Tachyons and perturbative unitarity

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The Cutkosky rules are generalized to include tachyons. A consequence is that Lorentz-invariant interacting theories which possess tachyons cannot obey even the weakest possible form of unitarity beyond the tree level. The problem (although not the cutting rules) is shown to extend to bosonic string theory. Thus unitarity cannot be used to determine the range of modular integration in bosonic string loop amplitudes.

I. INTRODUCTION

In the evolution of bosonic string theory unitarity has been repeatedly invoked to determine features of the perturbative S matrix. The most recent case of this is the group-theoretic approach of Neveu and West,[1] but the particle is as old as string theory itself. To cite a few examples, (a) factorization on the poles of n-tachyon tree amplitudes determines the vertex functions for particles of higher mass;[2] (b) the critical dimension is required in order that states of negative norm should decouple;[3] (c) Chan-Paton factors are restricted;[4] (d) twisted open-string loops are required;[5,6] and (e) closed-string poles in open-string loop amplitudes require the inclusion of closed strings in the Hilbert space.[7] Perhaps the most bizarre instance of this practice arises in determining the range of parameter integration for loop amplitudes in the operator formalism. The procedure is first to compute an *Ansätz* for the amplitude using Feynman's tree theorem.[8] This method *guarantees* unitarity in theories which are both local and limited to a finite number of fields but it can fail when either property is absent as is the case for string theory. Upon checking the tree theorem *Ansätze* for unitarity it is found that while the open-string loops are correct, closed-string loops require modification. This modification, which takes the form of restricting the range of modular integration to the so-called "fundamental region," is also determined by unitarity.[9,10] The result agrees with the functional formalism of Polyakov[11] and with string field theory,[12] but the derivation leaves something to be desired.

That the one-loop amplitude can be partially determined by unitarity is especially curious in view of the fact that bosonic string theory is *not* actually unitary at one loop. This is because it has tachyons. As we will show, the presence of tachyons in an interacting theory spoils perturbative unitarity beyond tree order. Of course no one takes bosonic string theory seriously. And insofar as the theory exists at all, the range of integration inferred from unitarity is correct in the sense that it agrees with string field theory. Our motive here is not to question the answer but rather to highlight the role of string field theory in justifying it. Even for superstrings—which are free of tachyons—perturbative unitarity still breaks down above the threshold for massive particle creation. Thus we feel string field theory to be a generally superior method of defining (but not, of course, computing) the perturbative S matrix.

In Sec. II we discuss what is meant by the tachyonic S matrix and we identify the weakest sort of unitarity it might usefully possess. In Sec. III the Cutkosky rules are generalized to include tachyons. It is immediately obvious from the result that tachyonic loop diagrams cannot possess even the weak sort of unitarity discussed in Sec. II. This point is illustrated in Sec. IV with an example from tachyonic ϕ^4 theory. Section V extends the result to closed bosonic string theory. Our conclusions comprise Sec. VI. In examples from point-particle theory we shall always specialize to scalars. Our metric is spacelike and the mass shell of the tachyon is $p^2=m^2$.

II. TACHYONIC UNITARITY

The appearance of tachyons in an interacting quantum field theory wreaks far-reaching and often unpleasant changes in familiar theoretical structures. Since all particles, including the tachyon itself, are unstable against decay into tachyons, one might reasonably wonder whether scattering theory makes sense. The correct answer is that it does not. Thus there is really no S matrix at all, much less a unitary one. However, while this fact is technically beyond challenge, abandoning the subject on these grounds leaves one with the feeling that a more tolerant attitude might permit us to at least partially define a tachyonic S matrix. Requiring this to obey some form of "unitarity"—possibly a very weak one—might then serve to justify the range of parameter integration in bosonic string theory. Since the result of this paper is that one *cannot* do this, we shall have to be very generous about overlooking the various ill-defined expressions which are bound to arise in formally constructing a tachyonic S matrix. Our program in this section is first to give the standard prescription for generating the diagrams which comprise perturbative tachyonic scattering theory. Next we exclude the ill-defined ones and proceed to formulate a weak notion of unitarity on those which remain. This entails precisely specifying the allowed energies and momenta of single-particle states. We close with an important caveat concerning the difference between perturbative unitarity, which is what we are exploring, and the sort of nonperturbative unitarity which would pertain if a nontachyonic vacuum could be found.

What is usually meant by the tachyonic S matrix is the perturbative expansion of the Lehmann-Symanzik-Zimmerman (LSZ) reduction formula. In point-particle theories this corresponds to analytically continuing the conventional Feynman rules to negative mass squared. It also reproduces the Feynman rules which are actually used in bosonic string theory.[12] Note that one cannot *derive* the LSZ formula when tachyons are present, it must be assumed. This is because the interpolating field does not become weakly free at asymptotic times. In fact, the inner product between two states whose wave functionals are free eigenstates at fixed early and late times does not possess a smooth infinite-time limit. This is just another manifestation of the basic inconsistency of tachyonic scattering theory to which we have already alluded.

Since the limit upon which LSZ reduction is based actually fails to exist when tachyons are present, it has to be expected that certain of the resulting diagrams will be ill-defined. These are the ones which include tachyonic loop corrections to external legs; Fig. 1 is an example. The problem is that the loop induces an imaginary mass shift which cannot be renormalized away. When the external momenta go on mass shell the propagator connecting the loop to the rest of the diagram diverges. This is inevitable for an unstable particle and one must remember that *all* particles which interact with tachyons are unstable. The right thing to do upon noting this sort of thing in a nontachyonic theory is to let the unstable particle's mass shift into the complex plane and to stop including it as an asymptotic scattering state. Since all particles are unstable in an interacting theory which contains tachyons, we ought properly to reject the whole tachyonic S matrix as divergent nonsense. However, in the spirit of searching for a weak sense of unitarity we shall pursue a different course: to simply ignore any ill-defined graphs.

Although the decomposition of an amplitude into diagrams is unphysical in that it depends upon the choice of field variable and gauge, our procedure of ignoring ill-defined diagrams is not totally cavalier. This is because one can derive cutting rules which, for nontachyonic theories, permit us to express the absorptive part of each diagram *separately* as a sum over allowed intermediate states of lower-order processes corresponding to cuttings of the original diagram. Thus a sort of unitarity applies diagram by diagram. (In gauge theories this is only true for the sum of the ghost and gauge boson loops.) It is this relation, for well-defined diagrams, which might still be true in a tachyonic theory. Of course, it would have to break down for any diagram which could be cut into an ill-defined graph (see Fig. 2), but at least it has a chance of holding for those which cannot (see Fig. 3). Further, if the tachyonic S matrix could be shown to possess even this weak form of unitarity it would be sufficient to justify the range of parameter integration in bosonic string loops.

