New approach to one-loop calculations in gauge theories

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We propose using the technology of four-dimensional string theories to calculate amplitudes in gauge theories. Strings make such calculations much more efficient by summing a large number of Feynman diagrams all at once. We check the idea by constructing a string model reducing to a pure non-Abelian gauge theory in the infinite-tension limit and computing its β function with these techniques.

Feynman diagrams provide a powerful and universally used tool for calculating amplitudes in the perturbative expansion of quantum field theories. But the power of this technique is tempered by the rapid increase in the number of diagrams that must be computed at each order.

In QCD, the number of diagrams is quite large even for relatively simple tree-level processes (e.g., $gg \to ggg$), and becomes painful with a higher number of external particles, or for loop corrections. In spite of the proliferation of diagrams, the final answers one obtains for gauge-invariant amplitudes are often quite simple. This comes about because of the enormous cancellations that often occur between different Feynman diagrams; such cancellations are particularly remarkable in the case of gauge theories such as QCD. These cancellations suggest that a more efficient technique might actually exist.

In oriented closed-string theories, on the other hand, there is only one Feynman diagram at each order in perturbation theory; and string theories reduce to gauge theories when the string tension is taken to infinity. Each string diagram thus contains the sum of *all* gauge-theory Feynman diagrams contributing to a process at any given order in perturbation theory, and it suggests an alternative approach to computing amplitudes in a gauge theory: *compute the string amplitude in the limit that the string tension goes to infinity.* Lee, Nair, and one of the authors have presented a similar strategy for tree-level computations, using the open bosonic string[1] and have explicitly demonstrated the efficiency of string techniques.

In this paper we illustrate this approach by computing the β function for a non-Abelian gauge theory. This computation can be done very easily using the conventional Feynman-diagram approach, and so does not directly serve to illustrate the advantage of the string-based approach. But it does demonstrate its feasibility. The power of our approach lies in the fact that the difficulty and complexity of a calculation grows relatively slowly with an increasing number of external legs. A computation of the one-loop correction to $gg \to gg$ scattering will be presented elsewhere.[2]

Minahan has presented a similar calculation of the β function for a $N=1$ supersymmetric gauge theory, using an orbifold compactification of the ten-dimensional heterotic string theory.[3] Although the computation presented here is similar, we explicitly demonstrate that supersymmetry (SUSY) is not a relevant requirement to obtain a sensible low-energy limit; we obtain the pure Yang-Mills β function from an appropriate nonsupersymmetric string model. The dilaton singularity is not important for our purposes and may explicitly be shown to decouple (for $\alpha' \to 0$) (Ref. 2) when using the modular-invariant dimensional regularization of Green, Schwarz, and Brink.[4] This is, of course, expected since dilatons couple with the strength of gravity.

Although at tree level it is not crucial to construct a string theory whose low-energy limit is the particular four-dimensional gauge theory of interest, the situation is different for radiative corrections: unwanted particles will circulate in the loop unless they are eliminated from the spectrum. As we shall show, the technology of four-dimensional string theories[5-7] suffices to construct a string model whose infinite-tension limit is a pure gauge theory in four space-time dimensions. We will construct such a model, and then find the gauge-theory β function by taking the infinite-tension limit, $\alpha' \to 0$, of the three-point amplitude.

To construct an appropriate string theory, we follow the fermionic formulation of Kawai, Lewellen, and Tye[6] (KLT). We discuss in particular the construction of a heterotic string theory containing an SU(9) pure gauge theory in its infinite-tension limit. (There is no particular significance to nine colors; it just happens to be an easy model to construct and analyze.)

In the KLT formalism, the boundary conditions for the complex world-sheet fermions are represented by vectors $W_i=(l_1 \cdots l_{22} \mid r_1 \cdots r_{10})$, where the l_i component signifies that the ith left-mover fermion picks up an $\exp(-2\pi l_i)$ phase when going around the appropriate (world-sheet space or time) closed loop. A model specified by a set of basis vectors W_i is a consistent string theory if it satisfies certain constraint equations, Eqs. (3.33)–(3.35) of Ref. 6. The space-boundary-condition vectors specify the sectors, while the time-boundary-condition vectors determine the generalized Gliozzi-Scherk-Olive (GSO) projections that constrain the spectrum. The mass squared of a given state (in units of $1/\alpha'$) is determined by adding the quanta of the fermionic world-sheet oscillators to the vacuum energy with the usual left-right level matching. Modular invariance requires that in calculating the partition function (or scattering amplitudes) we sum over time- and space-boundary conditions, with coefficients given in Eq. (3.32) of Ref. 6.

