Structure of the singularities produced by colliding plane waves

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When gravitational plane waves propagating and colliding in an otherwise flat background interact, they produce spacetime singularities. If the colliding waves have parallel (linear) polarizations, the mathematical analysis of the field equations in the interaction region is especially simple. Using the formulation of these field equations previously given by Szekeres, we analyze the asymptotic structure of a general colliding parallel-polarized plane-wave solution near the singularity. We show that the metric is asymptotic to an inhomogeneous Kasner solution as the singularity is approached. We give explicit expressions which relate the asymptotic Kasner exponents along the singularity to the initial data posed along the wave fronts of the incoming, colliding plane waves. It becomes clear from these expressions that for specific choices of initial data the curvature singularity created by the colliding waves degenerates to a coordinate singularity, and that a nonsingular Killing-Cauchy horizon is thereby obtained. Our equations prove that these horizons are unstable in the full nonlinear theory against small but generic perturbations of the initial data, and that in a very precise sense, "generic" initial data always produce all-embracing, spacelike curvature singularities without Killing-Cauchy horizons. We give several examples of exact solutions which illustrate some of the asymptotic singularity structures that are discussed in the paper. In particular, we construct a new family of exact colliding parallel-polarized plane-wave solutions, which create Killing-Cauchy horizons instead of a spacelike curvature singularity. The maximal analytic extension of one of these solutions across its Killing-Cauchy horizon results in a colliding plane-wave spacetime, in which a Schwarzschild black hole is created out of the collision between two plane-symmetric sandwich waves propagating in a cylindrical universe.

I. INTRODUCTION

Gravitational plane waves are among the simplest nontrivial exact solutions to the vacuum Einstein field equations that describe time-varying gravitational fields. Although the existence and the quantitative structure of these solutions have been known since the early days of general relativity,[1] the surprisingly rich qualitative features that they possess were not fully understood until the mid-1960s when Penrose[2] carried out his investigations on their global structure. (In fact, Penrose proposed the plane wave spacetimes as counterexamples to a conjecture in global general relativity, which stated that any spacetime satisfying a sufficiently strong causality condition can be globally embedded in a high-dimensional Minkowski space.) The source of this rich global structure in plane-wave solutions is the focusing effect of gravitational plane waves, which is reviewed, for example, in Refs. 2 and 3, and in the introductory section of Ref. 4.

The presence of both spacelike and timelike nontrivial directions in exact (single) plane-wave solutions makes it possible to study interesting dynamical effects associated with the interaction of plane waves, without destroying the plane symmetry present in the original solutions. Thus, for example, it is not exceedingly difficult to write down solutions to the vacuum field equations that describe collisions of gravitational plane waves. The first such solution was discovered by Khan and Penrose[5] in their attempt to verify Penrose's earlier conjecture[2] that the focusing effect of single plane waves should cause the colliding waves to interact strongly and to eventually produce spacetime singularities. Several other solutions involving similar curvature singularities were obtained by Szekeres,[6] who formulated a general solution for the problem of colliding parallel-(linear)-polarized gravitational plane waves. Later Nutku and Halil[7] obtained a colliding plane-wave solution where the incoming waves had nonparallel linear polarizations; this solution too had a spacelike curvature singularity, similarly to the earlier solutions. The global structure of these early solutions is reviewed in Refs. 8 and 3.

The technique of generating colliding plane-wave solutions by the extension of suitable (but weakly restricted) plane-symmetric solutions to the field equations in the interaction region, pioneered by Khan and Penrose in Ref. 5, proved to be remarkably fertile in subsequent studies on colliding waves. Thus, using this technique, Chandrasekhar and Xanthopoulos[9] obtained many new solutions for both colliding purely gravitational plane waves and for colliding plane waves coupled with matter fields. Other solutions were obtained by the author in Ref. 10, where the Penrose-Khan prescription for generating colliding plane-wave solutions is reviewed, and compared with the direct method of solving the relevant initial-value problem, which, in the case of parallel-polarized waves, was worked out by Szekeres.[6]

A surprising result of the recent work on exact solutions for colliding plane waves was the discovery by Chandrasekhar and Xanthopoulos[11] of a solution, where the collision of the incoming waves (which are nonparallel-polarized) produces a nonsingular Killing-Cauchy horizon instead of a spacelike curvature singularity. The resulting metric can be analytically extended across this horizon to produce a maximal spacetime, whose singularities [which are timelike for the particular (i.e., maximal analytic) extension used by Chandrasekhar and Xanthopoulos[11]] could be avoided by observers traveling on timelike world lines, in striking contrast to the earlier solutions with their all-embracing, spacelike singularities which almost all observers are bound to encounter.[3] The structure, significance, and nongeneric nature of such Killing-Cauchy horizons in colliding plane-wave solutions (and in more general plane-symmetric spacetimes) are discussed extensively in Refs. 11, 4, and 3. The Chandrasekhar-Xanthopoulos[11] solutions are relevant to the subject matter of the present paper, in that they point to a hitherto unsuspected richness in the structure of singularities produced by colliding plane waves. In fact, it was more or less widely believed[11] before the discovery of these solutions, that the singularity structure exhibited by the earlier exact solutions[5,6,7,8] was universal for colliding plane-wave spacetimes. And even after this remarkable discovery, one might be tempted to believe that the unusual structure of Chandrasekhar-Xanthopoulos[11] spacetimes is a result of the nonparallel configuration of the incoming polarizations, and that colliding plane waves with parallel polarizations will always produce singularities with the same global structure as the earlier exact solutions. One of the specific results of this paper is that this is not the case; in particular, in Sec. IV we present examples of exact solutions for colliding parallel-polarized plane waves, which possess nonsingular Killing-Cauchy horizons that are very similar in local structure to the horizons of the Chandrasekhar-Xanthopoulos[11] spacetimes.

The overall purpose of this paper is to explore in detail the structure of the spacetimes that result from the collisions of parallel-polarized plane waves, especially their singularity and Cauchy-horizon structures. The plan of the paper is as follows.

In Sec. II A, we give a very brief review of Szekeres's[6] formulation of the field equations and the characteristic initial-value problem for colliding parallel-linear-polarized plane waves, in the (u,v,x,y) coordinate system which we call "Rosen type" and which is tuned to the plane-symmetry of the spacetime. Our presentation is necessarily brief, and the reader is referred to Ref. 6 for the full mathematical details, or to Ref. 10 for a short outline.

In Sec. II B, we perform a coordinate transformation to a new (α,β,x,y) coordinate system in which the mathematical analysis of the field equations simplifies considerably. Although this coordinate system and its properties were known to Szekeres,[6] he did not make extensive use of them since the coordinates (α,β) are badly behaved on the initial null surfaces where the initial data are posed. However, we will find this new coordinate system very useful both in discussing the general solution of the field equations (Section II B), and in discussing the asymptotic behavior of the resulting spacetime (subsequent sections).

It will become clear in Sec. II B that some kind of singularity is associated with the "surface" $\alpha=0$ in a general colliding plane-wave spacetime. (Note: α is a timelike coordinate which monotonically decreases to zero along the world lines of all observers running into the singularity.) We show in Sec. III A that the spacetime metric asymptotically approaches an inhomogeneous Kasner[12] solution as α approaches zero, where the time coordinate t of the asymptotic Kasner spacetime is monotonically related to α, and the Kasner singularity at $t=0$ corresponds to the singularity at $\alpha=0$. We give explicit expressions which relate the spatially inhomogeneous asymptotic Kasner exponents along the singularity to the initial data posed along the wave fronts of the incoming, colliding plane waves. In general, these exponents depend on β, the spacelike coordinate running along the nontrivial spatial (z) direction in the spacetime.

Our discussion in Sec. III A indicates that for some specific choices of the initial data, the Kasner exponents (either locally, or globally for a finite interval in the spatial coordinate β) may take on the values associated with a degenerate Kasner solution. A degenerate Kasner spacetime is flat, and instead of a spacelike curvature singularity, it possesses a Killing-Cauchy horizon at $t=0$. It is then natural to expect that, when the asymptotic limit of the metric as $\alpha\rightarrow 0$ is a degenerate Kasner solution, our colliding plane-wave spacetime possesses a nonsingular Killing-Cauchy horizon at $\alpha=0$, across which the metric can be extended smoothly. However, to demonstrate this rigorously, we need to study the behavior of the spacetime curvature near $\alpha=0$, and to show that the curvature is indeed well behaved when the metric approaches a degenerate Kasner limit at $\alpha=0$. This would give us information about the asymptotic behavior of the *derivatives* of the metric as $\alpha\rightarrow 0$, complementing our analysis in Sec. III A of the asymptotic behavior of the metric itself. Thus, in Sec. III B, we derive expressions for the Newman-Penrose[13,3] curvature quantities (with respect to the null tetrad that we set up earlier in Sec. II A) in terms of the metric components in the (α,β,x,y) coordinate system. We then read out from these expressions the asymptotic structure of the curvature quantities as $\alpha\rightarrow 0$. This analysis indeed shows that when the metric is asymptotic to a degenerate Kasner solution, the curvature remains well behaved as $\alpha\rightarrow 0$.

We begin Sec. III C by recapitulating the principal conclusion of the analysis of Sec. III B: When the asymptotic limit of the solution is a degenerate Kasner metric, the colliding plane-wave spacetime possesses a Killing-Cauchy horizon (a coordinate singularity) at $\alpha=0$ across which the curvature quantities are finite and well behaved. We note that spacetime can be extended through these horizons in infinitely many different ways; the geometry beyond the horizons cannot be determined from the initial data posed by the incoming, colliding plane waves. We then briefly recall our earlier work in Ref. 4, where we have proved general theorems stating the instability of such Killing-Cauchy horizons in any plane-symmetric spacetime against generic, plane-symmetric perturbations. In the specific case of the

Killing-Cauchy horizons which occur at $\alpha=0$ in our colliding plane-wave solutions, the existence of these instabilities is particularly clear: We discuss how our equations imply (i) that the horizons at $\alpha=0$ are unstable in the full nonlinear theory against small but generic perturbations of the initial data (since such perturbations drive the asymptotic Kasner exponents away from the degenerate values), and (ii) that in a very precise sense, "generic" initial data always produce all-embracing, spacelike spacetime singularities at $\alpha=0$ across which no extension of the metric is possible.

In Sec. IV, we give several examples of exact solutions for colliding parallel-polarized plane waves, which illustrate some of the different asymptotic singularity structures that are discussed in the previous sections. Most of the examples we consider are new, and are discussed here for the first time. However, all of our examples have asymptotic Kasner exponents which are uniformly constant across the whole range of the spatial coordinate β. It seems particularly difficult to write down a full solution, expressible in closed form, which would exhibit a truly inhomogeneous asymptotic structure near the singularity $\alpha=0$. By using the same line of reasoning that we have followed in Ref. 10, we construct exact colliding parallel-polarized plane-wave solutions, which produce Killing-Cauchy horizons at $\alpha=0$ instead of a curvature singularity. The maximal analytic extension of one of these solutions across the horizon produces a colliding plane-wave spacetime with a surprising global structure.

In the concluding section, we briefly list the major results of the paper, and discuss some suggestions and plans for future research.

The notation and conventions of this paper are the same as in Refs. 3, 4, and 10. In particular, we adopt the metric signature $(-,+,+,+)$, and we use the "rationalized" Newman-Penrose equations appropriate to this signature, which can be found, e.g., in Refs. 14 and 3.

II. THE FIELD EQUATIONS FOR COLLIDING PARALLEL-POLARIZED GRAVITATIONAL PLANE WAVES AND THEIR SOLUTION

A. Formulation of the problem in the Rosen-type (u,v,x,y) coordinate system

In any plane-symmetric spacetime (see Sec. IIIB of Ref. 3, or Sec. II of Ref. 4 for a careful definition of plane symmetry), there exists a canonical null tetrad[13] whose construction is described in Sec. IIIB of Ref. 3. In this null tetrad, which we call the standard tetrad, l and $\mathbf{n}$ are tangent to the two null geodesic congruences everywhere orthogonal to the plane-symmetry generating Killing vector fields ξ_1 and ξ_2, and $\mathbf{m}$ and its complex conjugate are linear combinations of the ξ_i, $i=1$, 2. As is shown by Szekeres,[6] it follows from the presence of only two nontrivial dimensions in the spacetime, that we can find a local coordinate chart (u,v,x,y) in which $\xi_i=\partial/\partial x^i$ $[(x^1,x^2)\equiv(x,y)]$, and in which the standard tetrad can be expressed as

$$\begin{aligned} l&=2e^{M(u,v)}\frac{\partial}{\partial u}+P^i(u,v)\frac{\partial}{\partial x^i} , \\ \mathbf{n}&=\frac{\partial}{\partial v}+Q^i(u,v)\frac{\partial}{\partial x^i} , \\ \mathbf{m}&=\frac{1}{F(u,v)}\frac{\partial}{\partial x}+\frac{1}{G(u,v)}\frac{\partial}{\partial y} . \end{aligned} \tag{2.1}$$

Here P^i,Q^i,M are real, and F,G are complex functions of (u,v), with $F^*G-G^*F\neq 0$ throughout the region on which strict plane symmetry[3,4] holds and on which the tetrad (2.1) and the coordinate chart (u,v,x,y) are well behaved. In the specific plane-symmetric spacetimes which describe gravitational plane waves propagating and colliding in an otherwise flat background, there will be a region, corresponding to the spacetime before the arrival of either plane wave, where the metric is flat. It is shown by Szekeres[6] (see also Sec. IIIB of Ref. 4), that the presence of such a flat region makes it possible to find a new coordinate system, which we still denote by (u,v,x,y), in which $P^i=Q^i=0$ and the standard tetrad (2.1) takes the simpler form

$$\begin{aligned} l&=2e^{M(u,v)}\frac{\partial}{\partial u} , \quad \mathbf{n}=\frac{\partial}{\partial v} , \\ \mathbf{m}&=\frac{1}{F(u,v)}\frac{\partial}{\partial x}+\frac{1}{G(u,v)}\frac{\partial}{\partial y} . \end{aligned} \tag{2.2}$$

Finally, when the colliding plane waves have parallel linear polarizations, the tetrad components in Eq. (2.2) can be further restricted[6,10] to give

$$\begin{aligned} l&=2e^{M}\frac{\partial}{\partial u} , \quad \mathbf{n}=\frac{\partial}{\partial v} , \\ \mathbf{m}&=N_1\frac{\partial}{\partial x}+N_2\frac{\partial}{\partial y} , \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} N_1&=\frac{1+i}{2}e^{(U-V)/2} , \\ N_2&=\frac{1-i}{2}e^{(U+V)/2} , \end{aligned} \tag{2.4}$$

with U and V real and with M, U, and V functions of u and v only. The presence of a difference between the linear polarizations of the incoming waves (or, the presence of a circular polarization component in any of the colliding waves) would manifest itself in the presence of a (u,v)-dependent relative phase factor between N_1 and N_2 in Eq. (2.4) above. The tetrad (2.3)–(2.4) gives rise to the metric

$$g=-e^{-M}du\,dv+e^{-U}(e^{V}dx^2+e^{-V}dy^2) . \tag{2.5}$$

Thus, the x-y part of the metric is in diagonal form uniformly at all points in the spacetime, and the Killing vector fields $\partial/\partial x$ and $\partial/\partial y$ are everywhere hypersurface orthogonal; each of these facts being equivalent to the assumption that the colliding plane waves have parallel (linear) polarizations.[6,10] The coordinate system (u,v,x,y) is uniquely determined, up to transformations of the form $u=f(u')$, $v=g(v')$, by demanding (i) that the metric in it has the above form (2.5) (hence, in particular, that the plane-symmetry generators are $\partial/\partial x^i$), and (ii) that in the flat region describing the spacetime before the arrival of either wave, (u,v,x,y) reduce to Minkowski coordinates. [Here, f and g are functions which are constrained to be of the form $f(u')=cu'$, $g(v')=v'/c$ in the *flat Minkowski region*, but which are completely arbitrary elsewhere. We will use this coordinate freedom below when we discuss the initial-value problem for the field equations.] Therefore, the coordinate system (u,v,x,y) is the direct analogue of the Rosen-type coordinates associated with each of the incoming, colliding plane waves. (For a discussion of different coordinate systems associated with plane-wave spacetimes, see Ref. 2, Sec. II of Ref. 3, and Sec. I of Ref. 4.) We will thus call (u,v,x,y) the Rosen-type coordinates on the colliding plane-wave spacetime.