The key point we have yet to define is the range over which one sums "allowed intermediate states" in attempting to check the unitarity of a diagram. There is no very good answer for tachyons. The states are labeled by momenta satisfying $k^2=m^2$, which are spacelike. To preserve Lorentz invariance we must therefore include negative-energy as well as positive-energy states. Aside from its aesthetic appeal, this is probably necessary for unitarity. To see why, note that the amplitude T is Lorentz invariant.[13] Hence, so too is its absorptive part. Thus if anything such as the usual unitarity relation,

$$-i\langle \alpha | T-T^{\dagger} | \beta\rangle = \sum_{\gamma} \frac{1}{\langle \gamma | \gamma \rangle} \langle \alpha | T | \gamma \rangle \langle \gamma | T^{\dagger} | \beta \rangle \,, \tag{2.1}$$

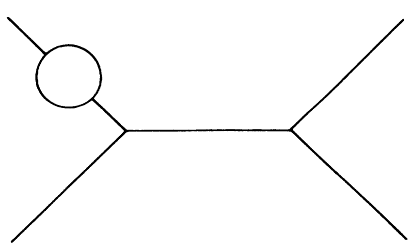

FIG. 1. Divergent tachyonic loop correction to an external leg.

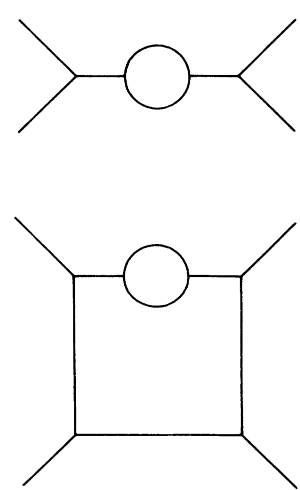

FIG. 2. Diagrams which can be cut to give ill-defined graphs.

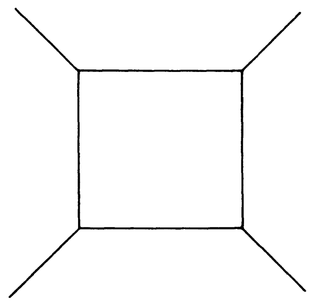

FIG. 3. Diagram which does not cut into ill-defined graphs.

is true, it must be that the sum over states is also Lorentz invariant.

Note that in order for the energy of a tachyon to be real, its $(D-1)$-momentum must obey

$$\mathbf{k}\cdot\mathbf{k}\geq m^2 . \tag{2.2}$$

We do not include complex energies (or momenta) for the same reasons as with ordinary particles, i.e., because (1) fields with complex wave vectors grow exponentially at infinity, and (2) complex energies must be excluded if even the *free* theory is to be unitary.

Although we have adduced strong arguments to support our choice for the single tachyon sector of the space of states, a compelling objection might be raised. Our prescription requires the usual creation operator to do double duty as both the creator of positive-energy particles and the annihilator of negative-energy ones. A similar arrangement must hold for the usual annihilation operator. Thinking canonically, it is very difficult to conceive of a vacuum state which could be consistent with this sort of thing. This must be regarded as yet another indication that no really satisfactory description of tachyons is possible.

Before proceeding to the tachyonic cutting rules we should emphasize that it is perturbative unitarity about a tachyonic vacuum which we are discussing. The theory might be quite unitary when formulated around a different vacuum. Consider, for example, the symmetry-breaking Lagrangian

$$\mathcal{L}=-\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi+\tfrac{1}{2}m^2\phi^2-\frac{1}{4!}\lambda\phi^4 . \tag{2.3}$$

The corresponding Hamiltonian is bounded below and at least naively Hermitian. Hence the (renormalized) evolution operator $\exp(-iHt)$ should be unitary. Each of these statements is true, of course, but they do not imply perturbative unitarity of the tachyonic S matrix. This is because they ignore the role of the space of states. Unitarity of the S matrix means that the asymptotic "in" and "out" Fock spaces are unitarily equivalent. While this is true for the scattering theories based upon vacua $\phi_0=\pm(6m^2/\lambda)^{1/2}$, it is certainly not the case when the tachyonic vacuum $\phi_0=0$ is used. Although evolution is still "unitary" in a different Hilbert space when interactions are included, incoming tachyonic Fock space states are not generally carried to the outgoing tachyonic Fock space.

III. TACHYONIC CUTTING RULES

In this section we establish the fundamental result of the paper, namely, that even the weak sense of unitarity that we defined in Sec. II cannot be realized beyond tree order for interacting theories which possess tachyons. This is done by deriving a set of cutting rules for computing the absorptive parts of tachyonic diagrams following the method of Veltman.[14] It turns out that "cutting" is actually a misnomer since for tachyons the "cut" propagators (1) have support for momenta which are not on mass shell and (2) do *not* have support on the negative-energy half of the mass shell. An immediate consequence is that tachyonic loops cannot be unitary. This section closes with a discussion of this point.

The method of Veltman involves an underlining operation that acts on a position-space diagram $F(x_1,\ldots;y_1,\ldots;z_1,\ldots)$. One obtains the scattering amplitude from F by Fourier-transforming the incoming (x_i) and outgoing (z_i) points and integrating the interaction points (y_i):

$$iT(p_1p_2\cdots\to q_1q_2\cdots)=\int d^Dx_1e^{+ip_1\cdot x_1}\cdots\int d^Dy_1\cdots\int d^Dz_1e^{-iq_1\cdot z_1}\cdots F(x_1,\ldots;y_1,\ldots;z_1,\ldots) . \tag{3.1}$$

If the diagram described by F involves a line joining the points u_i and u_j, then F contains a factor of the propagator $\Delta(u_i-u_j)$ which for tachyons is given by

$$\Delta(x)\equiv\int\frac{d^Dk}{(2\pi)^D}\frac{-ie^{ik\cdot x}}{k^2-m^2-i\epsilon} . \tag{3.2}$$

The action of underlining replaces this propagator by one of four terms, depending upon which, if any, of the points u_i or u_j is underlined. In enumerating the possibilities it is convenient to suppress the coordinates somewhat:

$$\Delta_{ij}\equiv\Delta(u_i-u_j) , \tag{3.3a}$$

$$\theta_{ij}\equiv\theta(u_i^0-u_j^0) , \tag{3.3b}$$

The four cases are

$$\Delta_{ij} \rightarrow \Delta_{ij} \ , \tag{3.4a}$$

$$\Delta_{i\underline{j}} \rightarrow \theta_{ij}\Delta_{ij} + \theta_{ji}\Delta_{ij}^{*} \ , \tag{3.4b}$$

$$\Delta_{\underline{i}j} \rightarrow \theta_{ij}\Delta_{ij}^{*} + \theta_{ji}\Delta_{ij} \ , \tag{3.4c}$$

$$\Delta_{\underline{ij}} \rightarrow \Delta_{ij}^{*} \ . \tag{3.4d}$$

Cases (3.4b) and (3.4c) look different from the usual prescriptions Δ^{+} and Δ^{-} but are in fact equivalent to the latter for nontachyonic particles.