The model at hand is specified by the five "basis" vectors:

$$W_0=(\tfrac{1}{2}^{22} \mid \tfrac{1}{2}^{10}), \quad W_1=(\tfrac{1}{3}^{18}0^4 \mid 0^{10}) ,$$

$$W_2=(0^9\tfrac{1}{2}^9\tfrac{1}{2}000 \mid \tfrac{1}{2}(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2})(\tfrac{1}{2}00)(\tfrac{1}{2}00)) ,$$

$$W_3=(0^9\tfrac{1}{2}^9 0\tfrac{1}{2}00 \mid \tfrac{1}{2}(0\tfrac{1}{2}0)(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2})(0\tfrac{1}{2}0)) ,$$

$$W_4=(0^9\tfrac{1}{2}^9 00\tfrac{1}{2}0 \mid \tfrac{1}{2}(00\tfrac{1}{2})(00\tfrac{1}{2})(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2})) ,$$

where l^n signifies n contiguous components with value l. The gauge group we will be interested in corresponds to the first nine left-mover oscillators, while the space-time index for vectors is carried by the first right-mover complex fermion. This model is a modular-invariant string theory, as can be shown by solving the KLT constraint equations.

We are interested in the spectrum of massless particles, since only these will survive in the infinite-tension limit. Sectors containing massless particles must have both left- and right-vacuum energies that are zero or negative. There are seventeen such sectors; sixteen of these are easily eliminated, as exciting a gauge oscillator in those sectors would necessarily yield a massive state. These sectors also contain no tachyons. The remaining sector for us to analyze is the W_0 sector, which is the one containing the graviton and the gauge bosons.

Before proceeding with this analysis, it will be convenient to develop some notation to refer to different groups of oscillators. The first nine nonzero components of W_1 we will refer to as G, corresponding to the gauge group of interest, and the next nine as G'. The nonzero elements of $W_{\{2,3,4\}}$ amongst the components representing left movers we will refer to as $L_{\{2,3,4\}}$; the nonzero components amongst the right movers we will refer to as $R_{\{2,3,4\}}$.

In the W_0 sector, the left-mover vacuum energy is -1 and the right-mover vacuum energy is $-\frac{1}{2}$. The projection equations[6] are (all mod 1)

$$W_0:\ \tfrac{1}{2}\sum_{\text{all}}(n-\bar{n})\equiv\tfrac{1}{2}, \quad W_1:\ \tfrac{1}{3}\sum_{G\oplus G'}(n-\bar{n})\equiv 0 ,$$

$$W_a:\ \tfrac{1}{2}\sum_{L_a}(n-\bar{n})+\tfrac{1}{2}\sum_{R_a}(n-\bar{n})\equiv\tfrac{1}{2} \quad (a=2,3,4) ,$$

where n counts the number of particle raising operators $b^\dagger$, and $\bar{n}$ the number of antiparticle raising operators $d^\dagger$.

The W_0 equation is in fact weaker than the level-matching condition. The W_1 equation forbids raising with a single $b_G^\dagger$ or $d_G^\dagger$, and thereby eliminates any would-be tachyons transforming nontrivially under G or G'; it also forces massless states generated via a $b^\dagger_{\{G,G'\}}$ or $d^\dagger_{\{G,G'\}}$ to have exactly the form $b^\dagger_{\{G,G'\}}d^\dagger_{\{G,G'\}}\mid 0\rangle$. The level-matching condition requires raising with a $b^\dagger$ or $d^\dagger$ amongst the right movers, so at this stage there are potentially both massless vectors and massless scalars transforming nontrivially under the gauge group of interest.

