r m)^2\right]=\frac{1}{\sqrt{2\tau_i}}\sum_{m,n\in Z}\exp\left[-\frac{\pi}{2\tau_i}(4m^2|\tau|^2+n^2-4mn\tau_r)\right]. \qquad \text{(A6)}$$

For A^2, m,n are half integers. They can be parametrized as

$$m=\alpha+\tfrac{1}{2},\quad n=\beta+\tfrac{1}{2},\quad \text{and } \alpha,\beta\in Z\;. \qquad \text{(A7)}$$

Then

$$A^2\begin{bmatrix}0\\0\end{bmatrix}=\sum_{\alpha\in Z}\exp\left[-4\pi i\tau_r\left[\frac{\alpha}{2}+\frac{1}{4}\right]-2\pi\tau_i\left[\alpha+\tfrac{1}{2}\right]^2-\frac{\pi\tau_i}{2}\right]\times\sum_{\beta\in Z}\exp[-2\pi\tau_i(\beta^2+\beta)-4\pi i\tau_r\beta(\alpha+\tfrac{1}{2})]\;. \qquad \text{(A8)}$$

Using the Poisson resummation formula on index β, Eq. (A8) becomes

$$A^2\begin{bmatrix}0\\0\end{bmatrix}=\frac{1}{\sqrt{2\tau_i}}\sum_{\alpha,\beta\in Z}\exp\left[-\frac{\pi}{2\tau_i}[(2\alpha+1)^2|\tau|^2+\beta^2-2\beta(2\alpha+1)\tau_r]\right]\times\exp(-\pi i\beta)\;. \qquad \text{(A9)}$$

Adding Eqs. (A6) and (A10) one gets

$$A\begin{bmatrix}0\\0\end{bmatrix}=\frac{1}{\sqrt{2\tau_i}}\sum_{m,n\in Z}(-)^{mn}\exp\left[-\frac{\pi}{2\tau_i}(n^2+|\tau|^2m^2-2mn\tau_r)\right] \qquad \text{(A10)}$$

Equation (A10) is the expression used in Eq. (3.20a). Other expressions in Eqs. (3.20) can be obtained in a similar way.

APPENDIX B

In this appendix, Riemann theta functions are reviewed.[15] It is shown that a generalization of theta functions can be done through the SS mixing such that all the modular-transformation properties remain unchanged.

In our notations Riemann theta functions can be written as[15]

$$\Theta\begin{bmatrix} v \\ u \end{bmatrix}(z,\tau)=\sum_{n\in Z}\exp\left[\pi i\tau\left[n+\frac{v}{2}\right]^2+2\pi i\left[n+\frac{v}{2}\right]+\frac{i\pi}{2}uv\right] . \quad \text{(B1)}$$

Equation (B1) is closely related to the PF of a complex left-moving fermion. The PF of a free (complex) left-moving fermion as given by Eqs. (3.16) is

$$\begin{bmatrix} v_L \\ u_L \end{bmatrix}(0,\tau)\equiv \mathrm{Tr}\exp\left[2\pi i\tau H_{v_L}-\pi i u_L Q_L(v_L)+\frac{\pi i}{2}u_L v_L\right] \quad \text{(B2)}$$

$$=\frac{1}{\eta(\tau)}\sum_{n_L\in z} p^{(n_L+v_L/2)^2}e^{-\pi i n_L u_L} . \quad \text{(B3)}$$

This definition of the PF can be generalized to

$$\begin{bmatrix} v_L \\ u_L \end{bmatrix}(z_L,\tau)\equiv \mathrm{Tr}\exp\left[2\pi i\tau H_{v_L}+2\pi i z_L Q_L(v_L)-\pi i u_L Q_L(v)+\frac{\pi i}{2}u_L v_L\right] \quad \text{(B4)}$$

$$=\frac{\Theta\begin{bmatrix} v_L \\ u_L \end{bmatrix}(z_L,\tau)}{\eta(\tau)} . \quad \text{(B5)}$$