The largest time equation follows immediately from these underlining conventions. That is, if the coordinate u happens to have the largest 0 component of all the arguments of F, then

$$F(u) - F(\underline{u}) = 0 \ . \tag{3.5}$$

Since there is always a largest time, the sum over all underlinings of F must vanish:

$$\sum_{\text{underlinings}} (-1)^{\#} F = 0 \ , \tag{3.6}$$

where # is the number of coordinates underlined. [If the largest time is not unique then (3.5) is not true, however, (3.7) still holds because (3.6) holds except on a set of measure zero.] Integrating this equation in the manner of (3.1) and specializing to scalars we obtain

$$2\,\text{Im}(T) = \sum_{\text{underlinings}}{}' \ (-1)^{\#} \int d^{D}x_1 e^{ip_1 \cdot x_1} \cdots F \ , \tag{3.7}$$

where the sum runs over underlinings of one or more, but not all, of the arguments of F (and we have used the fact that there is a factor of i for each vertex in F). This is the usual result, however, now a "cut" line, represented by Fig. 4 with the momentum positive into the underlined vertex (denoted by a circle), carries with it a modified propagator which is the Fourier transform of (3.4b):

$$\tilde{\Delta}_{\text{cut}}(k) = \tilde{\Delta}_A(k) + \tilde{\Delta}_B(k) \ , \tag{3.8a}$$

$$\tilde{\Delta}_A(k) = 2\pi\theta(k^0)\delta(k^2 - m^2) \ , \tag{3.8b}$$

$$\tilde{\Delta}_B(k) = \frac{k^0}{k^2 - m^2} \frac{\theta(m^2 - \mathbf{k}^2)}{\sqrt{m^2 - \mathbf{k}^2}} \ . \tag{3.8c}$$

$\tilde{\Delta}_A$ is familiar from the positive-mass-squared situation. It obviously decomposes diagrams into sums over cuttings; just as obviously, it can have trouble describing intermediate states which contain both positive-energy and negative-energy particles. The second term, $\tilde{\Delta}_B$, has no conventional analog. Since it does not even put the cut line on shell, any nonzero contribution from this term endangers unitarity.[15]

It is easy to see that nothing goes very wrong at tree order. The reason is that the $\tilde{\Delta}_B$'s from conjugate underlinings cancel, while the noncovariant $\theta(\pm k^0)$ terms in $\tilde{\Delta}_A$'s add to give unity. A simple example is depicted in Fig. 5. The first term on the right-hand side contributes a factor of

$$\lambda^2[\tilde{\Delta}_A(p_1 + p_2) + \tilde{\Delta}_B(p_1 + p_2)] \ , \tag{3.9a}$$

while the second gives

$$\lambda^2[\tilde{\Delta}_A(-p_1 - p_2) + \tilde{\Delta}_B(-p_1 - p_2)] \ . \tag{3.9b}$$

The sum obviously reproduces the result of direct calculation:

$$2\,\text{Im}\left[\frac{-\lambda^2}{s + m^2 + i\epsilon}\right] = \lambda^2 2\pi\delta(s + m^2) \ . \tag{3.10}$$

It also agrees, up to a factor of 2, with the sum over states. The graphs which contribute are depicted in Fig. 6. If we normalize tachyonic states in the conventional way,

$$\langle \mathbf{k}, E \mid \mathbf{k}', E' \rangle = 2 \mid E \mid \theta(EE')(2\pi)^{D-1}\delta^{D-1}(\mathbf{k} - \mathbf{k}') \ , \tag{3.11}$$

then each of these terms separately contributes a factor of (3.10) to the absorptive part and we seem to have a minor violation of unitarity.

All tachyonic theories violate unitarity at tree order by this same factor of 2. The reason is that the cutting rules which would be consistent with the sum over the intermediate tachyonic-state space require cut lines such as Fig. 4 to acquire a factor of

$$2\pi\delta(k^2 - m^2) \ . \tag{3.12}$$

At tree order the actual cutting rules (3.8) do produce this factor, but only in the sum of two conjugate lines. Hence tree diagrams have exactly half the absorptive part they should according to the sum over states. This cannot be compensated by modifying the single state normalization (3.11) because the conjugate diagrams typically involve different numbers of intermediate particles (one and five for Fig. 6). Nor can one simply discard disconnected contributions. This is because certain diagrams obtain their *entire* absorptive part from disconnected processes.

FIG. 4. Cut line connecting a vertex which is not underlined with one that is.

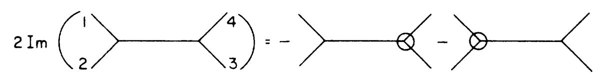

FIG. 5. Graphical expansion for the absorptive part of the s-channel tree amplitude in tachyonic ϕ^3 theory.

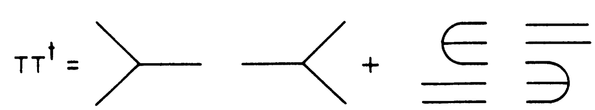

FIG. 6. Processes contributing to the sum over states for the tachyonic tree of Fig. 5.

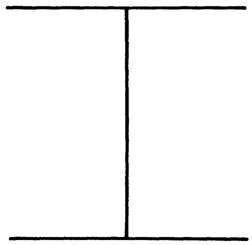

FIG. 7. The t-channel tree graph of tachyonic ϕ^3 theory.

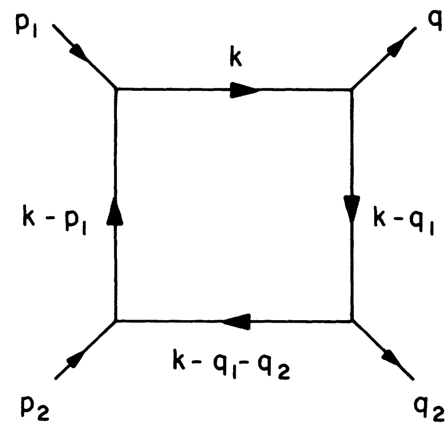

FIG. 9. A four-point, 1PI loop in tachyonic ϕ^3 theory.

For an example, consider the t-channel graph of Fig. 7 and the corresponding sum over states in Fig. 8. (Note that the t-channel-exchange propagator *can* go on mass shell, even in the s channel, for tachyons.) The factor of 2 is probably related to the fact that the same free oscillators must both create and destroy tachyons, as explained in Sec. II. One might plausibly argue that the whole sum over states, therefore, ought to be diluted by a factor of one-half. Even if this prescription is rejected, the discrepancy in unitarity is still very slight at tree order. Really impressive violations occur at one loop.

Two general phenomena frustrate unitarity beyond tree order.

(1) The $\tilde{\Delta}_B$ terms always contribute.

(2) The positive-energy θ functions in $\tilde{\Delta}_A$ prevent the absorptive part from capturing certain processes in the sum over states involving positive-energy and negative-energy intermediate states.