For particles transforming nontrivially under G or G' exclusively, the W_a equations require that $\sum_{R_a}(n-\bar{n})/2\equiv\frac{1}{2}$, so that we must raise with a $b^\dagger$ or $d^\dagger$ contained in all the R_a. However, this intersection contains only the space-time index, so would-be scalars are projected out of the spectrum. For particles transforming nontrivially under both G and G', the W_a equations require that $\sum_{R_a}(n-\bar{n})/2\equiv 0$, so we must raise with a $b^\dagger$ or $d^\dagger$ contained in none of the R_a. There are no such oscillators, and so we are left with the gauge bosons of SU(9)$\times$SU(9). [There are other massless particles, such as additional U(1)'s and the graviton, but these decouple in the infinite-tension limit, and are irrelevant to our considerations.] As far as tachyons are concerned, the only possible tachyons are those with $m^2=-2/\alpha'$; these come from raising with a single oscillator on the left, and none on the right. However, the W_a equations require that such a single oscillator be in all of the L_a, but not in G', and no such oscillator exists, so all the tachyons in this sector are projected out. Thus, this string model yields a tachyon-free pure gauge model in the infinite-tension limit.

We turn now to the computation of the β function as determined by the coefficient of $\ln\alpha'$ in the three-point one-loop string amplitude ($\alpha'^{-1/2}$ acts as a momentum cutoff). This amplitude is given by the expectation value on a torus of three gluon vertex operators, appropriately integrated over moduli space.

One can show that the terms proportional to $\ln\alpha'$ do in fact have the same structure as the tree-level three-point coupling. In order to compute the overall coefficient, it is sufficient to study the $\epsilon_1\cdot\epsilon_2$ term. The coefficient of this kinematic factor is (leaving off the group-theory factor $\mathrm{Tr}([T^a,T^b]T^c)$ and integrating by parts to eliminate double derivatives of the bosonic Green's function)

$$\mathcal{A}_3=\frac{g^3}{256\pi^7}\alpha'\int_{\mathcal{F}_1}\frac{d^2\tau}{(\mathrm{Im}\tau)^2}\int_T d^2\nu_1 d^2\nu_2(\mathrm{Im}\tau)Z_B(\tau)$$

$$\times\sum_{\alpha,\beta}C^{\alpha}_{\beta}Z_F\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau)\Bigg\{k_1\cdot k_2\left[-\tfrac{1}{16}G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\bar\nu_{12})^2+\dot G_B(\bar\nu_{12})^2+\dot G_B(\bar\nu_{12})^2\right]$$

$$\times[k_1\cdot\epsilon_3\dot G_B(\bar\nu_{13})+k_2\cdot\epsilon_3\dot G_B(\bar\nu_{23})]$$

$$-\dot G_B(\bar\nu_{12})\dot G_B(\bar\nu_{13})\dot G_B(\bar\nu_{23})(k_2\cdot k_3 k_1\epsilon_3-k_1\cdot k_3 k_2\cdot\epsilon_3)$$

$$-\tfrac{1}{64}(k_2\cdot k_3 k_1\cdot\epsilon_3-k_1\cdot k_3 k_2\cdot\epsilon_3)$$

$$\times G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\bar\nu_{12})G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\bar\nu_{13})G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\bar\nu_{23})\Bigg\}$$

$$\times\exp\{-\alpha'[k_1\cdot k_2 G_B(\nu_{12})+k_1\cdot k_3 G_B(\nu_{13})+k_2\cdot k_3 G_B(\nu_{23})]\}$$

$$\times G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu_{12})G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu_{23})G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu_{13}),$$

where G_B and G_F are, respectively, the bosonic and fermionic Green's functions on the torus. The modular parameter is τ while the ν's represent the locations of the vertex operator insertions on the world sheet; ν_3 is fixed to be $i\,\mathrm{Im}\tau$. The fermionic partition function is given by the sum over sectors as discussed earlier with coefficients C^{α}_{β}. The overall normalization has been determined by Polchinski[8] and we follow the Green's-function normalization conventions of Minahan,[3] but with α' explicit. The first right-mover component of a space-boundary-condition vector keeps track of the space-time statistics; we will refer to it as the "spin" component, $\beta_\uparrow$. The component in a position corresponding to the gauge group of interest we label with a subscript G.

String computations are ordinarily on shell, but on shell the three-point amplitude vanishes. In order to extract the renormalization, we will make use of Minahan's modular-invariant off-shell continuation,[3] which relaxes momentum conservation to $(\sum k_i)^2=0$, so that $k_1\cdot k_2+k_1\cdot k_3+k_2\cdot k_3=0$.