Similarly the PF of a complex right-moving fermion is given by

$$\begin{bmatrix} v_R \\ u_R \end{bmatrix}(z_R,\bar\tau)=\frac{\Theta\begin{bmatrix} v_L \\ u_L \end{bmatrix}(z_R,\bar\tau)}{\eta(\bar\tau)} . \quad \text{(B6)}$$

To make connection with the Scherk-Schwarz mixing, note that the quantities $\mathbf{e}_L(n-\tau m)$ and $\mathbf{e}_R(n-\bar\tau m)$ have exactly the same transformation properties as $\mathbf{z}_L$ and $\mathbf{z}_R$ (Ref. 15), i.e., under

$$\tau\to\frac{a\tau+b}{c\tau+d},\quad \mathbf{z}_L\to\frac{\mathbf{z}_L}{c\tau+d} . \quad \text{(B7)}$$

Therefore we take

$$\mathbf{z}_L=\mathbf{e}_L(n-\tau m),\quad \mathbf{z}_R=\mathbf{e}_R(n-\bar\tau m) . \quad \text{(B8)}$$

The transformation properties of m,n are the same as in Eq. (3.24) and $\mathbf{e}_L,\mathbf{e}_R$ are arbitrary parameters. One also needs to introduce quantities [see Eq. (4.19)]

$$Y_L=\mathbf{e}_L^2 m(n-\tau m),\quad Y_R=\mathbf{e}_R^2 m(n-\bar\tau m) , \quad \text{(B9)}$$

which transform as

$$Y_L\to Y_L+\frac{c}{c\tau+d}\mathbf{z}_L^2,\quad Y_R\to Y_R+\frac{c}{c\bar\tau+d}\mathbf{z}_R^2 . \quad \text{(B10)}$$

Then a generalized PF of a fermion with SS mixing can be written as

$$\begin{bmatrix} v_L \\ u_L \end{bmatrix}(Y_L,\mathbf{z}_L,\tau)\equiv e^{-i\pi Y_L}\begin{bmatrix} v_L \\ u_L \end{bmatrix}(\mathbf{z}_L,\tau) ,$$
$$Z\begin{bmatrix} v_R \\ u_R \end{bmatrix}(Y_R,\mathbf{z}_R,\tau)\equiv e^{i\pi Y_R}\begin{bmatrix} v_R \\ u_R \end{bmatrix}(\mathbf{z}_R,\bar\tau) . \quad \text{(B11)}$$

The PF

$$\begin{bmatrix} v_L \\ u_L \end{bmatrix}(Y_L,\mathbf{z}_L,\tau)$$

transforms exactly as the free fermionic PF

$$\begin{bmatrix} v_L \\ u_L \end{bmatrix}(0,0,\tau)$$

under the modular transformation.[5,6] The same is true for the right-moving fermions. As a consequence of this generalization, it is found that given a modular-invariant theory with $e_L^i=e_R^i=0$ for each fermion, one can always obtain another theory by taking nonzero e_L^i and e_R^i.

APPENDIX C

In this appendix, we prove various statements we made about the Thirring fermion in the text. The statements apply to the case of real fermionic construction. In the case of complex fermionic construction, some of these statements may not be valid.

(a) First of all, we shall show that in a string solution with space-time supersymmetry, if one labels the states created by the fermions in T, as defined in Sec. IV, by their modes m,n as defined in Eq. (4.8) and the σ-boundary condition v, then $m+n=0$ and $v=0$ for all the massless states. A consequence of this statement is that the mass-shift after the Thirring interaction among fermions in T are introduced can be written as

$$\Delta(M^2)=\tfrac{1}{4}m^2\left[R-\frac{1}{R}\right]^2 , \quad \text{(C1)}$$

where $\Delta(M^2)$ is nothing but the difference between the $L_0+\bar L_0$ in Eqs. (5.10) and (4.8). Therefore, the mass shift due to the interaction is positive-semidefinite and no tachyon will be generated.