The first problem is really a consequence of the genesis of the $\tilde{\Delta}_B$ terms: they come from imaginary energy poles of the propagator. These are prevented from contributing at tree order by the simple fact that all tree momenta are fixed and real. In loops certain momenta are integrated. Regarding the integrand as meromorphic function, it is clear that complex poles can contribute to the absorptive part. Of course, the same might be said about unstable particle poles in a theory without tachyons. The difference is that unstable particle poles lie *off* the physical sheet[14] whereas the $\tilde{\Delta}_B$ poles must lie *on* it. This is because they are connected to the real poles of $\tilde{\Delta}_A$ by continuous variation of k. In fact the contribution of either $\tilde{\Delta}_A$ or $\tilde{\Delta}_B$ alone would not give a Lorentz-invariant absorptive part. A somewhat similar effect has been discussed by Coleman in relation to unstable ghost particles.[16]

Another was of seeing the first problem, which also illustrates the second, is that conjugate diagrams no longer add to give (3.12) on cut lines. This is because loops connect vertices in more than one way. Thus conjugation generally affects *products* of propagators. To see this, consider the absorptive part of the diagram in Fig. 9. The only terms for which just the propagators carrying momenta k and $k-p_1$ are cut appear in Fig. 10. Up to some factors they contribute

$$\int \frac{d^Dk}{(2\pi)^D}\tilde{\Delta}_{\text{cut}}(-k)\tilde{\Delta}(k-q_1)\times\tilde{\Delta}(k-q_1-q_2)\tilde{\Delta}_{\text{cut}}(k-p_1) \quad (3.13a)$$

and

$$\int \frac{d^Dk}{(2\pi)^D}\tilde{\Delta}_{\text{cut}}(k)\tilde{\Delta}^*(k-q_1)\times\tilde{\Delta}^*(k-q_1-q_2)\tilde{\Delta}_{\text{cut}}(-k+p_1)\ . \quad (3.13b)$$

The $\tilde{\Delta}_B$ terms no longer cancel, they add. Further, although the graphs of Fig. 10 can represent part of the sum over states depicted in Fig. 11, they are incapable of representing the cases

$$k^0>0,\ \ p_1^0-k^0<0\ , \quad (3.14a)$$

$$k^0<0,\ \ p_1^0-k^0>0\ . \quad (3.14b)$$

Although this is a special case it is always possible to shift to a frame in which the intermediate states emerging from a given vertex must have opposite signs of the energy. Thus unitarity, even in the weak sense described in Sec. II, is well and truly gone at one loop.

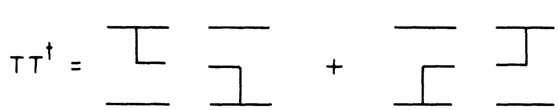

FIG. 8. Processes contributing to the sum over states for the tachyonic tree of Fig. 7.

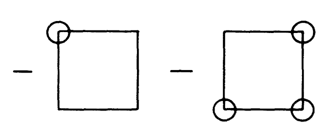

FIG. 10. Part of the graphical expansion for the absorptive part of Fig. 9. Shown are terms in which the k and $k-p_1$ lines are cut.

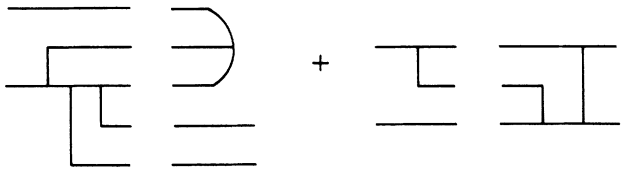

FIG. 11. Processes contributing to the sum over states corresponding to Fig. 10.

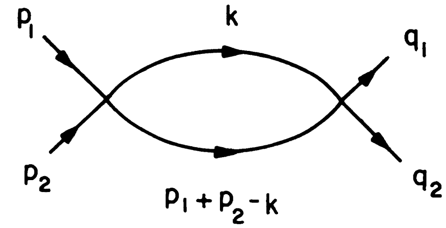

FIG. 12. A four-point loop in tachyonic ϕ^3 theory.

IV. A SIMPLE EXAMPLE

The two problems identified at the end of the preceding section invalidate unitarity in any tachyonic loop on quite general grounds. However, we feel there is merit in actually witnessing them arise in a specific diagram. The ϕ^3 box diagram (Fig. 9) is proverbially unwieldy,[17] so we have chosen instead the ϕ^4 loop of Fig. 12 in $D=4$. We first compute its absorptive part directly from the full answer. This is compared with the sum over states and shown to violate unitarity. The discrepancy is then analyzed by applying out cutting rules.

The diagram in Fig. 12 contributes the following expression to the amplitude:

$$iT^{1\,\text{loop}}(p_1p_1\rightarrow q_1q_2)=(2\pi)^D\delta^D(p_1+p_2-q_1-q_2)\tfrac{1}{2}(-i\lambda)^2\int\frac{d^Dk}{(2\pi)^D}\frac{-i}{k^2-m^2-i\epsilon}\,\frac{-i}{(p_1+p_2-k)^2-m^2-i\epsilon}\;. \tag{4.1}$$

This is straightforward to evaluate using dimensional regularization. Defining the amplitude A by $iT=i(2\pi)^D\delta^D(p_1+p_2-q_1-q_2)A$, the answer is

$$A^{1\,\text{loop}}=\frac{\lambda^2}{2^{D+1}\pi^{D/2}}\Gamma(2-D/2)\int_0^1 d\alpha[-m^2-\alpha(1-\alpha)s-i\epsilon]^{D/2-2}\;, \tag{4.2}$$

where, as usual, $s=-(p_1+p_2)^2$. Although the emissive part diverges for $D=4$, the absorptive part is finite and equal to

$$2\,\text{Im}(A^{1\,\text{loop}})=\text{Im}\left[\frac{-\lambda^2}{16\pi^2}\int_0^1 d\alpha\ln[-m^2-\alpha(1-\alpha)s-i\epsilon]\right] \tag{4.3a}$$

$$=\frac{\lambda^2}{16\pi}[\theta(s+4m^2)+\theta(-s-4m^2)(1-\sqrt{1+4m^2/s}\,)]\;. \tag{4.3b}$$

Note that because of the tachyonic mass-shell condition, there is no region of s for which the absorptive part vanishes.