Because of the explicit power of α' in front, any contribution to $\mathcal{A}_3$ that survives as $\alpha'\to 0$ must come from a region of the integral which diverges in that limit. Aside from poles whenever two or more ν_i's coincide, the integrand is everywhere finite, so such contributions can arise only from the aforementioned poles or from the divergence of the $\mathrm{Im}\tau$ integral, at its large-$\mathrm{Im}\tau$ end. All contributions in fact arise from that region only.

It is convenient to split up the integral into two contributions: $\mathcal{W}_3$ coming from the regions where $\nu_i\simeq\nu_j$ ("wave-function renormalization") and $\mathcal{V}_3$, coming from the region where all ν's are far apart ("vertex renormalization"). Integration by parts alters this split up; the wave-function renormalization and vertex renormalization do not have independent physical significance. Indeed, it turns out that the integration by parts performed above shifts all of the β-function contributions to the wave-function renormalization term. One can show that the only contribution to the wave-function renormalization for this particular kinematic term comes from the region where $\nu_1\simeq\nu_2$, but neither is close to ν_3.

Let us then consider this contribution. Both G_F and $\dot G_B$ have simple poles in the limit $\nu\to 0$, the former with residue 2, and the latter with residue $-\frac{1}{2}$. Several of the terms are subdominant in α'; dropping these, keeping only the leading pole terms, and performing the ν_2 integration, we obtain

$$\mathcal{W}_3=-\frac{g^3}{256\pi^6}(k_1-k_2)\cdot\epsilon_3\int_{\mathcal{F}_1}\frac{d^2\tau}{(\mathrm{Im}\tau)^2}\int_T d^2\nu_1(\mathrm{Im}\tau)Z_B(\tau)\sum_{\alpha,\beta}C^{\alpha}_{\beta}Z_F\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau)\left[\tfrac{1}{16}G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\bar\nu_{13})^2-\dot G_B(\bar\nu_{13})^2\right]$$

$$\times G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu_{13})+G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu_{13})\exp[\alpha' k_1\cdot k_2 G_B(\nu_{13})]\ .$$

The terms in which we are interested—those that go as $\ln(\alpha' k_i\cdot k_j)$—arise only from the large-$\mathrm{Im}\tau$ region. This also means we need retain only contributions which do not include decaying exponentials in $\mathrm{Im}\nu$.

In this limit we may expand the integrand, retaining only terms which are not exponentially suppressed. The first simplification this brings is the disappearance of all sectors other than the W_0 sector. This comes about because the other sectors have no massless particles transforming nontrivially under the gauge group, and so will give only exponentially suppressed contributions to the integral.

Defining $\hat{q}=e^{-2\pi\,\mathrm{Im}\tau}$, we want to expand the Green's functions and partition function as $\hat{q}\to 0$. In this limit $(\alpha_\uparrow=\alpha_G=\frac{1}{2})$,

$$\dot{G}_B(\bar{v})\to\frac{i\pi}{2}\left[1+\frac{2\,\mathrm{Im}v}{\mathrm{Im}\tau}\right],$$

$$G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\bar{v})\to -4\pi i\left(e^{\pi i\,\mathrm{Re}v}e^{\pi\,\mathrm{Im}v}-\hat{q}^{\,1/2}e^{-\pi i\,\mathrm{Re}\tau}e^{-\pi i\,\mathrm{Re}v}e^{-\pi\,\mathrm{Im}v}\cos 2\pi\beta_\uparrow\right),$$

$$Z_R(\tau)\to\hat{q}^{\,-1/2}e^{\pi i\,\mathrm{Re}\tau}\left[1-2\hat{q}^{\,1/2}e^{-\pi i\,\mathrm{Re}\tau}\sum_{i=1}^{10}\cos 2\pi\beta_{Ri}\right]$$

with similar expansions for the left movers, and where the full partition function in any sector is $Z_L(\tau)Z_R(\tau)/\mathrm{Im}\tau$.