To prove that $m+n=0$ for the massless states when $R=1$, we first observe that states with $m+n\neq 0$, say $m+n=1$, correspond in the fermionic language of Eq. (3.2) to the states created by the modes of free fermions β^1 or γ^1. Therefore, to prove $m+n=0$ one only has to show that in the creation of all the massless states, no modes of β^1 or γ^1 are used. To show this, consider first a sector B in which the fermions in T are antiperiodic, i.e., $v=0$ and $T\cap B=\varnothing$. Since $C(S\mid B)=-1$ by supersymmetry, it implies the GSO projection $(-)^S=-1$. Therefore one has to use at least one creation mode from the fermions in S and that saturates the energy allowed for the massless state. Therefore no modes of β^1,γ^1 are used. Next consider $v=1$ and $T\subset B$. Here, β^1,γ^1 are periodic and α^1 and ψ^1 must share the same boundary condition. If α^1 is periodic then the world-sheet supercurrent requires that for each I, one or three of the elements in the set $\{\alpha^I,\beta^I,\gamma^I\}$ must be in B. Since $n_L(S\cap B)=0$ (mod4), we have either $B\cap S=\varnothing$ or $n_L(B)\geq 10$ and no massless state results. The superpartner of the states in sector B are contained in the sector SB. When $B\cap S=\varnothing$, $n_L(BS)=n_L(B)+n_L(S)\geq 10$. The sector SB therefore contains no massless state and neither does sector B by supersymmetry. Q.E.D.

We have proved that in a theory with a massless gravitino, massless states are labeled by $v=0$, $m+n=0$ for the mode numbers of the fermions in T. Therefore, no tachyon will be generated by the Thirring interaction. When SS mixing is introduced, e^i can be chosen to be sufficiently small so that the massive states with $m+n\neq 0$ remain massive when $e'\neq 0$ are introduced.

(b) Here we prove a statement in Sec. V B that in an $N=1$ superstring theory, one cannot find a set of four fermions such as T to be contained in the same minimal intersection. As shown in Ref. 6, an $N=1$ superstring solution contains at least four basis elements, F,S,S_1,S_2, where S_1,S_2 can be chosen so that they both contain $\{\psi^i\}$ and each share four elements with S. In addition, $S\cap S_1\cap S_2=\{\psi^i\}$. This immediately implies that $\{\psi^i\}$ cannot be in the same minimal intersection with an α^I. In fact $\{\psi^i\}$ forms a minimal intersection all by itself. However, by world-sheet supercurrent, $\{\beta^I,\gamma^I\}$ for some I can belong to a minimal intersection only if α^I is in the same minimal intersection with ψ^i. This is not possible in an $N=1$ theory as shown. Q.E.D.

APPENDIX D

In this appendix the condition for the free fermionic construction to have a torus factor has been obtained by using a factorization theorem. This theorem states the following: If a set α contained in the spin structure Ξ is also a minimal intersection of Ξ, then the partition function will factorize into a part containing the contributions of the fermions in α and a part for the rest of the fermions, $F\alpha$, if and only if for every β in Ξ that is disjoint from α the coefficient $\delta_\beta C(\alpha\mid\beta)$ is independent of β. Recall that $\delta_\beta=-1$ if β contains (ψ^i) and $\delta_\beta=+1$ if otherwise