It is simple to show that (4.3) does not agree with the sum over states. The two processes which contribute are represented in Fig. 13. Since they each give the same factor and the whole result must be divided by two for tree unitarity (cf. Sec. III), the net contribution is

$$\langle q_1q_2\,|\,T^{\text{tree}}(T^{\text{tree}})^\dagger\,|\,p_1p_2\rangle=\tfrac{1}{2}\lambda^2\int\frac{d^Dk_1}{(2\pi)^{D-1}}\delta(k_1^2-m^2)\int\frac{d^Dk_2}{(2\pi)^{D-1}}\delta(k_2^2-m^2)(2\pi)^D\delta^D(k_1+k_2-q_1-q_2)$$
$$\times(2\pi)^D\delta^D(k_1+k_2-p_2-p_2) \tag{4.4a}$$

$$=[(2\pi)^D\delta^D(p_1+p_2-q_1-q_2)\tfrac{1}{2}\lambda^2(2\pi)^{2-D}]\int d^Dk\,\delta(k^2-m^2)\delta[(k-p_1+p_2)^2-m^2]\;. \tag{4.4b}$$

Making the change of variables

$$k=p/2+l\;, \tag{4.5}$$

$$p\equiv p_1+p_2\;, \tag{4.6}$$

the integral in (4.4b) becomes

$$\int d^D l \, \delta(2p\cdot l)\delta(l^2 - s/4 - m^2) \,. \tag{4.7}$$

The product of the δ functions in (4.7) has support on the intersection of a hyperboloid with the hyperplane orthogonal to p. For $s>0$ (p timelike) the intersection is a $(D-2)$-sphere. We can choose the frame in which $p=(\sqrt{s},\mathbf{0})$ to obtain

$$(4.7) = \frac{1}{2\sqrt{s}}\frac{1}{2}(s/4+m^2)^{(D-3)/2}\int d^{D-2}\hat{l} \,. \tag{4.8}$$

For $s<0$ (p spacelike) the intersection is a $(D-2)$-hyperboloid. Note that in this case the support is on a noncompact region, so the sum over states *diverges*. Choosing a frame in which $p=(0,\sqrt{-s},0,\ldots,0)$ we find

$$(4.7) = \frac{1}{2\sqrt{-s}}\left[\int_0^\infty dv\, v^{D-3}\frac{\theta(v^2-s/4-m^2)}{(v^2-s/4-m^2)^{1/2}}\right]\int d^{D-3}\hat{v} \,. \tag{4.9}$$

Combining the two cases and specializing to $D=4$ we obtain

$$T^{\text{tree}}(T^{\text{tree}})^\dagger = (2\pi)^4\delta^4(p_1+p_2-q_1-q_2)\frac{\lambda^2}{16\pi}\left[\theta(s)\sqrt{1+4m^2/s}+\theta(-s)\frac{2}{\sqrt{-s}}\int_0^\infty\frac{\theta(v^2-s/4-m^2)v\,dv}{(v^2-s/4-m^2)^{1/2}}\right] . \tag{4.10}$$

Note that for no value of s does this agree with (4.3), even up to a constant factor. For $s<0$ the discrepancy is not even finite.

We now apply the cutting rules of Sec. III with the object of seeing how each of the two problems discussed there manifests itself for this diagram. The graphical expansion of the absorptive part is given in Fig. 14. The corresponding formula is

$$\frac{1}{2}\lambda^2\int\frac{d^Dk}{(2\pi)^D}[\tilde{\Delta}_{\text{cut}}(k)\tilde{\Delta}_{\text{cut}}(p-k)+\tilde{\Delta}_{\text{cut}}(-k)\tilde{\Delta}_{\text{cut}}(-p+k)] \,. \tag{4.11}$$

Since neither $\tilde{\Delta}_{\text{cut}}$ nor either of its two components is Lorentz invariant, it is advantageous to do the calculation in special frames. For $s>0$ the best choice to make is $p=(\sqrt{s},\mathbf{0})$. This causes the $\tilde{\Delta}_A$-$\tilde{\Delta}_B$ cross terms to vanish. Only the first $\tilde{\Delta}_A$-$\tilde{\Delta}_A$ term contributes. The answer for it is

$$\frac{1}{2}\lambda^2\int\frac{d^Dk}{(2\pi)^D}2\pi\theta(k^0)\delta(k^2-m^2)2\pi\theta(\sqrt{s}-k^0)\delta(-s+2\sqrt{s}k^0) = \frac{\lambda^2}{8}\frac{1}{(4\pi)^{(D-3)/2}\Gamma\left[\frac{D-1}{2}\right]} \times\frac{1}{\sqrt{s}}(s/4+m^2)^{(D-3)/2} \,. \tag{4.12}$$

Since this is just the result obtained from the sum over states, the diagram would be unitary if the $\tilde{\Delta}_B$-$\tilde{\Delta}_B$ terms were to cancel. Unfortunately they add. After performing the k^0 and $\hat{k}$ integrations we obtain

$$-\frac{\lambda^2}{4}\frac{1}{(4\pi)^{(D-3)/2}\Gamma\left[\frac{D-1}{2}\right]}\int_0^m dk\frac{k^{D-2}}{2\pi}\frac{1}{(m^2-k^2)^{1/2}(s/4+m^2-k^2)} \,. \tag{4.13a}$$

The final integral is simple to evaluate for $D=4$ and gives

$$(4.13a) = -\frac{\lambda^2}{16\pi}(\sqrt{1+4m^2/s}-1) \,. \tag{4.13b}$$

Addition of (4.12) and (4.13b) reproduces the result (4.3b) for $s>0$. This is an example of how the $\tilde{\Delta}_B$ terms can spoil unitarity.[18]

For $s<0$ we can choose a frame in which $p=(0,\sqrt{-s},0,\ldots,0)$. Since the incoming energy is then zero the energy θ functions in the $\tilde{\Delta}_A$-$\tilde{\Delta}_A$ terms conflict

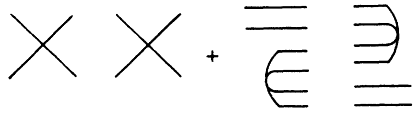

FIG. 13. Processes which contribute to the sum over states for the tachyonic loop of Fig. 12.

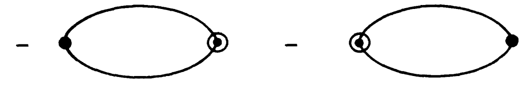

FIG. 14. Graphical expansion for the absorptive part of the tachyonic loop of Fig. 12.

and they make no contribution. Note that it was intermediate states of opposite energy which led to the divergence we found for $TT^\dagger$ in (4.10). This sort of thing can never happen using our cutting rules. On the other hand, one gets contributions from $\tilde{\Delta}_A$-$\tilde{\Delta}_B$ and $\tilde{\Delta}_B$-$\tilde{\Delta}_B$ terms which cannot even be interpreted as a sum over on-shell states.

V. BOSONIC STRING THEORY

The simplest way to argue that bosonic string loops are not unitary would be to invoke the cutting rules of Sec. III. Unfortunately, these do not quite apply to string theory because the infinite number of states can lead to singularities which contribute to the absorptive part. Therefore, we shall instead exploit the close relation which exists between string amplitudes (even superstring ones as it turns out) and those of ϕ^3 theory, where we know how to compute the absorptive part.