Let us now consider separately the contributions of the left and right movers (including contributions from the partition function). What is the effect of summing over time-boundary conditions? (The space-boundary conditions, as fixed above, are given by W_0.) In this sector, the coefficients are simply given by $-(\cos 2\pi\beta_\uparrow)/\mathcal{N}$, where $\mathcal{N}$ normalizes the generalized GSO projection properly. In our model, $\mathcal{N}=48$. For the left movers, we may group the time-boundary conditions into triplets $(W+0W_1, W+1W_1, W+2W_1)$. Each of these triplets gives the same coefficient $C_\beta^{W_0}$ since W_1 has a zero in the spin component. The complex exponentials $e^{-2\pi i\beta_G}$ and $e^{-4\pi i\beta_G}$ are simply the cube roots of either 1 or -1, and so will vanish when summed over all time-boundary conditions; only terms which give an "interference"—terms independent of β_G—can survive. Furthermore, we need not consider contributions which contain no powers of $\hat{q}^{\,1/2}$ coming from the propagators, since these are suppressed by exponentials of the $\mathrm{Im}v_i$. Thus the terms we are left with contain one power of $\hat{q}^{\,1/2}$ from the partition function, and one from the Green's functions. Note that the sum over time-boundary conditions also eliminates contributions coming from oscillators in the partition function that do not transform under the gauge group, since such terms do not give rise to an interference. This leaves us with just $32\pi^2N_c$, where N_c is the number of colors ($N_c=9$ for the model discussed earlier).

For the right movers, note that there are an equal number of sectors with coefficient $+1$ and -1, so only terms with an even number of powers of $\cos 2\pi\beta_\uparrow$ (including those from the coefficient) will survive. The other terms correspond to scalars which have been projected out. The two right-mover terms then simplify as follows:

$$\tfrac{1}{16}Z_R(\tau)G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\bar{v}_{13})^2\to 2\pi^2\cos 2\pi\beta_\uparrow,$$

$$-Z_R\dot{G}_B(\bar{v}_{13})^2\to -\frac{\pi^2}{2}\left[1+\frac{2\,\mathrm{Im}v_{13}}{\mathrm{Im}\tau}\right]^2\cos 2\pi\beta_\uparrow.$$

The $\mathrm{Re}\tau$ and $\mathrm{Re}v_1$ integrations are now trivial; defining $x=\mathrm{Im}v_1/\mathrm{Im}\tau$ we are left with

$$\mathcal{W}_3=\frac{2g^3N_c}{32\pi^2}(k_1-k_2)\cdot\epsilon_3\sum_\beta\frac{\cos^2 2\pi\beta_\uparrow}{\mathcal{N}}\int_0^1 dx[4-(2x-1)^2]\int^\infty\frac{d\,\mathrm{Im}\tau}{\mathrm{Im}\tau}e^{\pi\alpha' k_1\cdot k_2(2-x)(1-x)\mathrm{Im}\tau}$$
$$=\left[-\frac{11N_c}{3}\right]\frac{2g^3}{32\pi^2}\ln(\alpha' k_1\cdot k_2)(k_1-k_2)\cdot\epsilon_3+\cdots.$$

When we perform the same expansions in the vertex renormalization piece, we find (after rescaling $\mathrm{Im}v$ by $\mathrm{Im}\tau$) by counting powers of $\mathrm{Im}\tau$ that there is no integral of the form leading to a $\ln\alpha'$ term, which is necessary to generate a contribution to the β function.

The only contribution thus comes from the wave-function renormalization; comparing with the coefficient of the same kinematic term in the tree-level coupling, $2g\,\mathrm{Tr}([T^a,T^b]T^c)$, and noting that $1/\alpha'$ is the square of the momentum cutoff, we find the well-known result $\beta(g)=-(11N_c/3)(g^3/16\pi^2)$.

One loop-corrections to scattering amplitudes in a non-Abelian gauge theory are given by simple generalizations of the three-point amplitude, and their computation proceeds along the lines of the β-function computation presented here. Massless fermions transforming in the fundamental representation of the gauge group can also be incorporated, through appropriate modification of the string model. The characteristic infrared divergences of amplitudes in gauge theories can be handled[2] using the string-compatible dimensional regularization scheme of Ref. 4. The computation we have performed demonstrates the feasibility of computing such one-loop corrections to scattering amplitudes using string technology. We believe this approach will make such computations far more tractable than does the standard technology of Feynman diagrams.

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