The vacuum field equations for the metric (2.5) are[6,15]

$$2(U_{,uu}+M_{,u}U_{,u})-U_{,u}{}^2-V_{,u}{}^2=0\ , \tag{2.6a}$$

$$2(U_{,vv}+M_{,v}U_{,v})-U_{,v}{}^2-V_{,v}{}^2=0\ , \tag{2.6b}$$

$$U_{,uv}-U_{,u}U_{,v}=0\ , \tag{2.6c}$$

$$V_{,uv}-\tfrac{1}{2}(U_{,u}V_{,v}+U_{,v}V_{,u})=0\ , \tag{2.6d}$$

where the integrability condition for the first two equations is satisfied by virtue of the last two, and yields the remaining field equation

$$M_{,uv}-\tfrac{1}{2}(V_{,u}V_{,v}-U_{,u}U_{,v})=0\ . \tag{2.7}$$

Therefore, it is sufficient to solve Eqs. (2.6c) and (2.6d) first and to obtain M by quadrature from the first two equations (2.6a) and (2.6b) afterward, since Eq. (2.7) as well as the integrability condition for Eqs. (2.6a) and (2.6b) are automatically satisfied as a result of Eqs. (2.6c) and (2.6d).

The initial-value problem associated with the field equations (2.6) and (2.7) is best formulated in terms of initial data posed on null (characteristic) surfaces. A natural choice for the initial characteristic surface is the surface made up of the two intersecting null hyperplanes which form the past wave fronts of the incoming plane waves, and which, by a readjustment of the null coordinates u and v if necessary, can be arranged to be the surfaces $\{u=0\}$ and $\{v=0\}$. The geometry of the resulting characteristic initial-value problem is depicted in Fig. 1. The initial data supplied by the plane wave propagating in the v direction (to the right in Fig. 1) is posed on the $u\geq 0$ portion of the surface $\{v=0\}$, and the initial data supplied by the plane wave propagating in the u direction (to the left in Fig. 1) is posed on the $v\geq 0$ portion of the surface $\{u=0\}$. In region IV, which represents the spacetime before the passage of either plane wave, the geometry is flat and all metric coefficients M, U, and V vanish identically. Now recall that there is a remaining coordinate freedom in the choice of the (u,v,x,y) coordinate system, given by the transformations of the form $u=f(u')$, and $v=g(v')$. This gauge freedom also manifests itself in the choice of initial data on the characteristic initial surface $\{u=0\}\cup\{v=0\}$: The choice of the initial data $\{M(u=0,v),\ M(u,v=0)\}$ for the metric function M is completely arbitrary, since, clearly, for a single plane wave [cf. Eq. (2.5)] $M(u)$ $[M(v)]$ can be adjusted freely by coordinate transformations of the form $u=f(u')$ $[v=g(v')]$. [This arbitrariness (gauge freedom) in the choice of initial data in the (u,v) coordinates disappears, when, as we will do in Sec. II B, one formulates the field equations in the (α,β) coordinate system. Then, any two different but *equivalent* choices of initial data for the functions V, U, and M in the (u,v) coordinates correspond, in the formalism of Sec. II B, to a *unique* choice of the functions $V(r,1)$ and $V(1,s)$ which determine the initial data. In fact, even the boost freedom (see below), which eventually remains in the choice of the (u,v) coordinates, is absent from the formalism based on the (α,β) coordinate system.] We will fix the above gauge freedom once and for all by posing our initial data so that

$$M(u=0,v)=M(u,v=0)\equiv 0\ . \tag{2.8}$$

Then the only remaining coordinate freedom in the problem is the scale (or boost) freedom given by the scaling (boost) transformations $u=cu'$, $v=v'/c$, where c is a positive constant. This remaining boost freedom is harmless however; in fact, it is even useful in carrying out computations involving colliding waves, when, for example, it is known from physical arguments that the results have to be scale invariant (see, e.g., the discussion in Refs. 15 and 16).

Our choice of gauge, Eq. (2.8), implies that the metric in region II (where $u\geq 0$, $v\leq 0$), describing the geometry of the incoming colliding wave that propagates in the v direction (to the right in Fig. 1), is given by

$$g_{\rm II}=-du\,dv+F_1{}^2(u)dx^2+G_1{}^2(u)dy^2\ , \tag{2.9}$$

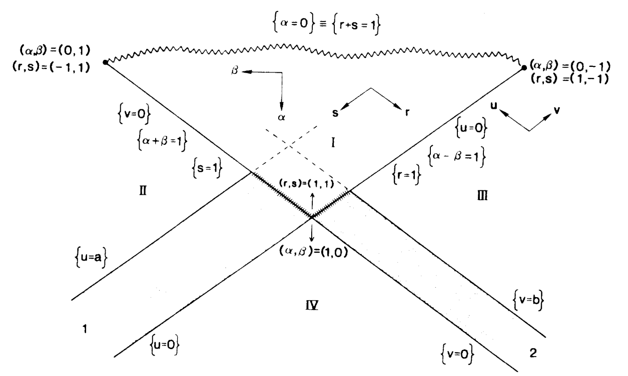

FIG. 1. The two-dimensional geometry of the characteristic initial-value problem for colliding plane waves. The null surfaces $\{u=0\}$ and $\{v=0\}$ are the past wave fronts of the incoming plane waves 1 and 2. Initial data corresponding to waves 1 and 2 are posed, respectively, on the upper portions of the surfaces $\{v=0\}$ and $\{u=0\}$ that are adjacent to the interaction region I. The geometry in the region IV is flat, and the geometry in regions II and III is given by the metric describing the incoming waves 1 and 2, respectively. The geometry of the interaction region I is uniquely determined by the solution of the above initial-value problem. The directions in which the various lines of constant coordinates u, v, α, β, r, and s run are also indicated, along with the descriptions of the initial null surfaces in these different coordinate systems.

and that the metric in region III (where $v \geq 0$, $u \leq 0$), describing the geometry of the incoming wave that propagates in the u direction (to the left in Fig. 1), is given by

$$g_{\rm III} = -du\,dv + F_2{}^2(v)dx^2 + G_2{}^2(v)dy^2 \ . \tag{2.10}$$

Here, F_1, G_1 are C^1 (and piecewise C^2) functions of u (for $u \geq 0$), and F_2, G_2 are C^1 (and piecewise C^2) functions of v (for $v \geq 0$), which satisfy the initial conditions $F_1(u=0)=G_1(u=0)=F_2(v=0)=G_2(v=0)=1$ (coordinates Minkowski in IV), and satisfy the differential equations

$$\frac{F_1''(u)}{F_1(u)} + \frac{G_1''(u)}{G_1(u)} = 0 \ , \quad \frac{F_2''(v)}{F_2(v)} + \frac{G_2''(v)}{G_2(v)} = 0 \tag{2.11}$$

for $u \geq 0$ and $v \geq 0$, respectively [these differential equations follow from the field equations (2.6)]. The initial data, induced on the characteristic initial surface $\{u=0\} \cup \{v=0\}$ by the colliding waves (2.9) and (2.10), are given by

$$U(u,v=0) \equiv U_1(u) = -\ln[F_1(u)G_1(u)] \ , \tag{2.12a}$$

$$V(u,v=0) \equiv V_1(u) = \ln\left[\frac{F_1(u)}{G_1(u)}\right] \ , \tag{2.12b}$$

$$U(u=0,v) \equiv U_2(v) = -\ln[F_2(v)G_2(v)] \ , \tag{2.12c}$$

$$V(u=0,v) \equiv V_2(v) = \ln\left[\frac{F_2(v)}{G_2(v)}\right] \ . \tag{2.12d}$$

If the colliding waves are sandwich plane waves (Sec. II of Ref. 3), we then have length scales f_1, f_2, a, f_1', f_2', and b such that

$$F_1(u) = \frac{F_1(a)}{a-f_1}(u-f_1) \ , \quad G_1(u) = \frac{G_1(a)}{a-f_2}(u-f_2) \quad \text{for } u \geq a \ , \tag{2.13}$$

and

$$F_2(v) = \frac{F_2(b)}{b-f_1'}(v-f_1') \ , \quad G_2(v) = \frac{G_2(b)}{b-f_2'}(v-f_2') \quad \text{for } v \geq b \ . \tag{2.14}$$

Although the initial data in the form of Eqs. (2.12) give the information about the incoming, colliding plane waves in an intuitively clear format [cf. Eqs. (2.9) and (2.10)], in the more precise mathematical description of the initial-value problem the initial data are completely determined by only the two freely specifiable functions $V_1(u)$, and $V_2(v)$. In other words, the initial data consist of

$$\{V_1(u),\ V_2(v)\} \ , \tag{2.15}$$

where $V_1(u)$ and $V_2(v)$ are C^1 (and picewise C^2) functions for $u \geq 0$ and $v \geq 0$, respectively, which are freely specified except for the initial conditions $V_1(u=0)=V_2(v=0)=0$. In the linearized regime (when V_1, $V_2 \ll 1$), the functions V_1 and V_2 correspond to the time-dependent physical amplitudes of the incoming, colliding gravitational waves [cf. Eqs. (2.9) and (2.10)]. The remaining functions $U_1(u)$ and $U_2(v)$ are uniquely determined, by the initial data (2.15), through the constraint equations [cf. Eqs. (2.6a) and (2.6b)]

$$2U_{1,uu} - U_{1,u}{}^2 = V_{1,u}{}^2 \ , \tag{2.16a}$$

$$2U_{2,vv} - U_{2,v}{}^2 = V_{2,v}{}^2 \ , \tag{2.16b}$$

with the initial conditions $U_1(u=0)=U_2(v=0)=0$, $U_{1,u}(u=0)=U_{2,v}(v=0)=0$. Note that, if we define two new functions $f(u)$ and $g(v)$ by

$$f(u) \equiv e^{-U_1(u)/2} \ , \quad g(v) \equiv e^{-U_2(v)/2} \ , \tag{2.17}$$

we can express Eqs. (2.16) in the form of "focusing" equations:

$$\frac{f_{,uu}}{f} = -\tfrac{1}{4}V_{1,u}{}^2 \ , \tag{2.18a}$$

$$\frac{g_{,vv}}{g} = -\tfrac{1}{4}V_{2,v}{}^2 \ , \tag{2.18b}$$

with the initial conditions $f(0)=g(0)=1$, $f'(0)=g'(0)=0$.

In Secs. III A and III B, when we discuss the asymptotic structure of the colliding plane-wave spacetime described by Eqs. (2.3)–(2.5), we will need the following equations which express the Newman-Penrose[13] curvature quantities in the null tetrad (2.3) and (2.4) in terms of the metric coefficients M, U, and V; the derivation of these equations can be found in Refs. 6 and 15:

$$\Psi_0 = 2ie^{2M}(M_{,u}V_{,u} + V_{,uu} - V_{,u}U_{,u}) \ , \tag{2.19a}$$

$$\Psi_2 = -e^{M}M_{,uv} \ , \tag{2.19b}$$

$$\Psi_4 = \frac{i}{2}[(U_{,v} - M_{,v})V_{,v} - V_{,vv}] \ , \tag{2.19c}$$

$$\Psi_1 = \Psi_3 = 0 \ . \tag{2.19d}$$

B. The field equations and their solution in the (α,β) coordinates

We now construct a new coordinate system (α,β,x,y), in which the field equations and the initial-value problem associated with them take simpler forms. This coordinate system is constructed as follows.

Consider the interaction region (region I in Fig. 1) where $u \geq 0$ and $v \geq 0$. This region is the domain of dependence[17] of the characteristic initial surface $\{u=0\} \cup \{v=0\}$, on which the initial-value problem defined by Eqs. (2.6), (2.8), (2.15), and (2.16) is to be solved. Consider the field equation (2.6c) in the interaction region. It follows from this equation that if we define

$$\alpha(u,v) \equiv e^{-U(u,v)} \ , \tag{2.20}$$

then, throughout the interaction region, $\alpha(u,v)$ satisfies

$$\alpha_{,uv} = 0 \ , \tag{2.21}$$

the flat-space wave equation in two dimensions. Equation (2.21) suggests that we define another function, $\beta(u,v)$, such that

$$\beta_{,u}=-\alpha_{,u}\ ,\quad \beta_{,v}=\alpha_{,v}\ , \tag{2.22}$$

since, clearly, the integrability condition for Eqs. (2.22) is satisfied by virtue of Eq. (2.21). The general solution of Eq. (2.21) is

$$\alpha(u,v)=a(u)+b(v)\ , \tag{2.23}$$

where $a(u)$ and $b(v)$ are arbitrary functions. With this solution for α, Eqs. (2.22) yield

$$\beta(u,v)=-a(u)+b(v)+c\ , \tag{2.24}$$

where c is an arbitrary constant. Note that Eq. (2.20) defines α not only throughout the interior of the interaction region I where $u>0$, $v>0$, but also along the boundary $\{u=0\}\cup\{v=0\}$ of this region, which is the characteristic initial surface. Hence, the boundary values (2.12a), (2.12c) for the function $U(u,v)$ yield, through Eq. (2.20), the following boundary values for α:

$$\alpha(u,v=0)=e^{-U_1(u)}\ ,\quad \alpha(u=0,v)=e^{-U_2(v)}\ . \tag{2.25}$$

These initial values (2.25), when combined with the general solution (2.23) and the initial condition $U(u=0,v=0)=0$, immediately yield the unique solution

$$\alpha(u,v)=e^{-U_1(u)}+e^{-U_2(v)}-1 \tag{2.26}$$

for $\alpha(u,v)$, which holds throughout the interaction region. This solution, combining with Eq. (2.24) and setting the arbitrary constant c equal to zero, yields the solution

$$\beta(u,v)=e^{-U_2(v)}-e^{-U_1(u)} \tag{2.27}$$

for $\beta(u,v)$ and completes the construction of the new variables (α,β). To see that these variables actually define a new coordinate system, consider the two-form given by the exterior product $d\alpha\wedge d\beta$. When this two-form is nonzero throughout some region $\mathcal{U}$, it follows from the inverse function theorem[18] that the functions $\alpha(u,v)$ and $\beta(u,v)$ (together with the usual spatial coordinates x, y) constitute a regular coordinate system throughout $\mathcal{U}$. Now, Eqs. (2.26) and (2.27) give

$$d\alpha\wedge d\beta\ =\ 2U_1{}'(u)U_2{}'(v)e^{-[U_1(u)+U_2(v)]}\,du\wedge dv\ . \tag{2.28}$$

On the other hand, it immediately follows from Eqs. (2.16), or more clearly from the "focusing" equations (2.18), that as long as the initial data (2.15) are nontrivial for both incoming waves [i.e., as long as neither $V_1(u)$ nor $V_2(v)$ is identically zero], and as long as the initial surfaces $\{u=0\}$ and $\{v=0\}$ correspond to the true past wave fronts of the colliding waves [i.e., as long as $V_1(u)\neq0$ and $V_2(v)\neq0$ for *all* sufficiently small but positive u and v], we have

$$\begin{aligned}&U_1{}'(u)>0\ ,\ f'(u)<0\quad \forall u>0\ ,\\&U_2{}'(v)>0\ ,\ g'(v)<0\quad \forall v>0\ ,\end{aligned} \tag{2.29}$$

whereas $U_1{}'(u=0)=U_2{}'(v=0)=0$ because of the initial conditions [cf. Eqs. (2.16)]. Therefore we conclude [Eq. (2.28)], that as long as the initial data (2.15) are nontrivial for both colliding waves, and as long as the null surfaces $\{u=0\}$ and $\{v=0\}$ are the true past wave fronts, the functions (α,β,x,y) constitute a coordinate system which is regular wherever the coordinate system (u,v,x,y) is regular in the interior of the interaction region, $u>0$, $v>0$. On the other hand, the coordinates α,β are singular along the initial null surfaces $\{u=0\}$ and $\{v=0\}$. In other words, the singularities of the coordinate system (α,β,x,y) consist of the singularities of the (u,v,x,y) coordinates (when there are any), and the singularity along the initial characteristic surface $\{u=0\}\cup\{v=0\}$. Since the only place in the interaction region where the coordinates (u,v,x,y) can develop singularities is the "surface" $\{\alpha=0\}$ (see Sec. III A), it follows that the coordinate system (α,β,x,y) covers the domain of dependence of the initial surface $\{u=0\}\cup\{v=0\}$ regularly except for the singularities on $\{u=0\}$ and $\{v=0\}$.