Consider the Lagrangian

$$\mathcal{L}=-\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi-\tfrac{1}{2}\mu^2\phi^2-\frac{1}{3!}\lambda\phi^3 . \tag{5.1}$$

It is straightforward to show that the one-loop, one-particle-irreducible (1PI) contribution to the N-particle invariant amplitude is

$$iA(1,\ldots,N)=\tfrac{1}{2}\pi^{D/2}\lambda^N\int_0^1 dt_1\cdots\int_0^1 dt_{N-1}\left[\theta(E)\int_0^\infty d\tau\,\tau^{N-1-D/2}e^{-\tau E}+\theta(-E)e^{i(N-D/2)\pi}\int_0^\infty d\tau\,\tau^{N-1-D/2}e^{\tau E}\right] , \tag{5.2a}$$

$$E\equiv\mu^2-\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}p_i\cdot p_j[\,|t_i-t_j|-(t_i-t_j)^2] . \tag{5.2b}$$

It is apparent that A derives its absorptive part entirely from regions of parameter space where E is negative. Although one can *always* compute Im(A) from (5.2a), a shortcut is often useful. The method is to consider A to be a meromorphic function of the $N(N-3)/2$ independent Mandestam parameters and to define the amplitude on that portion of $\mathbb{C}^{N(N-3)/2}$ for which E is always positive:

$$A(1,\ldots,N)\equiv\tfrac{1}{2}\pi^{D/2}\lambda^N\int_0^1 dt_1\cdots\int_0^1 dt_{N-1}\int_0^\infty d\tau\,\tau^{N-1-D/2}e^{-\tau E} ,\quad \forall p_i\cdot p_j\ni E>0 . \tag{5.3}$$

One then extends the result over all $\mathbb{C}^{N(N-3)/2}$ by analytic continuation. Note that for tachyons ($\mu^2=-m^2$) or massless particles ($\mu^2=0$) there is *no* region of $\mathbb{C}^{N(N-3)/2}$ for which E is everywhere positive. The correct result in this case is obtained by decomposing the dual amplitude into individual channels and then defining the various pieces by analytic continuation from the (different) regions of $\mathbb{C}^{N(N-3)/2}$ over which E is positive. Each piece has the general form (5.3) but with a different range of t_i integration.

Since the exponent E is real for physical momenta, the way (5.3) acquires a nonzero absorptive part is by the integrand becoming singular. This can only happen when E is negative as τ approaches infinity. We therefore learn two things. First, the reason (5.3) fails to exist for $\mu^2\le 0$ is that massless particles or tachyons can always be exchanged in one channel or another so the full 1PI amplitude will always have a nonzero imaginary part—even if not the right one for tachyons. Second, it is only the large-τ limit which affects the absorptive part. Moving the lower limit above zero can drastically alter the emissive part—for example, ultraviolet divergences are centered at $\tau=0$—but the absorptive part is unchanged.

We now turn to the closed bosonic string. The N-tachyon loop is[19]

$$A(1,\ldots,N)=\tfrac{1}{2}(\pi\kappa)^N\int d^2v_1\cdots d^2v_N[\tfrac{1}{2}\mathrm{Im}(v_N)]^{-D/2}\|f(e^{2\pi i v_N})\|^{-2(D-2)}\times e^{4\pi\mathrm{Im}(v_N)}\prod_{i=1}^{N-1}\prod_{j=i+1}^{N}[\chi(e^{2\pi i(v_j-v_i)},e^{2\pi i v_N})]^{p_i\cdot p_j/2} , \tag{5.4a}$$

$$f(w)\equiv\prod_{n=1}^{\infty}(1-w^n) , \tag{5.4b}$$

$$\chi(z,w)\equiv\exp\left[\frac{\ln^2\|z\|}{2\ln\|w\|}\right]\|z^{1/2}-z^{-1/2}\|\left\|\prod_{n=1}^{\infty}\frac{(1-w^nz)(1-w^nz^{-1})}{(1-w^n)^2}\right\| . \tag{5.4c}$$

The range of integration is

$$-\tfrac{1}{2}\le\mathrm{Re}(v_i)\le+\tfrac{1}{2} , \tag{5.5a}$$

$$0\le\mathrm{Im}(v_i)\le\mathrm{Im}(v_N)\ge\{1-[\mathrm{Re}(v_N)]^2\}^{1/2} . \tag{5.5b}$$

This expression is ill-defined for two reasons. First, it is a dual amplitude and therefore the sum of all one-loop Feynman diagrams in string field theory. Some of these, for example, Fig. 1, are ill-defined and must be excluded.

Others, such as the first graph of Fig. 2, cut to give ill-defined graphs and must also be excluded from the discussion of unitarity. In this respect tachyonic string theories are no different from tachyonic point-particle theories. The second problem is that (5.4a) is a real integral, yet the presence of massless particles and tachyons in the theory forces the correct amplitude to have an imaginary part. A consequence is that the integral diverges just as (5.3) does for $\mu^2 \leq 0$. We emphasize that this occurs even for superstring amplitudes owing to the presence of massless particles.

Both problems can be solved by appealing to string field theory. Since any string field theory provides a triangulation of moduli space, each contributing Feynman diagram corresponds to a known portion of the modular integration for the dual amplitude. The correct prescription for excluding reducible graphs from (5.4) is to excise small regions (whose precise shape we need not determine) around the surfaces defined by $v_i = v_j$. We could also define the amplitude by using a field-theory triangulation to divide the modular integration up into portions over which the integrand would remain finite when the Mandelstam variables were confined to different regions of $\mathbb{C}^{N(N-3)/2}$. However, a much simpler alternative is available if only the absorptive part is desired. As with (5.3), the imaginary part of (5.4) derives from regions where the integrand becomes singular. This can happen when two or more source points coincide ($v_i = v_j$) or when the imaginary part of v_N approaches infinity. The first case gives the isolated poles of reducible diagrams. These terms have been excluded in order to obtain a well-defined amplitude. The second case gives the cut structure with which we can test unitarity. It turns out that in this limit the integrand becomes a sum of field theory expressions of the same type as (5.3).

To take the appropriate limit it is useful to change variables:

$$v_k = \alpha_k + i\frac{2}{\pi}\tau t_k \ , \tag{5.6a}$$

$$t_N \equiv 1 \ . \tag{5.6b}$$

The amplitude then becomes

$$A^{1\mathrm{PI}}(1,\ldots,N) = \tfrac{1}{2}\pi^{D/2}(2\kappa)^N \int_0^1 dt_1 \cdots \int_0^1 dt_{N-1} \int_{-1/2}^{1/2} d\alpha_1 \cdots \int_{-1/2}^{1/2} d\alpha_N \Theta(t_1,\ldots,t_{N-1};\alpha_1,\ldots,\alpha_N)$$
$$\times \int_{(\pi/2)(1-\alpha_N^2)^{1/2}}^{\infty} d\tau\, \tau^{N-1-D/2} \| f(e^{2\pi i\alpha_N} e^{-4\tau}) \|^{-2(D-2)} e^{8\tau}$$
$$\times \prod_{i=j}^{N-1} \prod_{j=i+1}^{N} [\chi(e^{2\pi i(\alpha_j-\alpha_i)} e^{-4\tau(t_j-t_i)}, e^{2\pi i\alpha_N} e^{-4\tau})]^{p_i\cdot p_j/2} \ , \tag{5.7}$$

where the function Θ is zero near $t_i = t_j$, $\alpha_i = \alpha_j$ and unity everywhere else. For $t_i \neq t_j$ and τ large the χ's have a particularly simple form:

$$\chi(e^{2\pi i(\alpha_j-\alpha_i)} e^{-4\tau(t_j-t_i)}, e^{2\pi i\alpha_N} e^{-4\tau}) = \exp[-2\tau(t_j-t_i)^2 + 2\tau\,|\,t_j-t_i\,|\,][1+O(e^{-4\tau})] \ . \tag{5.8}$$