To prove the factorization theorem for the free fermionic construction, consider a spin structure Ξ generated by a basis $\{\alpha,\beta_1,\beta_2,\ldots,\beta_N\}$. Note that a set in Ξ which is also a minimal intersection of Ξ can always be used as a basis element. Therefore Ξ can be written as $\Xi=\{\beta,\alpha\beta\mid\beta\in\Xi'\}$, where $\Xi'=(\prod_{i=1}^n\beta_i^{m_i}\mid m_i=0,1)$; Ξ' can further be split into two parts $\Xi'=\Xi_1+\Xi_2$, where $\Xi_1=\{\beta\mid\beta\in\Xi',\alpha\cap\beta=\varnothing\}$ and $\Xi_2=\{\beta\mid\beta\in\Xi',\alpha\subset\beta\}$ $=\{\alpha\beta\mid\beta\in\Xi_1\}$. This splitting is possible because α is a minimal intersection. It implies that $\Xi=\{\beta,\alpha\beta\mid\beta\in\Xi_1\}$, that is, those β in Ξ_2 only reproduce the same set in Ξ. The partition function in Eq. (2.5) can now be written as

$$Z=\sum_{\beta_1,\beta_2\in\Xi_1}([\beta_1\mid\beta_2]+[\alpha\beta_1\mid\beta_2]+[\beta_1\mid\alpha\beta_2]+[\alpha\beta_1\mid\alpha\beta_2])\ ,$$

where we have used the shorthand notation $[\alpha\mid\beta]=C(\alpha\mid\beta)Z(\alpha\mid\beta)$. By using the modular-invariant properties for the C coefficients,

$$C(\alpha\beta\mid\gamma)=\delta_\gamma C(\alpha\mid\gamma)C(\beta\mid\gamma)\ ,$$

$$C(\gamma\mid\alpha\beta)=\delta_\gamma C(\gamma\mid\alpha)C(\gamma\mid\beta)\ ,$$

and

$$C(\alpha\mid\beta)=C(\beta\mid\alpha)\quad\text{if } \alpha\cap\beta=\varnothing\ ,$$

we can rewrite Z as

$$\sum_{\beta_1,\beta_2\in\Xi_1}[Z_\alpha(0\mid 0)+\delta_{\beta_2}C(\alpha\mid\beta_2)Z_\alpha(1\mid 0)+\delta_{\beta_1}C(\alpha\mid\beta_1)Z_\alpha(0\mid 1)+\delta_{\beta_1}\delta_{\beta_2}C(\alpha\mid\beta_1)C(\alpha\mid\beta_2)Z_\alpha(1\mid 1)]C(\beta_1\mid\beta_2)Z_{F\alpha}(\beta_1'\mid\beta_2')\ ,$$

where $Z_\alpha(u\mid v)$ is the partition function for the fermion in α with boundary condition u,v, and β_1',β_2' are the boundary conditions in β_1,β_2 for the fermions in $F\alpha=F-\alpha$. If the coefficients $\delta_\beta C(\beta\mid\alpha)=C_\alpha$ is independent of $\beta\in\Xi$, then Z is factorized into a product of

$$Z_\alpha(0\mid 0)+Z_\alpha(1\mid 1)+C_\alpha[Z_\alpha(1\mid 0)+Z_\alpha(0\mid 1)]$$

and

$$\sum_{\beta_1,\beta_2\in\Xi_1}C(\beta_1\mid\beta_2)Z_{F\alpha}(\beta_1'\mid\beta_2')\ .$$

Q.E.D.

Note that in this proof we have not used the explicit form of $Z(\alpha\mid\beta)$. The only property we used is the algebraic relation between the C coefficients required by the modular invariance. These coefficients are by construction independent of the Thirring interaction and the SS mixing. Therefore, as long as α is not the set containing the decomposed (or Thirring) fermions the theorem also applies even when the Thirring interaction and SS mixing are present. However, in this case, the factorization should be viewed as happening to each mode (m,n) under the summation sign in Eq. (4.21). That is, the factorization applies to the sum over the spin structure and not to the sum over m,n. This provides a connection between the present four-dimensional formulation and the five-dimensional formulation of FKP. The condition for the existence of a torus factor (with $R=1$) is that there must be a set, in Ξ, such as T which contains two left movers and two right movers and is also a minimal interaction of Ξ. Since the torus compactification of $d=5$ supersymmetric models always results in $N=2$ or $N=4$ models, this provides another manifestation of the statement and proof we made in Appendix C.

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