The coordinates (α,β,x,y) enjoy a number of properties which make them useful in studying the field equations for colliding plane waves. First, the functions $\alpha(u,v)$ and $\beta(u,v)$ satisfy the wave equation in the two-dimensional Minkowski metric $-du\,dv$ (and, by conformal invariance, also in the two-dimensional metric $-e^{-M}du\,dv$). Hence, it follows that the $du\,dv$ part of the metric (2.5) will be in diagonal form [Eq. (2.43)] in the new coordinate system (α,β,x,y). Second, by performing the transformation (2.26) and (2.27) from the variables (u,v) to the new variables (α,β), we have eliminated one of the metric coefficients [namely, the function $U(u,v)$] from Eq. (2.5), and absorbed it into the definition of the coordinate α. Therefore, the field equations in the new coordinate system [Eqs. (2.44)] will involve only two unknown variables, instead of the three functions M, V, and U involved in Eqs. (2.6). Finally, the Eqs. (2.26) and (2.27), which together with Eq. (2.20) yield the unique solution to the initial-value problem for $U(u,v)$ [Eq. (2.41)], also provide expressions for the new variables α and β purely in terms of the initial data on $\{u=0\}\cup\{v=0\}$. In other words, it is not necessary to solve any of the remaining field equations to perform the transformation from the (u,v,x,y) coordinates to the new (α,β,x,y) coordinate system.

We now proceed with the mathematical analysis of the initial-value problem defined by Eqs. (2.5), (2.6), (2.8), (2.15), and (2.16), in the new coordinate system (α,β,x,y). First we note the transformation rules

$$\partial_u=\alpha_{,u}(\partial_\alpha-\partial_\beta)\ , \tag{2.30a}$$

$$\partial_v=\alpha_{,v}(\partial_\alpha+\partial_\beta)\ , \tag{2.30b}$$

and their inverses

$$\partial_\alpha=\frac{1}{2}\left[\frac{1}{\alpha_{,u}}\partial_u+\frac{1}{\alpha_{,v}}\partial_v\right]\ , \tag{2.31a}$$

$$\partial_\beta=\frac{1}{2}\left[\frac{1}{\alpha_{,v}}\partial_v-\frac{1}{\alpha_{,u}}\partial_u\right]\ , \tag{2.31b}$$

which are derived using Eq. (2.22). Here, ∂_u, ∂_v, ∂_α, and ∂_β denote, respectively, the differential operators ($\equiv$vector fields) $\partial/\partial u$, $\partial/\partial v$, $\partial/\partial\alpha$, and $\partial/\partial\beta$. A short computation involving Eqs. (2.30) and (2.31) now gives

$$-du\,dv=\frac{1}{4\alpha_{,u}\alpha_{,v}}(-d\alpha^2+d\beta^2)\ . \tag{2.32}$$

When inserted into the expression (2.5) for the metric and combined with Eq. (2.20), Eq. (2.32) yields the expression

$$g\;=\;\frac{e^{-M}}{4\alpha^2 U_{,u}U_{,v}}(-d\alpha^2+d\beta^2)\;+\;\alpha(e^V dx^2+e^{-V}dy^2) \tag{2.33}$$

for the spacetime metric, which is valid throughout the interaction region (region I in Fig. 1). Next, another short calculation using Eqs. (2.30) and (2.31) together with Eq. (2.21) gives

$$\partial_\alpha{}^2-\partial_\beta{}^2=\frac{1}{\alpha_{,u}\alpha_{,v}}\partial_u\partial_v\ , \tag{2.34}$$

where $\partial_\alpha{}^2$, $\partial_\beta{}^2$, and $\partial_u\partial_v$ denote the second-order differential operators $\partial^2/\partial\alpha^2$, $\partial^2/\partial\beta^2$, and $\partial^2/\partial u\partial v$, respectively. Combining Eq. (2.34) with the field equation (2.6d) and using Eq. (2.31a) yields

$$V_{,\alpha\alpha}+\frac{1}{\alpha}V_{,\alpha}-V_{,\beta\beta}=0\ , \tag{2.35}$$

which is one of the field equations in the (α,β,x,y) coordinate system. To obtain the remaining field equations, we proceed as follows: First we note that we can rewrite the field equations (2.6a) and (2.6b) in the form

$$\frac{2}{e^M U_{,u}U_{,v}}(e^M U_{,u}U_{,v})_{,u}=U_{,u}+\frac{V_{,u}{}^2}{U_{,u}}+2U_{,u}\ ,$$

$$\frac{2}{e^M U_{,u}U_{,v}}(e^M U_{,u}U_{,v})_{,v}=U_{,v}+\frac{V_{,v}{}^2}{U_{,v}}+2U_{,v}\ .$$

Thus, if we define a new function P by

$$e^P\equiv 4c\,e^M U_{,u}U_{,v}\ , \tag{2.36}$$

where c is an arbitrary constant having the dimensions of $(\text{length})^2$ [we will fix c later with our normalization condition Eq. (2.40)], then P satisfies

$$2P_{,u}=3U_{,u}+\frac{V_{,u}{}^2}{U_{,u}}\ , \tag{2.37a}$$

$$2P_{,v}=3U_{,v}+\frac{V_{,v}{}^2}{U_{,v}}\ . \tag{2.37b}$$

Combining Eqs. (2.37) with Eqs. (2.30) and using Eq. (2.20), we obtain

$$2\alpha_{,u}(P_{,\alpha}-P_{,\beta})=-\frac{3\alpha_{,u}}{\alpha}-\alpha\alpha_{,u}(V_{,\alpha}{}^2+V_{,\beta}{}^2-2V_{,\alpha}V_{,\beta})\ ,$$

$$2\alpha_{,v}(P_{,\alpha}+P_{,\beta})=-\frac{3\alpha_{,v}}{\alpha}-\alpha\alpha_{,v}(V_{,\alpha}{}^2+V_{,\beta}{}^2+2V_{,\alpha}V_{,\beta})\ ,$$

which, after some rearrangements, can be written in the form

$$(2P+3\ln\alpha)_{,\alpha}=-\alpha(V_{,\alpha}{}^2+V_{,\beta}{}^2)\ , \tag{2.38a}$$

$$(2P+3\ln\alpha)_{,\beta}=-2\alpha V_{,\alpha}V_{,\beta}\ . \tag{2.38b}$$

Equations (2.38) suggest that it will be convenient to define the combination $2P+3\ln\alpha$ as a new variable, which, together with the variable V, would uniquely determine the metric in the (α,β,x,y) coordinate system. Thus, after first introducing the two "normalization" length scales l_1 and l_2 by the equations

$$l_1\equiv\frac{1}{2U_{,u}(u_0,v_0)}\ ,\quad l_2\equiv\frac{1}{2U_{,v}(u_0,v_0)}\ , \tag{2.39a}$$

where (u_0,v_0), $u_0>0$, $v_0>0$ is an arbitrary, fixed point in the *interior* of the interaction region, we define a new function $Q(\alpha,\beta)$ by the relation

$$e^{Q/2}\equiv 4l_1l_2\,e^M U_{,u}U_{,v}\,\alpha^{3/2}\ . \tag{2.39b}$$

Using Eqs. (2.39a), we can now fix the constant c which occurs in Eq. (2.36):

$$c\equiv l_1l_2\ . \tag{2.40}$$

Note that the length scales l_1 and l_2 are determined by Eqs. (2.39a) in a well-defined manner, since by Eqs. (2.20) and (2.26)

$$U(u,v)=-\ln\alpha(u,v)=-\ln(e^{-U_1(u)}+e^{-U_2(v)}-1)\ , \tag{2.41}$$

so that

$$U_{,u}(u,v)=\frac{1}{\alpha(u,v)}U_1{}'(u)e^{-U_1(u)}\ ,$$

$$U_{,v}(u,v)=\frac{1}{\alpha(u,v)}U_2{}'(v)e^{-U_2(v)}\ ;$$

and therefore, by Eqs. (2.29), $U_{,u}(u,v)>0$, $U_{,v}(u,v)>0$ for any point (u,v) in the interior of the interaction region, where $u>0$, $v>0$, and where [as long as (u,v) is in the domain of dependence of the initial surface $\{u=0\}\cup\{v=0\}$ (cf. Secs. III A–III C)] $\alpha(u,v)>0$. It is now easy to obtain the remaining field equations, satisfied by the new variable $Q(\alpha,\beta)$: Combining Eq. (2.39b) with Eqs. (2.40) and (2.36), and then using Eqs. (2.38), we find

$$Q_{,\alpha}=-\alpha(V_{,\alpha}{}^2+V_{,\beta}{}^2)\ , \tag{2.42a}$$

$$Q_{,\beta}=-2\alpha V_{,\alpha}V_{,\beta}\ , \tag{2.42b}$$

where the integrability condition for Eqs. (2.42) is satisfied by virtue of the field equation (2.35) for $V(\alpha,\beta)$.

We are now in a position to write down the complete formulation, in the (α,β,x,y) coordinate system, of the metric and the field equations in the interaction region of a colliding parallel-polarized plane-wave spacetime. For this, we first combine Eq. (2.39b) with the expression (2.33) for the metric in the interaction region. This gives us the expression of the interaction region metric in terms of the two unknown variables V and Q. Then, we recall the field equation (2.35) for $V(\alpha,\beta)$, and combine it with the unique solution of the field equations (2.42) for $Q(\alpha,\beta)$, which we obtain by using the initial value of Q that follows from the normalization conditions Eqs. (2.39). As a result, we obtain the following expressions for the metric and the field equations in the interaction region of a colliding plane-wave spacetime:

$$g = e^{-Q(\alpha,\beta)/2}\frac{l_1 l_2}{\sqrt{\alpha}}(-d\alpha^2+d\beta^2) + \alpha(e^{V(\alpha,\beta)}dx^2+e^{-V(\alpha,\beta)}dy^2) , \quad (2.43)$$

where V and Q satisfy the following field equations:

$$V_{,\alpha\alpha}+\frac{1}{\alpha}V_{,\alpha}-V_{,\beta\beta}=0 , \quad (2.44a)$$

$$Q(\alpha,\beta) = \int_{C:(\alpha_0,\beta_0)}^{(\alpha,\beta)}[-\alpha(V_{,\alpha}{}^2+V_{,\beta}{}^2)d\alpha - 2\alpha V_{,\alpha}V_{,\beta}d\beta] + 2M(\alpha_0,\beta_0)+3\ln\alpha_0 . \quad (2.44b)$$

Here, $\alpha_0\equiv\alpha(u_0,v_0)$, $\beta_0\equiv\beta(u_0,v_0)$, $M(\alpha_0,\beta_0)\equiv M(u_0,v_0)$, and C is *any* (differentiable) curve in the (α,β) plane that starts at the initial point (α_0,β_0), and ends at the field point (α,β) at which Q is to be computed. The result of the integral in Eq. (2.44b) depends only on the end points of the curve C, since the integrability condition for Eqs. (2.42) is satisfied by virtue of the field equation (2.44a).

Equations (2.43) and (2.44) summarize the mathematical problem of colliding parallel-polarized plane waves in a remarkably compact form. In particular, the only unknown to be solved for is the function $V(\alpha,\beta)$ which satisfies the *linear* field equation (2.44a). Once $V(\alpha,\beta)$ is known, Q is determined by the explicit expression (2.44b) up to an unknown additive constant, which—by suitably choosing the initial point (u_0,v_0) [or (α_0,β_0)]—can be made arbitrarily small. The only disadvantage of this formalism based on the (α,β,x,y) coordinates is the coordinate singularity that the (α,β) chart develops on the characteristic initial surface $\{u=0\}\cup\{v=0\}$. This coordinate singularity causes, among other things, the function $Q(\alpha,\beta)$ to be logarithmically divergent (to $-\infty$) on the surfaces $\{u=0\}$ and $\{v=0\}$. Nevertheless, it is still possible to set up a well-defined initial-value problem for the function $V(\alpha,\beta)$, involving the initial data posed on the same characteristic surface $\{u=0\}\cup\{v=0\}$.

It becomes clear from Eqs. (2.43) and (2.44), that the "surface" $\{\alpha=0\}$ represents some kind of a singularity [either a spacetime singularity or (at least) a coordinate singularity] of the colliding plane-wave solution described by the metric (2.43). Since we are primarily interested in the behavior of the spacetime near this "surface" $\{\alpha=0\}$, which is bounded away from the coordinate singularity on the initial null surfaces, the formalism based on the new (α,β) variables is well suited to our objectives.

In the remaining two paragraphs of this section, we will describe the initial-value problem for the metric function $V(\alpha,\beta)$ and its solution. First, in the next paragraph, we explain how to pose the initial data given by Eq. (2.15), in the new formalism based on the (α,β) coordinates. Then, in the following paragraph, we give the explicit solution of this initial-value problem for $V(\alpha,\beta)$.