Similarly the partition function is just 1 to leading order. Hence the leading term is

$$A^{1\mathrm{PI,\ leading}}(1,\ldots,N) = \tfrac{1}{2}\pi^{D/2}(2\kappa)^N \int_0^1 dt_1 \cdots \int_0^1 dt_{N-1} \int_{-1/2}^{1/2} d\alpha_1 \cdots \int_{-1/2}^{1/2} d\alpha_N$$
$$\times \Theta(t_1,\ldots,t_{N-1};\alpha_1,\ldots,\alpha_N) \int_{(\pi/2)(1-\alpha_N^2)^{1/2}}^{\infty} d\tau\, \tau^{N-1-D/2}$$
$$\times e^{8\tau} \exp\left\{ \tau \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} p_i\cdot p_j [\,|\,t_i - t_j\,| - (t_i-t_j)^2] \right\} . \tag{5.9}$$

The integrand of this expression is just that of (5.3) with $\mu^2 = -8$ and $\lambda = 2\kappa$. Although the ranges of integration differ somewhat this is not significant for the absorptive part. (The lower limit of the τ integration is irrelevant and the effect of Θ can be made arbitrarily small.) Hence the absorptive parts are equal.

It turns out that each of the subdominant terms in the large-τ expansion of the integrand has a similar interpretation but with different particles propagating along the N internal lines.[20] Hence the expansion decomposes the absorptive part of (5.7) into a sum of the form

$$2\,\mathrm{Im}(A^{1\mathrm{PI}}) = \sum_{\substack{\text{particle} \\ \text{on line 1}}} \cdots \sum_{\substack{\text{particle} \\ \text{on line } N}} 2\,\mathrm{Im}(A^{\text{particle loop}}) \ . \tag{5.10}$$

Since our cutting rules apply to point particle theories, we can employ them to evaluate each term on the right-hand side of (5.10). Now although Sec. III dealt with pure tachyon theories, it is clear that a mixed-loop diagram containing even a single tachyon line succumbs to the same problems with unitarity. Barring the possibility that these individual violations of unitarity cancel in the sum we can therefore conclude that the right-hand side of (5.10) does not equal the corresponding sum over states:

$$\sum_{\substack{\text{particle}\\\text{on line 1}}} \cdots \sum_{\substack{\text{particle}\\\text{on line } N}} 2\,\mathrm{Im}(A^{\text{particle loop}}) \neq \sum_{\substack{\text{particle}\\\text{on line 1}}} \cdots \sum_{\substack{\text{particle}\\\text{on line } N}} \sum_{\text{states}} (A^{\text{particle tree}})(A^{\text{particle tree}})^\dagger \;. \tag{5.11}$$

That this is fatal to unitarity becomes apparent upon interchanging the sum over states with the sums over particle types. The latter can be done to recover full string trees:[21]

$$\sum_{\substack{\text{particle}\\\text{on line 1}}} \cdots \sum_{\substack{\text{particle}\\\text{on line } N}} \sum_{\text{states}} (A^{\text{particle tree}})(A^{\text{particle tree}})^\dagger = \sum_{\text{states}} (A^{\text{string tree}})(A^{\text{string tree}})^\dagger \;. \tag{5.12}$$

Hence we conclude:

$$2\,\mathrm{Im}(A^{1\mathrm{PI}}) \neq \sum_{\text{states}} (A^{\text{string tree}})(A^{\text{string tree}})^\dagger \;, \tag{5.13}$$

where we remind the reader that this is contingent upon the assumption that the unitarity violations which are certainly present in individual particle theory loops containing tachyons do not somehow cancel one another.

Even were such cancellations to occur, unitarity would fail to hold in the form in which it has previously been invoked, or indeed in any useful form, because neither the absorptive part nor the sum over states is very well defined. The problem with $\mathrm{Im}(A^{1\mathrm{PI}})$ is that there is no threshold below which only a finite number of terms contribute. This derives from the fact that the Δ_B terms do not force cut lines onto a real mass shell. Hence no diagram with one cut line propagating a tachyon and the other an arbitrary particle can be made to vanish by adjusting the incoming momenta. There are, of course, an infinite number of particle types in string theory and hence an infinite number of Δ_B-anything terms which contribute to $\mathrm{Im}(A^{1\mathrm{PI}})$. It is not known whether the sum even converges, much less what it converges to. This is not as vexing as it would otherwise be because we are similarly unable to evaluate the sum over states. The problem here is that many terms in the series are ill-defined. This is because one must allow intermediate states in which a negative-energy tachyon and a positive-mass-squared particle are exchanged. We saw in Sec. IV that the volume of phase space for this process is not compact, hence the momentum integral tends to diverge. The better ultraviolet behavior of string amplitudes ameliorates this but not, it turns out, enough to prevent divergences from happening for certain incoming momenta.

It is amusing to note that unitarity could actually be invoked to obtain the *wrong* region of modular integration. The usual argument is that the contribution from one modular region gives a finite, nonzero absorptive part. Therefore the result obtained from integrating over all regions must diverge. It is then asserted that this can never agree with the sum over states. Of course, the tree-tree sum over states is always finite for positive mass-squared particles because their phase space is compact. However we have seen that the phase space for *tachyonic* exchanges is noncompact. There is also the problem that negative-energy tachyons can be exchanged with any of an infinite number of positive mass-squared particles. Thus one might actually expect the sum over states to diverge and so welcome a similar divergence in the absorptive part. It is now known from string field theory that this is not the case;[12] our point is merely that the result does not follow from unitarity.

VI. CONCLUSIONS

We have shown that interacting, Lorentz-invariant theories with tachyons obey no perturbative form of unitarity beyond tree level. This statement applies in spite of our having done everything possible to find a weak sense of unitarity which might pertain. Ill-defined diagrams, and those which cut to give ill-defined diagrams, were simply discarded. Furthermore, we overlooked the factor of 2 that spoils tree-level unitarity due to the presence of negative-energy states. Even so, nothing can be done to save unitarity from complete collapse at one loop and beyond.