We begin by noting that [cf. Eqs. (2.26) and (2.27)] in the α,β coordinates the initial null surfaces $\{u=0\}$ and $\{v=0\}$ are expressed as (Fig. 1)

$$\{u=0\}\equiv\{\alpha-\beta=1\} , \quad \{v=0\}\equiv\{\alpha+\beta=1\} . \quad (2.45)$$

Equations (2.45), together with Eq. (2.44a), suggest introducing the "characteristic" coordinates

$$r\equiv\alpha-\beta , \quad s\equiv\alpha+\beta , \quad (2.46)$$

so that the initial null surfaces become

$$\{u=0\}\equiv\{r=1\} , \quad \{v=0\}\equiv\{s=1\} . \quad (2.47)$$

In the new (r,s) coordinate system [Eqs. (2.46)], the field equation (2.44a) takes the form

$$V_{,rs}+\frac{1}{2(r+s)}(V_{,r}+V_{,s})=0 , \quad (2.48)$$

which is a partial differential equation for the function $V(r,s)$. The initial-value problem for $V(r,s)$ consists of Eq. (2.48), and the initial data on the characteristic initial surface $\{r=1\}\cup\{s=1\}$ given by the freely specifiable functions $V(r,s=1)$ and $V(r=1,s)$. More precisely, the initial data consist of

$$\{V(r,1),V(1,s)\} , \quad (2.49)$$

where $V(r,1)$ and $V(1,s)$ are C^1 (and piecewise C^2) functions for $r\in(-1,1]$ and $s\in(-1,1]$, respectively, which are freely specified except for the initial conditions $V(r=1,1)=V(1,s=1)=0$. Once the initial-value problem (2.48) and (2.49) is solved for the function $V(r,s)$, the function $V(\alpha,\beta)$ is determined by the obvious expression

$$V(\alpha,\beta)\equiv V(r=\alpha-\beta,s=\alpha+\beta) . \quad (2.50)$$

There is a one-to-one correspondence between the initial data of the form (2.15), and initial data of the form (2.49), for the initial-value problem of colliding parallel-polarized plane waves. When initial data are given in the form of Eq. (2.15), i.e., when the functions $V_1(u)$ and $V_2(v)$ are specified, initial data in the form of Eq. (2.49) are uniquely determined in the following way: First, Eqs. (2.16) are solved with the given data $V_1(u)$ and $V_2(v)$, and the functions $U_1(u)$ and $U_2(v)$ are obtained as the unique solutions [cf. Eqs. (2.16) and the discussion following them]. Then, using the identities [cf. Eqs. (2.26) and (2.27) and Eq. (2.46)]

$$r=2a(u)=2e^{-U_1(u)}-1 , \quad s=2b(v)=2e^{-U_2(v)}-1 \quad (2.51)$$

along the initial null surfaces $\{u=0\}$ and $\{v=0\}$, $u(r)$ and $v(s)$ are defined as the unique solutions to the implicit equations

$$r=2e^{-U_1[u(r)]}-1 , \quad s=2e^{-U_2[v(s)]}-1 . \quad (2.52)$$

Finally, the initial data $\{V(r,1),V(1,s)\}$ in the form (2.49) are determined uniquely from the data $\{V_1(u),V_2(v)\}$ by

$$V(r,1)=V_1[u=u(r)]\ ,\quad V(1,s)=V_2[v=v(s)]\ . \tag{2.53}$$

Conversely, when initial data are given in the form of Eq. (2.49), i.e., when the functions $V(r,1)$ and $V(1,s)$ are specified, initial data in the form of (2.15) are uniquely determined in the following way: First, the differential equations

$$2U_{1,uu}-U_{1,u}{}^2=4e^{-2U_1}\,U_{1,u}{}^2\,[V_{,r}(r=2e^{-U_1}-1,1)]^2\ , \tag{2.54a}$$

$$2U_{2,vv}-U_{2,v}{}^2=4e^{-2U_2}\,U_{2,v}{}^2\,[V_{,s}(1,s=2e^{-U_2}-1)]^2\ , \tag{2.54b}$$

for the functions $U_1(u)$ and $U_2(v)$ are solved with the initial conditions $U_1(u=0)=U_2(v=0)=U_{1,u}(u=0)=U_{2,v}(v=0)=0$ [cf. Eqs. (2.16)]. Then, using Eqs. (2.52), the initial data $\{V_1(u),V_2(v)\}$ in the form (2.15) are determined uniquely from the data $\{V(r,1),V(1,s)\}$ by

$$\begin{aligned} V_1(u)&=V(r=2e^{-U_1(u)}-1,1)\ ,\\ V_2(v)&=V(1,s=2e^{-U_2(v)}-1)\ . \end{aligned} \tag{2.55}$$

This completes the formulation of the initial-value problem for the function $V(\alpha,\beta)$, or, equivalently, for the function $V(r,s)$ [cf. Eq. (2.50)].

The solution to a two-dimensional linear hyperbolic initial-value problem of the form (2.48) and (2.49) is obtained by using the appropriate Riemann function (Sec. 4.4 of Ref. 18). Specifically, the Riemann function for Eq. (2.48) is a two-point function $A(r,s;\xi,\eta)$, which satisfies the adjoint[18] equation to Eq. (2.48);

$$A_{,rs}-\frac{1}{2(r+s)}(A_{,r}+A_{,s})+\frac{1}{(r+s)^2}A=0\ , \tag{2.56}$$

with the initial values

$$\begin{aligned} A(r,\eta;\xi,\eta)&=\left[\frac{r+\eta}{\xi+\eta}\right]^{1/2}\ ,\\ A(\xi,s;\xi,\eta)&=\left[\frac{\xi+s}{\xi+\eta}\right]^{1/2}\ . \end{aligned} \tag{2.57}$$

Once the Riemann function A is known, the solution $V(r,s)$ of the initial-value problem (2.48) and (2.49) is given by (Sec. 4.4 of Ref. 18)

$$\begin{aligned} V(r,s)=\ &A(1,1;r,s)V(1,1)\\ &+\int_1^s\left[V_{,s'}(1,s')+\frac{V(1,s')}{2(1+s')}\right]A(1,s';r,s)ds'\\ &+\int_1^r\left[V_{,r'}(r',1)+\frac{V(r',1)}{2(1+r')}\right]A(r',1;r,s)dr'\ . \end{aligned} \tag{2.58}$$

It is found by Szekeres in Ref. 6, that the unique Riemann function which solves Eq. (2.56) with the boundary values (2.57) is

$$A(r,s;\xi,\eta)=\left[\frac{r+s}{\xi+\eta}\right]^{1/2}P_{-1/2}\left[1+2\frac{(r-\xi)(s-\eta)}{(r+s)(\xi+\eta)}\right]\ , \tag{2.59}$$

where $P_{-1/2}$ is the Legendre function P_ν for $\nu=-\frac{1}{2}$ [Ref. 19, Eqs. (8.820)–(8.222)]. Combining Eq. (2.59) with Eq. (2.58), and noting that $V(1,1)=0$ [Eq. (2.49)], we obtain the following explicit solution $V(r,s)$ of the initial-value problem (2.48) and (2.49):

$$\begin{aligned} V(r,s)=\ &\int_1^s\left[V_{,s'}(1,s')+\frac{V(1,s')}{2(1+s')}\right]\left[\frac{1+s'}{r+s}\right]^{1/2}P_{-1/2}\left[1+2\frac{(1-r)(s'-s)}{(1+s')(r+s)}\right]ds'\\ &+\int_1^r\left[V_{,r'}(r',1)+\frac{V(r',1)}{2(1+r')}\right]\left[\frac{1+r'}{r+s}\right]^{1/2}P_{-1/2}\left[1+2\frac{(1-s)(r'-r)}{(1+r')(r+s)}\right]dr'\ . \end{aligned} \tag{2.60}$$

We have thus completed the full solution of the initial-value problem for colliding parallel-polarized gravitational plane waves, expressed in the (α,β,x,y) coordinate system that we constructed in the beginning of this section. We are now ready to study the asymptotic structure of the colliding plane-wave spacetime near the singularity $\alpha=0$.

III. THE ASYMPTOTIC STRUCTURE OF SPACETIME NEAR $\alpha=0$

A. The behavior of the metric near $\alpha=0$: An inhomogeneous Kasner singularity

Before embarking on a full mathematical analysis of the asymptotic structure of the metric (2.43) near $\alpha=0$, we use the field equations (2.44) to make a few introductory observations about the asymptotic behavior of the functions $V(\alpha,\beta)$ and $Q(\alpha,\beta)$. These observations yield some preliminary insights into the asymptotic structure of the metric (2.43) which we will find useful both in this section and in the next one.

Our starting point is the solution of Eq. (2.44a) by the well-known method of separation of variables. Using this method, we easily find that the *formal* solution to Eq. (2.44a) can be written in the form

$$\begin{aligned} V(\alpha,\beta)=\int_{-\infty}^{+\infty}[&(A_k\sin k\beta+B_k\cos k\beta)N_0(k\alpha)\\ &+(C_k\sin k\beta+D_k\cos k\beta)J_0(k\alpha)]dk\ , \end{aligned} \tag{3.1}$$

where J_0 and N_0 are the Bessel functions of the first and second kind, respectively. Using the series representations for J_0 and N_0 given by Eqs. (8.441) and (8.444) of Ref. 19, we find that Eq. (3.1) yields the expression

$$V(\alpha,\beta)=\epsilon(\beta)\ln\alpha+\delta(\beta)+H(\alpha,\beta) \tag{3.2}$$

for $V(\alpha,\beta)$, where

$$\epsilon(\beta)=\frac{2}{\pi}\int_{-\infty}^{+\infty}(A_k\sin k\beta+B_k\cos k\beta)dk\ , \tag{3.3a}$$

$$\delta(\beta)=\int_{-\infty}^{+\infty}(C_k\sin k\beta+D_k\cos k\beta)dk + \frac{2}{\pi}\int_{-\infty}^{+\infty}[\gamma+\ln(\tfrac{1}{2}k)](A_k\sin k\beta+B_k\cos k\beta)dk\ , \tag{3.3b}$$

$$\lim_{\alpha\to 0}H(\alpha,\beta)\equiv 0\ , \tag{3.3c}$$

and where γ is Euler's constant.[19] From Eq. (3.2) it immediately follows, using the field equation (2.44b), that the asymptotic structure of the function $Q(\alpha,\beta)$ near $\alpha=0$ is determined by

$$Q(\alpha,\beta)=-[\epsilon(\beta)]^2\ln\alpha+\mu(\beta)+L(\alpha,\beta)\ , \tag{3.4}$$

where

$$\lim_{\alpha\to 0}L(\alpha,\beta)\equiv 0\ , \tag{3.5}$$

and $\mu(\beta)$ is a (C^1) function of β determined by an expression similar to Eq. (3.3b). Note that the functions $H(\alpha,\beta)$ and $L(\alpha,\beta)$ [although they remain finite (in fact, vanish) as $\alpha\to 0$] are *not* smooth functions near $\alpha=0$. In fact, it follows from the series expansions for J_0 and N_0 [Eqs. (8.441)–(8.444) in Ref. 19], that, for example, the function $H(\alpha,\beta)$ has the behavior

$$\begin{aligned}H(\alpha,\beta)=&c_1(\beta)\alpha^2\ln\alpha+c_2(\beta)\alpha^4\ln\alpha+\cdots\\&+c_k(\beta)\alpha^{2k}\ln\alpha+\cdots+d_1(\beta)\alpha^2\\&+d_2(\beta)\alpha^4+\cdots+d_k(\beta)\alpha^{2k}+\cdots\ ,\end{aligned} \tag{3.6}$$

where $c_k(\beta)$ and $d_k(\beta)$ are functions of β determined by expressions similar to Eq. (3.3b). Equation (3.6), when combined with Eq. (3.2), yields a more detailed expression for the asymptotic structure of $V(\alpha,\beta)$ near $\alpha=0$:

$$\begin{aligned}V(\alpha,\beta)=&\epsilon(\beta)\ln\alpha+c_1(\beta)\alpha^2\ln\alpha+c_2(\beta)\alpha^4\ln\alpha+\cdots\\&+\delta(\beta)+d_1(\beta)\alpha^2+d_2(\beta)\alpha^4+\cdots\ .\end{aligned} \tag{3.7}$$

Equations (3.7) and (3.4) summarize the asymptotic behavior of the metric functions V and Q near the singularity $\alpha=0$. But Eqs. (3.3) are not terribly useful for expressing the key functions $\epsilon(\beta)$ and $\delta(\beta)$ in terms of the initial data (2.15) or (2.49) for the colliding plane waves. For this purpose, it is better to use the explicit solution (2.60) for the function $V(r,s)$ that we obtained in the last section. Thus, in the next paragraph, we will analyze the asymptotic behavior of the solution (2.60) near the singularity $\alpha=0$, and obtain explicit formulas for the functions $\epsilon(\beta)$ and $\delta(\beta)$ expressed in terms of the initial data (2.49) for the incoming waves. Then, in the remainder of this section, we will use the analysis carried out so far to investigate the asymptotic structure of the spacetime metric (2.43) near the singular surface $\alpha=0$.

Note that in the (r,s) coordinate system of Sec. II B, the singularity $\alpha=0$ corresponds to $r+s=0$ [Eqs. (2.46)]. Combining Eq. (2.60) with Eq. (2.50), it is clear that the asymptotic structure of $V(\alpha,\beta)$ near $\alpha=0$ is determined by the asymptotic behavior of the function $P_{-1/2}(1+2z)$ as $z\to\infty$. To evaluate this asymptotic behavior, we first note that the integral representation [Eq. (8.822) in Ref. 19]

$$P_{-1/2}(z)=\frac{1}{\pi}\int_0^{\pi}(z+\sqrt{z^2-1}\cos\phi)^{-1/2}d\phi,\quad \mathrm{Re}\,z>0 \tag{3.8}$$

can be rewritten in the form

$$P_{-1/2}(z)=\frac{2}{\pi}(z+\sqrt{z^2-1})^{-1/2}K\left[\left(\frac{2\sqrt{z^2-1}}{z+\sqrt{z^2-1}}\right)^{1/2}\right]\ , \tag{3.9}$$

where K is the complete elliptic integral of the second kind.[19] Subsequently, the asymptotic expression [Eq. (8.113) in Ref. 19]

$$K(k)=-\ln(\sqrt{1-k^2})+\ln 4+O(1-k^2)\ ,$$

when combined with Eq. (3.9), yields the asymptotic relation

$$P_{-1/2}(z)=\frac{\sqrt{2}}{\pi}z^{-1/2}\ln z+\frac{3\sqrt{2}\ln 2}{\pi}z^{-1/2}+O\left(\frac{1}{z^2}\right)\ , \tag{3.10}$$

which in turn yields the desired asymptotic behavior

$$P_{-1/2}(1+2z)=\frac{1}{\pi}z^{-1/2}\ln z+\frac{3\ln 2}{\pi}z^{-1/2}+O\left(\frac{1}{z^{3/2}}\right) \tag{3.11}$$

as $z\to\infty$. Now we combine Eq. (3.11) with Eq. (2.60) and Eq. (2.50), and then compare the resulting asymptotic form of $V(\alpha,\beta)$ with Eqs. (3.2) and (3.7) to read out the following explicit expressions for the functions $\epsilon(\beta)$ and $\delta(\beta)$:

$$\epsilon(\beta)=\frac{1}{\pi}\int_{\beta}^{1}\left[V_{,s}(1,s)+\frac{V(1,s)}{2(1+s)}\right]\frac{1+s}{\sqrt{(1+\beta)(s-\beta)}}ds+\frac{1}{\pi}\int_{-\beta}^{1}\left[V_{,r}(r,1)+\frac{V(r,1)}{2(1+r)}\right]\frac{1+r}{\sqrt{(1-\beta)(r+\beta)}}dr\ , \tag{3.12a}$$

$$\delta(\beta) = -\int_{\beta}^{1}\left[V_{,s}(1,s)+\frac{V(1,s)}{2(1+s)}\right]\frac{1+s}{\sqrt{(1+\beta)(s-\beta)}}\left[\frac{2\ln 2}{\pi}+\frac{1}{\pi}\ln\left[\frac{(1+\beta)(s-\beta)}{1+s}\right]\right]ds$$
$$-\int_{-\beta}^{1}\left[V_{,r}(r,1)+\frac{V(r,1)}{2(1+r)}\right]\frac{1+r}{\sqrt{(1-\beta)(r+\beta)}}\left[\frac{2\ln 2}{\pi}+\frac{1}{\pi}\ln\left[\frac{(1-\beta)(r+\beta)}{1+r}\right]\right]dr\ . \qquad (3.12b)$$

We note that Eq. (3.12a) can be rewritten in the simpler form

$$\epsilon(\beta) = \frac{1}{\pi}\frac{1}{\sqrt{1+\beta}}\int_{\beta}^{1}[(1+s)^{1/2}V(1,s)]_{,s}\left[\frac{s+1}{s-\beta}\right]^{1/2}ds+\frac{1}{\pi}\frac{1}{\sqrt{1-\beta}}\int_{-\beta}^{1}[(1+r)^{1/2}V(r,1)]_{,r}\left[\frac{r+1}{r+\beta}\right]^{1/2}dr\ . \qquad (3.13)$$

The timelike coordinate α is a parameter which monotonically decreases to zero along the world line of any observer approaching the singularity. Consider the spacetime metric (2.43) in the vicinity of such a world line as the observer approaches the singularity $\alpha=0$ at a fixed spatial coordinate β. According to Eqs. (3.4) and (3.7), the asymptotic behavior of the metric along the observer's world line as $\alpha\to 0$ can be expressed as

$$g(\beta)\sim e^{-\mu(\beta)/2}l_1 l_2\,\alpha^{q_1(\beta)}(-d\alpha^2+d\beta^2)+e^{\delta(\beta)}\alpha^{q_2(\beta)}dx^2+e^{-\delta(\beta)}\alpha^{q_3(\beta)}dy^2\ , \qquad (3.14)$$

where

$$q_1(\beta)=\tfrac{1}{2}[\epsilon^2(\beta)-1]\ ,\quad q_2(\beta)=1+\epsilon(\beta)\ ,\quad q_3(\beta)=1-\epsilon(\beta)\ . \qquad (3.15)$$

On the right-hand side of Eq. (3.14), all quantities that depend on β are to be regarded as constants when interpreting the metric $g(\beta)$ as the asymptotic limit of the metric (2.43); this asymptotic metric describes a region of spacetime which is arbitrarily large in the Killing x, y directions, but which extends (in general) very little [over a range in β small enough for the variation in $\epsilon(\beta)$ to be negligible] in the β direction, and which covers a range $(0,\eta)$ in the coordinate α where η is arbitrarily small $(\eta\to 0)$. Now, notice that the quantity $q_1(\beta)$ is always greater than -2 [$q_1\geq -\frac{1}{2}$ by Eqs. (3.15)]. Thus, we can introduce a new timelike coordinate t,

$$t\equiv\alpha^{(q_1+2)/2}\ ,\quad \alpha\equiv t^{2/(q_1+2)}\ , \qquad (3.16)$$

which is monotonically related to α, and in which the singularity $\alpha=0$ is located at $t=0$. In terms of this new timelike coordinate t, the asymptotic metric $g(\beta)$ of Eq. (3.14) takes the form

$$g(\beta)\sim -\frac{4l_1l_2e^{-\mu(\beta)/2}}{[q_1(\beta)+2]^2}dt^2+l_1l_2e^{-\mu(\beta)/2}t^{2p_3}d\beta^2+e^{\delta(\beta)}t^{2p_1}dx^2+e^{-\delta(\beta)}t^{2p_2}dy^2\ , \qquad (3.17)$$

where

$$p_3(\beta)=\frac{\epsilon^2(\beta)-1}{\epsilon^2(\beta)+3}\ , \qquad (3.18a)$$

$$p_1(\beta)=\frac{2[1+\epsilon(\beta)]}{\epsilon^2(\beta)+3}\ , \qquad (3.18b)$$

$$p_2(\beta)=\frac{2[1-\epsilon(\beta)]}{\epsilon^2(\beta)+3}\ . \qquad (3.18c)$$

It is easily seen from Eqs. (3.18) that the exponents $p_1(\beta)$, $p_2(\beta)$, and $p_3(\beta)$ satisfy the Kasner relations[12]

$$p_1(\beta)+p_2(\beta)+p_3(\beta)=p_1{}^2(\beta)+p_2{}^2(\beta)+p_3{}^2(\beta)=1\ , \qquad (3.19)$$

for all values of $\epsilon(\beta)$. Therefore, the asymptotic limit of the metric (2.43) as $\alpha\to 0$ at a fixed spatial position β is a vacuum Kasner solution, which, after absorbing the constant terms on the right-hand side of Eq. (3.17) into the definition of the coordinates, and for simplicity using units in which lengths are dimensionless, can be represented in the form

$$g(\beta)=-dt^2+t^{2p_1}dx^2+t^{2p_2}dy^2+t^{2p_3}dz^2\ , \qquad (3.20)$$

where the asymptotic Kasner exponents p_k, $k=1,2,3$, are given by Eqs. (3.18) and satisfy the relations (3.19). Equations (3.18), when combined with Eq. (3.13), provide the explicit formulas which express the asymptotic Kasner exponents $p_k(\beta)$ along the singularity in terms of the initial data (2.49) for the colliding waves.

The Kasner solution[12] defined by the global spacetime metric (3.20) has the following curvature tensor:

$$\mathbf{R}=\sum_{j=1}^{3}\frac{p_j(p_j-1)}{t^2}(X_0\otimes\omega^j+X_j\otimes\omega^0)\otimes\omega^0\wedge\omega^j+\sum_{j<k}\frac{p_jp_k}{t^2}(X_j\otimes\omega^k-X_k\otimes\omega^j)\otimes\omega^j\wedge\omega^k\ , \qquad (3.21)$$

where the orthonormal basis $\{X_\mu\}$ and its dual $\{\omega^\nu\}$ are given by

$$X_0=\frac{\partial}{\partial t}\ ,\quad X_1=t^{-p_1}\frac{\partial}{\partial x}\ ,\quad X_2=t^{-p_2}\frac{\partial}{\partial y}\ ,\quad X_3=t^{-p_3}\frac{\partial}{\partial z}\ , \qquad (3.22a)$$

$$\omega^0=dt\ ,\quad \omega^1=t^{p_1}dx\ ,\quad \omega^2=t^{p_2}dy\ ,\quad \omega^3=t^{p_3}dz\ . \qquad (3.22b)$$

A number of fundamental properties of the Kasner spacetime can easily be deduced from the expression (3.21) of the curvature tensor. First, it becomes clear that the vacuum field equations are equivalent to the algebraic relations (3.19) for the exponents p_k. Next, a short computation using Eq. (3.21) gives

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = -\frac{16}{t^4}p_1p_2p_3 \tag{3.23}$$

provided that the vacuum conditions (3.19) are satisfied. Assuming that (3.19) hold, the following main conclusions are then obtained. (i) The "surface" $t=0$ is a curvature singularity of the vacuum Kasner solution *unless* one of the exponents is zero. This can only happen if (p_1,p_2,p_3) is equal to a permutation of (1,0,0), in which case we assume, without loss of generality, that $p_1=1$, $p_2=p_3=0$. For these values of the exponents (a degenerate Kasner solution) the metric (3.20) is flat [Eq. (3.21) gives $\mathbf{R}\equiv 0$]; the surface $\{t=0\}$ represents a Killing-Cauchy horizon[4] (a coordinate singularity) across which spacetime can be extended, e.g., to yield the maximal Minkowski space. The spacelike Killing vector $\partial/\partial x$ becomes null on this Killing-Cauchy horizon $\{t=0\}$. (ii) If all exponents are nonzero (the nondegenerate case), then one and only one of the exponents is strictly negative, while the other two are strictly positive. And finally, a straightforward application of the geodesic deviation equation with the curvature tensor (3.21) reveals that (iii) in a nondegenerate Kasner solution, timelike geodesic congruences which run into the singularity converge together in those spatial directions for which $p_k>0$, and diverge apart in the direction for which $p_k<0$. In other words, physical three-volumes get squashed in the two spatial directions with positive exponents, while they get infinitely stretched in the remaining direction with the negative exponent as the singularity is approached.[12]

After this brief interlude on the Kasner solution, we now return to the discussion of the asymptotic limit (3.17) and (3.18) of the colliding plane-wave metric (2.43). For much more detailed expositions on the Kasner solution (including its generalizations and their application to cosmology), the reader is referred to the literature listed as Ref. 12.

The following conclusions are easily obtained from Eqs. (3.17)–(3.20) combined with the results of our brief review of the Kasner solution: (i) If $|\epsilon(\beta)|<1$, then $p_1(\beta)$ and $p_2(\beta)$ are both positive and $p_3(\beta)$ is negative. This corresponds to an *anastigmatic*[2,3] singularity structure at $(\alpha=0,\beta)$; that is, focusing takes place in both the x and y directions. In particular, if the incoming plane waves are sandwich waves and either purely anastigmatic[2,3] or very nearly anastigmatic (i.e., if they have focal lengths f_1, f_2 [cf. Eqs. (2.13)–(2.14)] which are either equal $(f_1=f_2)$ or satisfy $|f_2-f_1|/f_1 \ll 1$), and if both incoming waves are sufficiently weak [i.e., if V_1, $V_2 \ll 1$, cf. Eq. (2.15)], then Eq. (3.13) implies that at least throughout a large subinterval of the range $(-1,1)$ of β, $|\epsilon(\beta)|$ will be much smaller than 1. Thus, under these circumstances, the structure of the singularity will be mostly anastigmatic. (ii) If, on the contrary, $|\epsilon(\beta)|>1$, then $p_3(\beta)$ is positive and one of $p_1(\beta)$, $p_2(\beta)$ is negative. This corresponds to an astigmatic singularity structure at $(\alpha=0,\beta)$; that is, focusing occurs in only one of the two transverse directions x, y, whereas in the other direction an infinite defocusing takes place. In particular, if the incoming plane waves are highly astigmatic $(|f_2-f_1|/f_1 \gg 1)$, or if they are sufficiently strong $(V_1, V_2 \sim 1)$, then it is possible to have an interval in β with $|\epsilon(\beta)|>1$, that is, an interval in β with an astigmatic singularity structure at $\alpha=0$. (See, however, our second example in Sec. IV in which colliding highly astigmatic plane waves create a purely anastigmatic singularity.) (iii) Finally, if $|\epsilon(\beta)|=1$, then $p_3(\beta)=0$ and one of $p_1(\beta)$, $p_2(\beta)$ is 1 whereas the other is zero. In this case the asymptotic metric $g(\beta)$ near $\alpha=0$ is a degenerate Kasner solution (3.20) with either $(p_1,p_2,p_3)=(1,0,0)$ or $(p_1,p_2,p_3)=(0,1,0)$.

It seems evident that if the quantity $\epsilon(\beta)$ is different from ± 1 (across an interval in β or at any point $\beta=\beta_0$), then the colliding plane-wave solution (2.43) has a curvature singularity at $(\alpha=0,\beta)$. On the other hand, in view of our conclusion (iii) in the above paragraph, it is also quite natural to expect that if $\epsilon(\beta)\equiv\pm 1$ throughout an interval (β_1,β_2) in β, then the portion $\{\alpha=0, \beta_1<\beta<\beta_2\}$ of the surface $\{\alpha=0\}$ is *not* a curvature singularity, but instead it represents a nonsingular Killing-Cauchy horizon of the colliding plane-wave spacetime on which one of the spacelike Killing vector fields $\partial/\partial x$, $\partial/\partial y$ becomes null, and across which the metric can be smoothly extended.[4] However, our analysis so far is not sufficient to reach these conclusions rigorously. The reason is that although we now know the asymptotic limit of the metric (2.43) explicitly [Eq. (3.17)], we do not yet have full control on the asymptotic behavior of the spacetime curvature near the singularity $\alpha=0$. In other words, in view of the presence of a whole series of logarithmic terms in the expansion (3.7) of $V(\alpha,\beta)$ [and similarly in the expansion (3.4) of $Q(\alpha,\beta)$], it is not clear *a priori* that the asymptotically Kasner nature of the metric as $\alpha\to 0$ implies the corresponding asymptotically Kasner (as $t\to 0$) behavior of the spacetime curvature (which involves the derivatives of the metric). Thus, in the following section, we are going to study the behavior of the curvature associated with the metric (2.43) near the singularity $\alpha=0$.

B. The behavior of curvature near $\alpha=0$

Since in most of the literature on colliding plane waves[5,6,7,9,11] the spacetime curvature is studied in terms of the Newman-Penrose curvature quantities, we will also find it convenient to carry out our analysis of curvature using the curvature quantities (2.19) with respect to the standard tetrad (2.3) and (2.4). Equations (2.19) express these curvature quantities in terms of the tetrad coefficients M, U, and V, and in the Rosen-type coordinate system (u,v,x,y) for the colliding plane-wave spacetime. In this section, we will first obtain the corresponding formulas expressing the *same functions* Ψ_0, Ψ_2, and Ψ_4 in terms of the metric functions $V(\alpha,\beta)$ and $Q(\alpha,\beta)$, and in our favorite (α,β,x,y) coordinate system. Then, using these expressions, we will read out the asymptotic behavior of the curvature quantities as $\alpha\to 0$.

Consider first Eq. (2.19b) for the quantity Ψ_2. Combining this equation with Eq. (2.34) and Eq. (2.39b), and using Eq. (2.20), we obtain

$$\Psi_2 = -\frac{e^{Q/2}}{4l_1 l_2}\alpha^{1/2}(\partial_\alpha{}^2 - \partial_\beta{}^2)M \ . \tag{3.24}$$

Now note the following identities

$$\ln(U_{,u}U_{,v}) = \ln(\alpha_{,u}) + \ln(\alpha_{,v}) - 2\ln\alpha \ , \tag{3.25a}$$

$$\partial_u\partial_v \ln(\alpha_{,u}) = 0 \ , \quad \partial_u\partial_v\ln(\alpha_{,v}) = 0 \ , \tag{3.25b}$$

which are derived by using Eqs. (2.20) and (2.21), respectively. If we take the logarithm of both sides in Eq. (2.39b) and apply the operator $\partial_\alpha{}^2 - \partial_\beta{}^2$ on both sides of the result, and if we then use Eqs. (2.34) and (3.25) to simplify, we obtain the identity

$$(\partial_\alpha{}^2 - \partial_\beta{}^2)M = \tfrac{1}{2}(Q_{,\alpha\alpha} - Q_{,\beta\beta} - \alpha^{-2}) \ , \tag{3.26}$$

which, when combined with Eq. (3.24), yields the desired expression for the quantity Ψ_2:

$$\Psi_2 = -\frac{e^{Q(\alpha,\beta)/2}}{8l_1 l_2}\alpha^{1/2}\left[Q_{,\alpha\alpha} - Q_{,\beta\beta} - \frac{1}{\alpha^2}\right] . \tag{3.27}$$

The calculation of the corresponding expressions for the remaining curvature quantities Ψ_0 and Ψ_4 proceeds along similar lines. Substituting Eq. (2.30a) in the expression (2.19a) for Ψ_0, and then making use of the identity

$$(\partial_\alpha - \partial_\beta)M = \frac{1}{2}\left[Q_{,\alpha} - Q_{,\beta} + \frac{1}{\alpha}\right] - \frac{\alpha_{,uu}}{\alpha_{,u}{}^2} \ ,$$

together with the Eqs. (2.20) and (2.21), we obtain

$$\Psi_0 = \frac{i^\alpha}{8l_1{}^2 l_2{}^2}\frac{e^{Q(\alpha,\beta)}}{\alpha_{,v}{}^2}\left[\tfrac{1}{2}(V_{,\alpha} - V_{,\beta})\left[Q_{,\alpha} - Q_{,\beta} + \frac{3}{\alpha}\right] + V_{,\alpha\alpha} + V_{,\beta\beta} - 2V_{,\alpha\beta}\right] . \tag{3.28}$$

And, substituting Eq. (2.30b) in the expression (2.19c) for Ψ_4, and then making use of the identity

$$(\partial_\alpha + \partial_\beta)M = \frac{1}{2}\left[Q_{,\alpha} + Q_{,\beta} + \frac{1}{\alpha}\right] - \frac{\alpha_{,vv}}{\alpha_{,v}{}^2} \ ,$$

together with the Eqs. (2.20) and (2.21), we obtain

$$\Psi_4 = -\frac{i}{2}\alpha_{,v}{}^2\left[\tfrac{1}{2}(V_{,\alpha} + V_{,\beta})\left[Q_{,\alpha} + Q_{,\beta} + \frac{3}{\alpha}\right] + V_{,\alpha\alpha} + V_{,\beta\beta} + 2V_{,\alpha\beta}\right] . \tag{3.29}$$

Note that the quantity $\alpha_{,v}$ that occurs in the expressions (3.28) and (3.29) is not fully expressed in terms of the metric functions $Q(\alpha,\beta)$ and $V(\alpha,\beta)$. However, by Eq. (2.26), $\alpha_{,v} = -U_{2,v}e^{-U_2(v)}$, and thus by the Eqs. (2.29) and the discussion preceding them, $\alpha_{,v}$ is nonzero and finite for all $-1 < \beta < 1$ in the limit $\alpha \to 0$. Therefore, the multiplicative factors involving $\alpha_{,v}$ in Eqs. (3.28) and (3.29) do not contribute to the qualitative asymptotic behavior of the curvature quantities near $\alpha = 0$. Hence, we will not attempt to further express the quantity $\alpha_{,v}$ in terms of the metric functions Q and V; instead, we will regard it as a (nonzero) constant, multiplying the asymptotic limits of Ψ_0 and Ψ_4 as $\alpha \to 0$ at a *fixed* spatial location β.