There are two reasons for this: (1) the tachyonic mass shell is a single-sheet hyperboloid, so Lorentz invariance requires negative-energy as well as positive-energy states, and (2) the tachyonic propagator has imaginary poles in the complex-energy plane. Both problems are apparent from the tachyonic Cutkosky rules we obtained. The "cut" propagator is given by Eq. (3.8):

$$\tilde{\Delta}_{\mathrm{cut}}(k) = \tilde{\Delta}_A(k) + \tilde{\Delta}_B(k) \;, \tag{6.1}$$

where the real poles contribute

$$\tilde{\Delta}_A(k) = 2\pi\theta(k^0)\delta(k^2 - m^2) \tag{6.2}$$

and the imaginary ones give

$$\tilde{\Delta}_B(k) = \theta(m^2 - \mathbf{k}^2)k^0/(k^2 - m^2)\sqrt{m^2 - \mathbf{k}^2} \;. \tag{6.3}$$

On the other hand, agreement with the sum over states would require instead simply

$$\tilde{\Delta}_{\mathrm{cut}}(k) = 2\pi\delta(k^2 - m^2) = \tilde{\Delta}_A(k) + \tilde{\Delta}_A(-k) \;.$$

Owing to the energy θ function in Δ_A, the absorptive part of tachyonic diagrams will miss certain contributions which would arise from exchanges involving negative-energy states. Because of the fact that Δ_B derives from imaginary poles, no contribution from it is interpretable as a sum over physical states.

At tree level the Δ_B terms cancel while the θ function in Δ_A gives rise to a discrepancy of a factor of 2 between the absorptive part and the sum over states which we have agreed to overlook. At loop order both problems occur in ways that cannot be ignored. We saw a vivid example at one loop in tachyonic ϕ^4 theory, where $\mathrm{Im}T$ is finite but $TT^\dagger$ can *diverge*.

The general nature of this problem indicates that bosonic string loop amplitudes must also fail to be unitary, due to the tachyon in the spectrum. Since we could not derive cutting rules for string diagrams (due to the infinite number of mass levels), we instead demonstrated the problem by establishing a correspondence between the absorptive part of a string diagram and those of a collection of cubic point-particle theories. This was sufficient to show that unitarity fails in the contribution from each mass level.

Our conclusion is that unitarity *per se* cannot be used to constrain the form of bosonic string amplitudes beyond tree level, despite a long tradition of belief to the contrary. In particular, the restriction to the fundamental region for the range of modular integration in closed-bosonic-string loops finds its proper justification not in unitarity but in string field theory.[22]

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[10]It was originally believed that the open, nonplanar, orientable loop also requires modification (Ref. 6). This resulted from the use of an old formalism—but the only one available in 1970—in which the intercept α_0 was left unspecified. For $\alpha_0 \neq 1$, only $L_{-1}-L_0$ creates spurious states so only a tiny fraction of the ghosts decouple. Permitting the remaining states to circulate around a nonplanar, orientable loop (but not a planar or nonorientable one) results in infinite overcounting by a simple duality argument. It was therefore necessary to restrict the range of loop integration to restore what was termed "unitarity." This was taken to mean agreement between $2\,\mathrm{Im}T$ and $TT^\dagger$ when the sum over states included the unprojected ghosts. The analogous condition would certaintly pertain in all known gauge theories without the inclusion of Faddeev-Popov ghosts. The reason it does not have to hold for strings is that the theory has an infinite number of particle types. However, if one specializes to $\alpha_0=1$, and projects out all the ghosts, then the tree theorem *Ansatz* is unitary in the strict sense. This was not seen by early workers because they failed to project out the extra states which became unphysical for $\alpha_0=1$ and $D=26$. Ironically, the procedure of modifying the range of integration was developed for open-string loops, where it is not really necessary, before being applied to closed ones, where it is. We speculate that the failure of the tree theorem for closed-string loops is telling us something profound about the structure of closed-string field theory in a covariant gauge.

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[13]Actually, "the amplitude" fails to exist in that it includes ill-defined graphs. However, if we develop perturbation theory using Lorentz-covariant fields and an invariant gauge then each of the resulting diagrams will be separately Lorentz invariant and the argument goes through as stated.

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[18]It is very tempting to try to regard (4.13) as arising from intermediate states of complex energies. If this could be done then the diagram would be "unitary" on the larger space of states. However, aside from its intrinsic ugliness, this proposal turns out to be inconsistent with the on-shell state condition. Even if we permit intermediate states of complex D momenta k_1 and k_2, momentum conservation (in this frame) implies:

$$k_1=(\omega+i\rho,\mathbf{k}+i\mathbf{l}),\quad k_2=(2E-\omega-i\rho,-\mathbf{k}-i\mathbf{l})\ .$$

Requiring $k_i^2=m^2$ then implies $\rho=0$. Thus the $\tilde{\Delta}_B$-$\tilde{\Delta}_B$ terms *cannot* be written as sums over on-shell states of complex energy. This is why any nonzero contribution they make spoils unitarity.

[19]J. H. Schwarz, Phys. Rep. **89**, 223 (1982).

[20]We shall not actually show this in detail but the basic mechanism is easy enough to describe. The higher-order terms from expansions of χ and the partition function give rise to integrals of the same form as (5.9) but containing an extra factor whose general form is

$$f(p_1,\ldots,p_N)g(\alpha_1,\ldots,\alpha_N)e^{-4\tau h(t_1,\ldots,t_{N-1})} .$$

The function f is a simple kinematic factor while g is a product of sines and cosines. The function h can be written as

$$h(t_1,\ldots,t_N)=I+\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}[J_{ij}\,|\,t_j-t_i\,|\,+K_{ij}(1-t_j+t_i)] ,$$

where I, J_{ij}, and K_{ij} are positive integers. It is relatively straightforward to see that the α_i integrations give zero for the absorptive part unless the integers I through K_{ij} are even. Hence the exponent in the first equation is really a multiple of -8τ. Since 8 is the fundamental mass of closed-string theory, it is perhaps not surprising that each of the higher-order terms in the large-τ expansion of the string integrand can be associated with the integrand of an analogous local-field-theory loop diagram in which more massive particles propagate over some or all of the internal lines. The precise correspondence depends upon the values of the integers I through K_{ij}. $I/2$ mass units are added to *all* internal lines; $J_{ij}/2$ additional units are added to those lines containing vertices i and j along the route which does not include vertex N; and $K_{ij}/2$ more mass units are added on the lines connecting i and j through N.

[21]Again we forbear to present the details, but the basic mechanism is familiar to anyone who has seen the local expansion of the Veneziano amplitude in open-string theory:

$$\frac{\Gamma(-s/2-1)\Gamma(-t/2-1)}{\Gamma(-s/2-t/2-2)}=\sum_{n=0}^{\infty}\frac{(t/2+2)\cdots(t/2+n+1)}{n!(n-1-s/2)} .$$

[22]Although unitarity *per se* cannot be used, one might still plausibly argue that one must restrict the range of integration to the fundamental region in order to avoid an infinite overall factor in the loop amplitude.