Before proceeding with the analysis of the asymptotic behaviors of Ψ_2, Ψ_0, and Ψ_4 near $\alpha = 0$, we rewrite Eqs. (3.27)–(3.29) in terms of the metric function $V(\alpha,\beta)$. After eliminating the terms that involve the derivatives of $Q(\alpha,\beta)$ by making use of Eqs. (2.42), we obtain

$$\Psi_2 = -\frac{e^{Q(\alpha,\beta)/2}}{8l_1 l_2}\alpha^{1/2}\left[-V_{,\alpha}{}^2 - V_{,\beta}{}^2 + 2\alpha V_{,\alpha}(V_{,\beta\beta} - V_{,\alpha\alpha}) - \frac{1}{\alpha^2}\right] , \tag{3.30}$$

$$\Psi_0 = \frac{i}{8l_1{}^2 l_2{}^2\alpha_{,v}{}^2}e^{Q(\alpha,\beta)}\alpha\left[\tfrac{1}{2}(V_{,\alpha} - V_{,\beta})\left[2\alpha V_{,\alpha}V_{,\beta} - \alpha V_{,\alpha}{}^2 - \alpha V_{,\beta}{}^2 + \frac{3}{\alpha}\right] + V_{,\alpha\alpha} + V_{,\beta\beta} - 2V_{,\alpha\beta}\right] , \tag{3.31}$$

$$\Psi_4 = -\frac{i}{2}\alpha_{,v}{}^2\left[\tfrac{1}{2}(V_{,\alpha} + V_{,\beta})\left[-2\alpha V_{,\alpha}V_{,\beta} - \alpha V_{,\alpha}{}^2 - \alpha V_{,\beta}{}^2 + \frac{3}{\alpha}\right] + V_{,\alpha\alpha} + V_{,\beta\beta} + 2V_{,\alpha\beta}\right] . \tag{3.32}$$

Now, substituting the asymptotic expressions (3.7) and (3.4) for $V(\alpha,\beta)$ and $Q(\alpha,\beta)$ into Eqs. (3.30)–(3.32), we obtain, after some lengthy algebra, the following equations revealing the asymptotic behavior of the curvature quantities Ψ_2, Ψ_0, and Ψ_4, as $\alpha \to 0$ at a fixed spatial coordinate β:

$$\Psi_2(\beta) \sim -\frac{e^{\mu(\beta)/2}}{8l_1 l_2}\alpha^{[1-\epsilon^2(\beta)]/2}\left[\frac{\epsilon^2(\beta) - 1}{\alpha^2} + \frac{1}{\alpha}O(\alpha)\right] , \tag{3.33}$$

$$\Psi_0(\beta) \sim \frac{ie^{\mu(\beta)}}{8l_1{}^2 l_2{}^2\alpha_{,v}{}^2}\alpha^{[1-\epsilon^2(\beta)]}\left[\frac{\epsilon(\beta)[1-\epsilon^2(\beta)]}{2\alpha^2} + \frac{3[\epsilon^2(\beta)-1][\epsilon'(\beta)\ln\alpha + \delta'(\beta)]}{2\alpha} - \frac{2\epsilon'(\beta)}{\alpha} + \frac{1}{\alpha}O(\alpha)\right] , \tag{3.34}$$

$$\Psi_4(\beta) \sim -\frac{i}{2}\alpha_{,v}{}^2\left[\frac{\epsilon(\beta)[1-\epsilon^2(\beta)]}{2\alpha^2} - \frac{3[\epsilon^2(\beta)-1][\epsilon'(\beta)\ln\alpha + \delta'(\beta)]}{2\alpha} + \frac{2\epsilon'(\beta)}{\alpha} + \frac{1}{\alpha}O(\alpha)\right] . \tag{3.35}$$

In Eqs. (3.33)–(3.35), $O(\alpha)$ denotes the remaining terms which are always of the form

$$O(\alpha)\equiv(\ )\alpha\ln\alpha+(\ )\alpha^2\ln\alpha+\cdots+(\ )\alpha(\ln\alpha)^2 +(\ )\alpha^2(\ln\alpha)^2+\cdots+(\ )\alpha+(\ )\alpha^2+\cdots\ , \tag{3.36}$$

where the ( ) denote well-behaved quantities that depend only on β.

Equations (3.33)–(3.35) provide a clear demonstration of our earlier statement (Sec. III A) that whenever $|\epsilon(\beta)|\neq 1$ (across an interval in β or at an isolated point $\beta=\beta_0$), the colliding plane-wave spacetime possesses a curvature singularity at $(\alpha=0,\beta)$. [The asymptotic form of the curvature invariant (3.23) can be computed using Eqs. (3.33)–(3.35) along with the identities $\Psi_1=\Psi_3\equiv 0$; it is easily seen that as $\alpha\to 0$ this invariant diverges in accordance with Eqs. (3.23) and (3.16), i.e., as $\sim\alpha^{-[\epsilon^2(\beta)+3]}$, whenever $|\epsilon(\beta)|\neq 1$.] In order to prove our second statement (Sec. III A), that when $|\epsilon(\beta)|\equiv 1$ throughout an interval (β_1,β_2) the surface $\{\alpha=0,\beta_1<\beta<\beta_2\}$ is a nonsingular Killing-Cauchy horizon,[4] we will need to perform a somewhat more detailed analysis of the asymptotic behavior of $V(\alpha,\beta)$ near $\alpha=0$. Thus, in the few remaining paragraphs of this section, we will present such an analysis and see that our conclusions indeed provide a proof for this second statement. Then, in the next section (Sec. III C), we will discuss the physical significance and the instabilities of these Killing-Cauchy horizons which occur at $\alpha=0$.

Before proceeding with our discussion, we note that when $|\epsilon(\beta)|\equiv 1$ across an interval in β, all divergent terms in the expressions (3.33)–(3.35) vanish except (possibly) for the logarithmically divergent terms which could be introduced by the remainders $O(\alpha)/\alpha$ [Eq. (3.36)]. To learn more about these logarithmic terms, consider the expansion (3.7) for $V(\alpha,\beta)$. Equations (3.12) and (3.13) give expressions for the two most important coefficients $\epsilon(\beta)$ and $\delta(\beta)$ which occur in this expansion, and the other coefficients $c_k(\beta)$ and $d_k(\beta)$ can be computed from the original field equation (2.44a) for V: The following expressions for the derivatives of V that occur in Eq. (2.44a) are obtained straightforwardly by using Eq. (3.7):

$$\frac{1}{\alpha}V_{,\alpha}=\frac{\epsilon(\beta)}{\alpha^2}+2c_1(\beta)\ln\alpha+4c_2(\beta)\alpha^2\ln\alpha +[c_1(\beta)+2d_1(\beta)]+[c_2(\beta)+4d_2(\beta)]\alpha^2 +O(\alpha^4)\ , \tag{3.37a}$$

$$V_{,\alpha\alpha}=-\frac{\epsilon(\beta)}{\alpha^2}+2c_1(\beta)\ln\alpha+12c_2(\beta)\alpha^2\ln\alpha +[3c_1(\beta)+2d_1(\beta)] +[7c_2(\beta)+12d_2(\beta)]\alpha^2+O(\alpha^4)\ , \tag{3.37b}$$

$$V_{,\beta\beta}=\epsilon''(\beta)\ln\alpha+c_1''(\beta)\alpha^2\ln\alpha+c_2''(\beta)\alpha^4\ln\alpha +\delta''(\beta)+d_1''(\beta)\alpha^2+d_2''(\beta)\alpha^4+O(\alpha^6)\ . \tag{3.37c}$$

Inserting Eqs. (3.37) in the field equation (2.44a) and collecting together the coefficients of identical terms in α, we obtain the identities

$$c_1(\beta)=\tfrac{1}{4}\epsilon''(\beta)\ ,\quad c_2(\beta)=\tfrac{1}{64}\epsilon''''(\beta)\ ,\quad \ldots\ , \tag{3.38a}$$

$$d_1(\beta)=\tfrac{1}{4}[\delta''(\beta)-\epsilon''(\beta)]\ ,\quad d_2(\beta)=\tfrac{1}{128}[2\delta''''(\beta)-3\epsilon''''(\beta)]\ ,\quad \ldots\ , \tag{3.38b}$$

which express all of the coefficients $c_k(\beta)$ and $d_k(\beta)$ in Eq. (3.7) in terms of the coefficients $\epsilon(\beta)$ and $\delta(\beta)$.

It now becomes clear that when $|\epsilon(\beta)|\equiv 1$ throughout an interval (β_1,β_2) [in fact, whenever $\epsilon(\beta)$ is constant across such an interval], all the coefficients $c_k(\beta)$, $k\geq 1$ vanish for $\beta\in(\beta_1,\beta_2)$. In that case, the expansion (3.7) of $V(\alpha,\beta)$ *does not contain any logarithmic terms except for the leading term* $\epsilon(\beta)\ln\alpha$. *In particular, the derivatives* $V_{,\alpha}$, $V_{,\beta}$, $V_{,\alpha\alpha}$, $V_{,\beta\beta}$, $V_{,\alpha\beta}$ *contain no logarithmic terms whatsoever in* α, for $\beta\in(\beta_1,\beta_2)$. Therefore, by Eqs. (3.30)–(3.32), the remainder terms $O(\alpha)$ in Eqs. (3.33)—(3.35) also do not involve *any* logarithmic terms in α across the same interval; in other words

$$O(\alpha)=(\ )\alpha+(\ )\alpha^2+\cdots\quad\forall\ \beta\in(\beta_1,\beta_2)\ . \tag{3.39}$$

Combining Eq. (3.39) with Eqs. (3.33)—(3.35), we find that we have proved the following result.

If $|\epsilon(\beta)|\equiv 1$ *throughout an interval* (β_1,β_2) *in* β, *then the curvature quantities* Ψ_0, Ψ_2, *and* Ψ_4 *are all bounded* ($\equiv$*finite, but in general nonzero) as* $\alpha\to 0$, *whenever* β *belongs to this interval* (β_1,β_2); *i.e., all curvature quantities are perfectly well behaved across the surface* $\{\alpha=0,\beta_1<\beta<\beta_2\}$.

Clearly, if $|\epsilon(\beta_0)|=1$ at an isolated point $\beta=\beta_0$, and furthermore if $\epsilon'(\beta_0)=\epsilon''(\beta_0)=0$, then by Eqs. (3.38a), $c_1(\beta_0)=\varnothing$, and consequently

$$O(\alpha)=(\ )\alpha^2\ln\alpha+(\ )\alpha^3\ln\alpha+\cdots+(\ )\alpha^2(\ln\alpha)^2 +\cdots+(\ )\alpha+(\ )\alpha^2+\cdots\quad\text{at }\beta=\beta_0\ . \tag{3.40}$$

Therefore, combining Eq. (3.40) with the Eqs. (3.33)–(3.35) as above, we obtain the following similar result.

If $|\epsilon(\beta)|=1$ *at an isolated point* $\beta=\beta_0$, *and if, in addition, the first two derivatives of* $\epsilon(\beta)$ *at the point* β_0 *vanish, then the curvature quantities* Ψ_0, Ψ_2, *and* Ψ_4 *are bounded* ($\equiv$*finite, but in general nonzero) as* $\alpha\to 0$ *at the point* $\beta=\beta_0$.

C. Instability of the Killing-Cauchy horizons that occur at $\alpha=0$

We begin this section by rephrasing, in a somewhat more precise format, the three fundamental conclusions of the preceding section (Sec. III B).

(i) When $|\epsilon(\beta_0)|\neq 1$, the two-surface $\{\alpha=0,\beta=\beta_0,-\infty<x<+\infty,-\infty<y<+\infty\}$ is a curvature singularity of the colliding plane-wave metric (2.43). This singularity is of asymptotically (nondegenerate) Kasner type, and it is in general inhomogeneous in the spatial β direction.

(ii) When $|\epsilon(\beta)|=1$ at an isolated point $\beta=\beta_0$, i.e., if the C^1 function $|\epsilon(\beta)|$ maps a small interval containing β_0 into a real interval containing 1 *in such a way that the inverse image of 1 is a single point (namely, β_0)*, then there are two possibilities: If $\epsilon'(\beta_0)=\epsilon''(\beta_0)=0$, then the two-surface $\mathcal{P}=\{\alpha=0,\beta=\beta_0,-\infty<x<+\infty,-\infty<y<+\infty\}$ is *not* a curvature singularity, but it still represents a spacetime singularity since no extension of the metric is possible across $\mathcal{P}$. Such an extension does not exist because in any spacetime neighborhood of $\mathcal{P}$ there are boundary points corresponding to true curvature singularities; consequently, any extension of the metric beyond the two-surface $\mathcal{P}$ would be incompatible with the topological manifold structure of the spacetime. [Note that, by Eqs. (3.14) and (3.15), β is a regular coordinate near β_0 when $|\epsilon(\beta_0)|=1$. Also note that similar topological singularities frequently occur in the exact solutions for colliding plane waves, see, e.g., Refs. 8, 5, and 3.] If, on the other hand, either one (or both) of $\epsilon'(\beta_0)$, $\epsilon''(\beta_0)$ are nonzero, then $\mathcal{P}$ is a genuine curvature singularity of the colliding plane wave metric (2.43).

(iii) Finally, when $|\epsilon(\beta)|\equiv 1$ throughout an interval (β_1,β_2), the three-surface $\mathcal{S}=\{\alpha=0,\ \beta_1<\beta<\beta_2,-\infty<x<+\infty,-\infty<y<+\infty\}$ is a nonsingular Killing-Cauchy horizon for the colliding plane-wave spacetime. The asymptotic Kasner exponents $[p_1(\beta),p_2(\beta),p_3(\beta)]$ take one of the degenerate values (1,0,0) [if $\epsilon(\beta)=+1$] or (0,1,0) [if $\epsilon(\beta)=-1$] for $\beta\in(\beta_1,\beta_2)$ [Eqs. (3.18)], and correspondingly one of the spacelike Killing vectors $\partial/\partial x$ or $\partial/\partial y$ becomes a null vector on $\mathcal{S}$. As can be seen easily by inspection of the metric (2.43), $\mathcal{S}$ is a null hypersurface in spacetime, and the Killing vector that becomes null on $\mathcal{S}$ is tangent to the null geodesic generators of $\mathcal{S}$. In fact, $\mathcal{S}$ is a "Killing-Cauchy horizon of type II" in the terminology of Ref. 4, where the reader can find a much more detailed description of such horizons. The spacetime curvature is perfectly well behaved across $\mathcal{S}$, and consequently $\mathcal{S}$ represents only a coordinate singularity of the (α,β,x,y) [or equivalently the (u,v,x,y)] coordinate system; it is possible to extend the metric and the spacetime beyond $\mathcal{S}$ after constructing a new admissible coordinate chart that covers $\mathcal{S}$ and its spacetime neighborhood regularly.

Now suppose $|\epsilon(\beta)|\equiv 1$ across some subinterval I of the range $(-1,1)$ of β. (Note that I might not be a connected interval.) Thus, the metric (2.43) can be extended beyond the null surface $\mathcal{S}=\{\alpha=0,\ \beta\in I,x,y\}$ in a perfectly smooth manner. This extension is not unique, however; the initial data posed by the incoming colliding plane waves do not uniquely single out a specific extension among the infinitely many possibilities. Therefore, the Killing-Cauchy horizon $\mathcal{S}$ is a future Cauchy horizon[17] for the initial characteristic surface $\{u=0\}\cup\{v=0\}$, i.e., $\mathcal{S}$ represents a future boundary for the domain of dependence $D^+[\{u=0\}\cup\{v=0\}]$ of this initial surface. Since this means a breakdown, beyond the surface $\mathcal{S}$, of the predictability of the spacetime geometry from the initial data posed on $\{u=0\}\cup\{v=0\}$ (or, equivalently, a breakdown of global hyperbolicity[17]), the occurrence of these Killing-Cauchy horizons in colliding plane-wave spacetimes may seem to contradict the cosmic censorship hypothesis,[20,21] or at least a version of this hypothesis suitably formulated for plane-symmetric spacetimes.[4] But recall that a careful formulation of cosmic censorship[20] always insists that the hypothesis holds only for "generic" spacetimes, where the notion of "genericity" is conveniently left unspecified so that it can be interpreted appropriately for specific examples. In fact, there are many "counter-examples" to cosmic censorship, which, in one way or another, fail to satisfy the criterion of "genericity."[20,21] Perhaps the best-known such examples are the maximal Reissner-Nordström and Kerr solutions; the inner horizons of these solutions constitute Cauchy horizons for all partial Cauchy surfaces located in the asymptotically flat region, and therefore cause the breakdown of global hyperbolicity in the corresponding maximal spacetimes. However, it is now well known[22] that these inner horizons are unstable against a large class of linearized perturbations (such as gravitational waves, electromagnetic radiation, . . .). It is therefore expected (but not yet fully proved), that in the interior of any rotating or charged black hole which is formed via "generic" gravitational collapse, the growth of these linear instabilities would destroy the inner horizon, turn it into a (spacelike) curvature singularity, and thereby restore the global hyperbolicity of the resulting spacetime.

Now, physically, though not in a formal mathematical way, the Killing-Cauchy horizon $\mathcal{S}$ of the colliding plane-wave solution (2.43) is similar to the inner Cauchy horizons of the Kerr and Reissner-Nordström solutions (which are also Killing-Cauchy horizons). To better understand the physical significance of the issue of the stability of the horizon $\mathcal{S}$, consider the geometry of the colliding plane-wave spacetime depicted in Fig. 2. For enhanced dramatical effect, we have assumed in this figure that the interval I [across which $|\epsilon(\beta)|\equiv 1$] is a disconnected interval made up of several connected pieces $I_1, I_2, \ldots, I_n$. Hence the Killing-Cauchy horizon $\mathcal{S}$ is also disconnected; it consists of several distinct horizons $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n$. The spacetime is extended beyond each of the horizons $\mathcal{S}_i$ in a different way; and there is also a large amount of freedom in the choice of each individual extension. In particular, one can choose the extensions in such a way that the n horizons $\mathcal{S}_i$ act as doorways (through the otherwise singular surface $\alpha=0$), which can be used by the observers living in the interaction region of the colliding plane-wave solution as "tunnels" into n different spacetimes, each *causally* disconnected from all the others. Or, any two distinct horizons $\mathcal{S}_i$, $\mathcal{S}_j$ may be joined, through suitable extensions, to the same spacetime but at different locations in time and space, thus giving the interaction region observers the possibility to influence the extended spacetime "simultaneously" at two different timelike-separated points (breakdown of global hyperbolicity in the extended spacetime). Although these possibilities are intriguing, clearly they cannot be realized (or at least they cannot be made physically plausible) unless the Killing-Cauchy horizon $\mathcal{S}$

is stable—unlike the unstable inner horizons of the Kerr and Reissner-Nordström solutions—against small perturbations of the colliding plane-wave spacetime. [In fact, similar speculations (such as using the interior regions as worm holes for spacetime travel) were made on the global structure of the maximal Kerr and Reissner-Nordström solutions;[17] these speculations were later rendered implausible by the instability results[22] we mentioned above (however, see Reference 23 in this context, where the worm-hole concept is revisited and resurrected in an unexpected direction).] Thus, for example, any "realistic" attempt to "build" a spacetime tunnel between two different universes by means of generating and colliding two gravitational plane waves would fail, unless the Killing-Cauchy horizons $\mathcal{S}_i$ at $\alpha=0$ are stable; in other words, unless the set of all initial data from which such horizons evolve constitutes an open subset (with respect to an appropriate topology[4]), or a subset with nonvanishing volume (with respect to an appropriate measure) in the set of all plane-symmetric initial data.

Now the horizons $\mathcal{S}_i$ are not the first examples of Killing-Cauchy horizons produced by colliding plane waves. As we have discussed in the Introduction, the occurrence of Killing-Cauchy horizons in colliding plane-

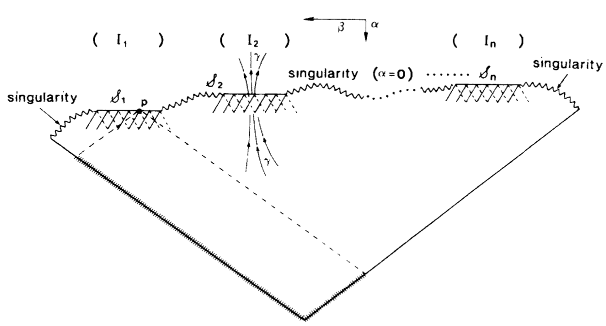

FIG. 2. The geometry of a colliding plane-wave solution (2.43) for which $|\epsilon(\beta)|\equiv 1$ throughout an interval I in β. The interval I is disconnected and is made up of several connected pieces $I_1, I_2, \dots, I_n$. Since the surface $\alpha=0$ corresponds to a Killing-Cauchy horizon whenever $\beta\in I$, the singularity at $\alpha=0$ is interrupted by the n Killing-Cauchy horizons $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n$ which are located along the intervals $I_1, I_2, \dots, I_n$, respectively. Through each of the horizons $\mathcal{S}_i$ the spacetime curvature is finite and well behaved. Consequently, across each horizon $\mathcal{S}_i$ the metric can be extended smoothly to a maximal spacetime (possibly a different one for each i) whose choice is essentially arbitrary. In particular, the observers living in the interaction region I can use these horizons $\mathcal{S}_i$ as tunnels along which they can travel (e.g., following the timelike world lines γ in the figure) into different universes. Equation (3.13) in the text, which relates the function $\epsilon(\beta)$ to the initial data for the colliding waves, can be regarded as describing a kind of superposition, at $\alpha=0$, of the two wave forms constituting the initial data for the incoming colliding plane waves. For example, to compute $\epsilon(\beta)$ at the point p in the figure, the initial data located in the cross-hatched portions of the two initial surfaces are superposed through Eq. (3.13). In order to have $|\epsilon(\beta)|$ constant across a connected interval like the interval I_1, it is necessary to adjust the initial data so as to cancel precisely the two separate contributions to $\epsilon(\beta)$ which would arise as the point p is moved across the interval I_1.

wave spacetimes was first discovered by Chandrasekhar and Xanthopoulos[11] when they produced several exact solutions which contained such horizons. Shortly after this work, Chandrasekhar and Xanthopoulos[24] discovered that the presence of a perfect fluid with (energy density) = pressure, or the presence of null dust, in their spacetime destroys the horizon in the full nonlinear Einstein theory. Independently and simultaneous with this discovery, the author[4] formulated and proved general theorems which established the instability of Killing-Cauchy horizons in *any* plane-symmetric spacetime against generic, linearized plane-symmetric perturbations. (In addition, there already exists a considerable amount of literature[22,21] on the instabilities of several particular examples of Killing-Cauchy horizons, and of general compact Cauchy horizons.[25]) However, except for the above-mentioned example of Chandrasekhar and Xanthopoulos[24] involving null fluids, the nonlinear growth of these linear instabilities and the subsequent transformation of the horizon into a singularity have remained only as plausible conjectures. Note that this situation is quite similar to the state of knowledge on the instability of the inner horizons of the Kerr and Reissner-Nordström spacetimes; there most of the convincing instability results are valid only for linearized perturbations, and we do not even have a Chandrasekhar-Xanthopoulos-type[24] analysis for special kinds of nonlinear perturbations.[22] (See, however, Ref. 26 where a qualitative argument is given for the full nonlinear instability of the Reissner-Nordström Killing-Cauchy horizon.)

It is therefore remarkable that the formalism which we have described thus far provides concise and rigorous proofs that (i) the Killing-Cauchy horizons $\mathcal{S}_i$ at $\alpha=0$ are unstable in the *full nonlinear theory* against small but generic (plane-symmetric) perturbations of the initial data for the colliding plane waves, and (ii) that in a very specific sense, "generic" initial data in the form (2.15) or (2.49) always produce all-embracing, spacelike spacetime singularities without Killing-Cauchy horizons at $\alpha=0$. Morover, the proofs of these statements are almost trivial. We shall demonstrate them in the following remaining few paragraphs of this section.

It is clear from our analysis thus far, that the structure of the singularities at $\alpha=0$ is completely determined by a single quantity: namely, the C^1 function $\epsilon(\beta)$. This function $\epsilon(\beta)$ is C^1 on the interval $(-1,1)$ because of Eq. (3.13) and because of our insistence [Eqs. (2.15) and (2.49)] that the initial data $V(r,1)$, $V(1,s)$ be C^1 functions. Now consider the space of all such C^1 functions on $(-1,1)$, which we will denote by F. This space F can be made into a Banach space[27] after endowing it with a suitable norm (such as the sup- or L^p norms[27]) and constructing its completion; but the precise choice of the norm is immaterial for the discussion that follows. Now the functional $\mathcal{E}$, which assigns a unique function $\epsilon(\beta)$ to each choice of initial data in the form (2.49) or (2.15), can be regarded as a mapping from the space of all possible initial data to the Banach space F of all possible functions $\epsilon(\beta)$. This mapping $\mathcal{E}$ is known to us in explicit form; using Eq. (3.13), we can write

$$\mathcal{E} : \{V(r,1),V(1,s)\} \rightarrow \epsilon(\beta) ,$$
$$\epsilon(\beta) = \frac{1}{\pi}\frac{1}{\sqrt{1+\beta}}\int_\beta^1 [(1+s)^{1/2}V(1,s)]_{,s}\left[\frac{s+1}{s-\beta}\right]^{1/2} ds + \frac{1}{\pi}\frac{1}{\sqrt{1-\beta}}\int_{-\beta}^1 [(1+r)^{1/2}V(r,1)]_{,r}\left[\frac{r+1}{r+\beta}\right]^{1/2} dr . \quad (3.41)$$

Or, using the Eqs. (2.51), we can write equivalently

$$\mathcal{E} : \{V_1(u),V_2(v)\} \rightarrow \epsilon(\beta) ,$$
$$\epsilon(\beta) = -\frac{1}{\pi}\left[\frac{2}{1+\beta}\right]^{1/2}\int_0^{v(\beta)}[e^{-U_2(v)/2}V_2(v)]_{,v}\left[\frac{2e^{-U_2(v)}}{2e^{-U_2(v)}-1-\beta}\right]^{1/2} dv$$
$$-\frac{1}{\pi}\left[\frac{2}{1-\beta}\right]^{1/2}\int_0^{u(-\beta)}[e^{-U_1(u)/2}V_1(u)]_{,u}\left[\frac{2e^{-U_1(u)}}{2e^{-U_1(u)}-1+\beta}\right]^{1/2} du , \quad (3.42a)$$

where $U_1(u)$, $U_2(v)$ are determined by Eqs. (2.16), and $v(\beta)$, $u(-\beta)$ are defined by

$$\beta=2e^{-U_2[v(\beta)]}-1 , \quad -\beta=2e^{-U_1[u(-\beta)]}-1 . \quad (3.42b)$$

Since it is obviously more transparent, we will use the representation of the mapping $\mathcal{E}$ given by Eq. (3.41) [instead of Eqs. (3.42)] throughout the present discussion. We will first prove the assertion (ii) that we have made in the last paragraph above, namely, that for generic initial data in the form (2.49), the surface $\alpha=0$ is an all-embracing spacetime singularity which does not involve any Killing-Cauchy horizons. We will then discuss the assertion (i) that the Killing-Cauchy horizons $\mathcal{S}_i$ are nonlinearly unstable against generic perturbations in the initial data, and will see that its proof follows very easily from the proof of assertion (ii).

Now Eq. (3.41) tells us that we can represent the map $\mathcal{E}$ symbolically as

$$\mathcal{E} : D \equiv F\oplus F \rightarrow F , \quad (3.43)$$

where D, the space of all initial data in the form $\{V(r,1),V(1,s)\}$, has been identified with the direct sum of Banach spaces $F\oplus F$ [cf. Eq. (2.49) and the discussion following it]. Consider, for each $\delta>0$, the subset H_δ of F given by

$$H_\delta \equiv \{\epsilon(\beta)\in F \mid \text{there exists a connected subinterval in } (-1,1) \text{ of length } \geq \delta \text{ across which } |\epsilon(\beta)| \equiv 1\} . \quad (3.44)$$

It is clear that H_δ is a closed subset of F with the property that its complement, H_δ', is dense in F with respect to the Banach space (norm) topology. We shall define a closed subset of a Banach space with this property as a *nongeneric subset;* i.e., a nongeneric subset in a Banach space B is a closed subset whose complement is dense in B. [This notion of a "nongeneric" subset intuitively corresponds to a physicist's notion of genericity. However, our notion does not necessarily coincide with the more frequently used notion of a "subset with measure zero." In fact, even in finite-dimensional Banach spaces there exist nongeneric subsets with nonzero Lebesgue measure (e.g., the fat Cantor set in the unit interval as a subset of R^1).[28] It is not yet clear whether our topological notion of genericity can be replaced with a measure theoretical alternative so as to leave the conclusions of this section intact. (See also the remarks at the end of the next paragraph in this connection.)] Thus, H_δ is a nongeneric subset of F for any $\delta>0$. Note also that $H_{\delta_1}\supset H_{\delta_2}$ whenever $\delta_1\leq\delta_2$.

It is now clear from the conclusions (i), (ii), and (iii) which we have listed in the beginning of this section, that if $\epsilon(\beta)$ is an element of F that does not belong to H_δ for any δ, then the corresponding colliding plane-wave solution possesses an all-embracing spacetime singularity at $\alpha=0$; this singularity is in general a curvature singularity, possibly crisscrossed with isolated (with respect to β) topological noncurvature singularities. Therefore, in order to prove our assertion (ii), we need only to prove that for all $\delta>0$, the inverse image $\mathcal{E}^{-1}(H_\delta)$ of H_δ under the map $\mathcal{E}$ is a nongeneric subset of the space of all initial data D. [The reader might be puzzled at this point as to why the subset $\mathcal{E}^{-1}(\cup_{\delta>0}H_\delta)$ of D is not what needs to be proved nongeneric. The answer lies in physics: From a physically realistic standpoint, there is always an absolute short-distance cutoff (a lower bound) on the length of a connected interval in β; this lower bound δ_c on δ is given by the Planck length l_P (or more precisely by $\delta_c=l_P/\sqrt{l_1 l_2}$). A Killing-Cauchy horizon that extends less than a Planck length δ_c in the β direction will almost certainly be indistinguishable, in its semiclassical manifestations, from a spacetime singularity. Furthermore, the subset $\cup_{\delta>0}H_\delta\subset F$ in question is *not* a nongeneric subset; in fact, it is easy to show that $\cup_{\delta>0}H_\delta$ is dense in F and hence is neither closed nor nongeneric. Thus, $\mathcal{E}^{-1}(\cup_{\delta>0}H_\delta)$ also is *not* nongeneric, and possibly it is dense in D. With our present somewhat naive (but physically satisfactory) notion of genericity, $\mathcal{E}^{-1}(\cup_{\delta>0}H_\delta)$ cannot be properly shown to be "nongeneric." In fact, it may be helpful to note that our genericity concept is similarly unable to identify the subset of all rational numbers in the unit interval as a "nongeneric" subset; the notion of the Lebesgue measure,[29] and not just a topological notion like ours, is needed to implement a formulation of genericity powerful enough to handle such questions effectively. It is conceivable that in our case too, a suitable extension of the notion of measure to infinite dimensional Banach spaces could yield both a more appropriate formulation and a proof for the "nongenericity" of the subset $\mathcal{E}^{-1}(\cup_{\delta>0}H_\delta)$.]

Turn now to the proof of the assertion that $\mathcal{E}^{-1}(H_\delta)\subset D$ is nongeneric for any $\delta>0$. For this proof, we need to consider some basic properties of the mapping $\mathcal{E}:D\to F$. First of all, it is easy to see that $\mathcal{E}$ is an onto map; that is, for any element $\epsilon(\beta)$ in F, there exists a choice (in fact infinitely many choices) of initial data in D which would yield, under the map $\mathcal{E}$, precisely the element $\epsilon(\beta)$. [In fact, the inverse image $\mathcal{E}^{-1}(q)$ of any point $q\equiv\epsilon(\beta)\in F$ is an infinite set in D; to find just one element in this set, take $V(1,s)$ to be any function and solve the resulting integral equation (3.41) for $V(r,1)$.] The second basic property of $\mathcal{E}$ is that $\mathcal{E}$ is defined on the whole Banach space D; that is, the domain of $\mathcal{E}$ is D. [To see this, apply a formal partial integration on both of the integrals in Eq. (3.41); the result can be written in a form which does not involve any differentiations of the functions $V(r,1)$ and $V(1,s)$.] And finally, $\mathcal{E}$ is a continuous linear mapping from D onto F. This follows (i) by first noting that $\mathcal{E}$ is a closed linear operator[27] [linearity of $\mathcal{E}$ is obvious from Eq. (3.41); closedness of $\mathcal{E}$ follows since $\mathcal{E}$ is essentially the composition of a differentiation operator (which is closed) and an integral operator (which is continuous)], and (ii) then using the closed graph theorem (Sec. II. 6 of Ref. 27) which says that a closed, onto linear mapping $\mathcal{E}:D\to F$ with domain $=D$ is continuous. Now we are ready to prove that $\mathcal{E}^{-1}(H_\delta)\subset D$ is nongeneric: Since $\mathcal{E}$ is continuous, $\mathcal{E}^{-1}(H_\delta)$ is a closed subset of D. To see that the complement of $\mathcal{E}^{-1}(H_\delta)$ in D is dense, use the open mapping theorem (Sec. II. 5 of Ref. 27) to conclude that $\mathcal{E}$ is an open map. If the complement of $\mathcal{E}^{-1}(H_\delta)$ were not dense in D, $\mathcal{E}^{-1}(H_\delta)$ would contain an open subset, and the open map $\mathcal{E}$ would send this open set onto an open subset of H_δ in F. This is impossible, since the subset H_δ is nongeneric and hence cannot contain an open set. This contradiction demonstrates that $\mathcal{E}^{-1}(H_\delta)$ is a nongeneric subset of D for any $\delta>0$.

It is now very easy to prove our remaining assertion: namely, that the Killing-Cauchy horizons $\mathcal{S}_i$ are nonlinearly unstable against generic perturbations of the initial data. Consider a given choice of initial data represented by a point p in the Banach space D. If the colliding plane-wave spacetime which evolves from these initial data p possesses Killing-Cauchy horizons $\mathcal{S}_i$ at $\alpha=0$, then there is a $\delta_0>0$ such that $p\in\mathcal{E}^{-1}(H_{\delta_0})$ (just take δ_0 as the size of the smallest horizon $\mathcal{S}_j$). Consequently, $p\in\mathcal{E}^{-1}(H_\delta)$ for each $\delta\le\delta_0$. Now for each such $\delta\le\delta_0$, no matter how small, the set $\mathcal{E}^{-1}(H_\delta)$ to which p belongs is a nongeneric subset of D. Therefore, for each $\delta\le\delta_0$, no matter how small, a generic perturbation of the point p representing the initial data will push p outside the subset $\mathcal{E}^{-1}(H_\delta)$; in other words, for any fixed but arbitrarily small $\delta>0$, a generic perturbation of the initial data p would destroy all horizons $\mathcal{S}_i$ at $\alpha=0$ that are of length $\ge\delta$, and turn them into spacetime singularities. In fact, since from a physically realistic standpoint all Killing-Cauchy horizons have to be of a size large compared to the Planck size δ_c, even the nongenericity of just the set $\mathcal{E}^{-1}(H_{\delta_c})$ is sufficient to conclude that the horizons $\mathcal{S}_i$ are nonlinearly unstable against plane-symmetric perturbations.

To get an intuitive feeling about these instabilities, it might be useful to think of Eq. (3.41) as describing some kind of a superposition, at $\alpha=0$, of the two wave forms described by the functions $V(r,1)$ and $V(1,s)$ which constitute the initial data (see Fig. 2). Killing-Cauchy horizons form at $\alpha=0$ only when this superposition results in a "perfectly destructive interference" ($|\epsilon(\beta)|\equiv1$) across some interval in the spatial coordinate β. Any generic perturbation in the wave forms $V(r,1)$ and $V(1,s)$ causes small imperfections in the precision of this destructive interference [$|\epsilon(\beta)|$ slightly deviates from 1]; and any small deviation from perfect destructive interference is sufficient to turn the Killing-Cauchy horizons into spacetime singularities (Fig. 2).

IV. EXAMPLES OF EXACT SOLUTIONS WHICH EXHIBIT SOME OF THE ABOVE-DISCUSSED ASYMPTOTIC SINGULARITY STRUCTURES

Our first example is the well-known Khan-Penrose[5] solution for colliding impulsive plane waves. The reader is referred to the original references[5,6,8] for comprehensive descriptions of the Khan-Penrose solution; here we will only discuss it from the point of view of our analysis in Sec. III above. The initial data for the Khan-Penrose solution written in the form of Eq. (2.15) are

$$V_1(u)=\ln\left[\frac{1+(u/a)}{1-(u/a)}\right],\quad V_2(v)=\ln\left[\frac{1+(v/b)}{1-(v/b)}\right],\tag{4.1}$$

which give, by Eqs. (2.16),

$$U_1(u)=-\ln\left[1-\frac{u^2}{a^2}\right],\quad U_2(v)=-\ln\left[1-\frac{v^2}{b^2}\right].\tag{4.2}$$

From Eqs. (4.2), (2.26), (2.27), and (2.51), we obtain the explicit forms of the various coordinate transformations we have discussed in Sec. II B,

$$\alpha=1-\frac{u^2}{a^2}-\frac{v^2}{b^2},\quad \beta=\frac{u^2}{a^2}-\frac{v^2}{b^2},$$
$$r=1-2\frac{u^2}{a^2},\quad s=1-2\frac{v^2}{b^2},\tag{4.3}$$

which, when combined with the Eq. (4.1), yield the Khan-Penrose initial data in the form (2.49):

$$V_{KP}(r,1)=\ln\left[\frac{1+\sqrt{(1-r)/2}}{1-\sqrt{(1-r)/2}}\right],$$
$$V_{KP}(1,s)=\ln\left[\frac{1+\sqrt{(1-s)/2}}{1-\sqrt{(1-s)/2}}\right]. \tag{4.4}$$

When computed with the initial data (4.4), the explicit solution (2.60) gives an expression for $V_{KP}(\alpha,\beta)$ in closed form. If we take the normalization point (u_0,v_0) as $[\frac{1}{2}(\sqrt{3}-1)a, \frac{1}{2}(\sqrt{3}-1)b]$, and insert $V_{KP}(\alpha,\beta)$ into Eq. (2.44b), the function $Q_{KP}(\alpha,\beta)$ can also be evaluated in closed form. Finally, by combining these results with Eq. (2.43), the Khan-Penrose metric is found to have the following expression in the (α,β,x,y) coordinate system:

$$g_{KP}=\frac{ab}{\sqrt{\alpha}}\frac{\alpha^2}{[(1-\alpha)^2-\beta^2]^{1/2}[(1+\alpha)^2-\beta^2]^{1/2}[\sqrt{(1-\alpha)^2-\beta^2}+\sqrt{(1+\alpha)^2-\beta^2}]^2}(-d\alpha^2+d\beta^2)$$
$$+\alpha\left[\frac{\sqrt{1+\alpha-\beta}+\sqrt{1-\alpha-\beta}}{\sqrt{1+\alpha-\beta}-\sqrt{1-\alpha-\beta}}\right]\left[\frac{\sqrt{1+\alpha+\beta}+\sqrt{1-\alpha+\beta}}{\sqrt{1+\alpha+\beta}-\sqrt{1-\alpha+\beta}}\right]dx^2$$
$$+\alpha\left[\frac{\sqrt{1+\alpha-\beta}-\sqrt{1-\alpha-\beta}}{\sqrt{1+\alpha-\beta}+\sqrt{1-\alpha-\beta}}\right]\left[\frac{\sqrt{1+\alpha+\beta}-\sqrt{1-\alpha+\beta}}{\sqrt{1+\alpha+\beta}+\sqrt{1-\alpha+\beta}}\right]dy^2 . \tag{4.5}$$

After expressing α and β in terms of u and v as in Eq. (4.3), Eq. (4.5) reduces to the standard expression[5] of the Khan-Penrose metric in the Rosen-type (u,v,x,y) coordinate system. Inspection of Eq. (4.5) shows that $q_1(\beta)$, $q_2(\beta)$, and $q_3(\beta)$ [cf. Eqs. (3.14) and (3.15)] for the Khan-Penrose solution are equal to the *constant* values $\frac{3}{2}$, -1, and 3, respectively. This implies that $[p_1(\beta),p_2(\beta),p_3(\beta)]$ [Eqs. (3.16) and (3.17)] are equal to the constant values $(-\frac{2}{7},\frac{6}{7},\frac{3}{7})$, and using the inverse relations to Eqs. (3.18) given by

$$\epsilon(\beta)=-1-2\frac{p_3(\beta)}{p_2(\beta)} \text{ if } p_2(\beta)\neq 0 ,$$
$$\epsilon(\beta)=1+2\frac{p_3(\beta)}{p_1(\beta)} \text{ if } p_1(\beta)\neq 0 , \tag{4.6}$$

these equations in turn imply that $\epsilon(\beta)\equiv -2$. Thus, for the Khan-Penrose solution

$$\epsilon_{KP}(\beta)\equiv -2 , \quad p_1(\beta)\equiv -\tfrac{2}{7} , \quad p_2(\beta)\equiv \tfrac{6}{7} ,$$
$$p_3(\beta)\equiv \tfrac{3}{7} \quad \forall \beta\in(-1,1) . \tag{4.7}$$

Numerical computation of the integrals in Eq. (3.13) with the Khan-Penrose initial data (4.4) indicates that both of the two terms on the right-hand side of Eq. (3.13) [involving the integrals of $V_{KP}(r,1)$ and $V_{KP}(1,s)$] are separately constant, and equal to -1 for all β. Note that since $|\epsilon(\beta)|>1$, the Khan-Penrose singularity is an astigmatic one [cf. the discussion following Eq. (3.23) in Sec. III A]. This is not surprising, since the incoming plane waves of the Khan-Penrose solution [which are described by the initial data (4.1) and Eqs. (4.2)] are (i) highly astigmatic (one of the focal lengths is infinite whereas the other is a or b), and (ii) very strong (both V_1 and V_2 are of order unity).

Turn now to our second example; a colliding plane-wave spacetime described by the initial data

$$V^{\times}(r,1)=-V_{KP}(r,1) , \quad V^{\times}(1,s)=V_{KP}(1,s) . \tag{4.8}$$

Since the integrals in Eq. (3.13) are both linear in their respective arguments $V(r,1)$ and $V(1,s)$, and since for the Khan-Penrose initial data (4.4) these integrals both take the constant value -1, it follows that, for the colliding plane-wave solution which evolves from the initial data (4.8),

$$\epsilon^{\times}(\beta)\equiv 0 , \quad p_1(\beta)\equiv \tfrac{2}{3} , \quad p_2(\beta)\equiv \tfrac{2}{3} ,$$
$$p_3(\beta)\equiv -\tfrac{1}{3} \quad \forall \beta\in(-1,1) . \tag{4.9}$$

Therefore, as $|\epsilon^{\times}(\beta)|<1$, the solution developing from (4.8) has a purely anastigmatic singularity structure at $\alpha=0$. This is interesting, because the incoming waves described by the data (4.8) are highly astigmatic. In fact, both incoming plane waves are impulsive waves identical in structure to the incoming waves of the Khan-Penrose solution [it is easy to see that $U_1(u)$, $U_2(v)$, α, β, r, and s for the initial data (4.8) have exactly the same forms as in the Khan-Penrose solution where they are given by Eqs. (4.2) and (4.3)]. The only exception to this identical structure is that one of the waves (namely, the wave that propagates in the v direction, see Fig. 1) has its direction of astigmatism "twisted" with respect to the other; in other words, one of the waves focuses in the x direction and defocuses in the y direction, whereas focusing by the other wave occurs with the roles of the x and y directions interchanged. Now, the solution $V^{\times}(\alpha,\beta)$ of the initial-value problem given by the field equation (2.44a) and the initial data (4.8) is easy to find: It is immediately seen after a short calculation that $V_{KP}(\alpha,\beta)$ is the sum of two pieces that separately satisfy Eq. (2.44a); and therefore, by the linearity of Eq. (2.44a), taking the difference of these pieces instead of their sum produces the unique solution of Eq. (2.44a) which satisfies the initial conditions (4.8). However, with this solution for $V^{\times}(\alpha,\beta)$, the integral in Eq. (2.44b) cannot be computed analytically to yield an expression for the function $Q^{\times}(\alpha,\beta)$ in closed form. Nevertheless, since the coordinate transformations between the (α,β) and (u,v) coordinates are known explicitly [Eqs. (4.3)], we can still write down the interaction-region metric for our solution in the following semiclosed form:

$$g^{\times} = -8\frac{u'v'e^{-Q^{\times}[\alpha(u,v),\beta(u,v)]/2}}{\sqrt{1-u'^2-v'^2}}du\,dv + (1-u'^2-v'^2)\left[\frac{\sqrt{1-u'^2}+v'}{\sqrt{1-u'^2}-v'}\right]\left[\frac{\sqrt{1-v'^2}-u'}{\sqrt{1-v'^2}+u'}\right]dx^2$$
$$+(1-u'^2-v'^2)\left[\frac{\sqrt{1-u'^2}-v'}{\sqrt{1-u'^2}+v'}\right]\left[\frac{\sqrt{1-v'^2}+u'}{\sqrt{1-v'^2}-u'}\right]dy^2 , \tag{4.10}$$

where $u'\equiv u/a$, $v'\equiv v/b$, $Q^{\times}$ is given by Eq. (2.44b), and $\alpha(u,v)$, $\beta(u,v)$ are determined by Eqs. (4.3). The metric in the remaining regions II, III, and IV (Fig. 1) of the solution (4.10) is found by extending (4.10) via the Penrose prescription.[5,9,10] Inspection of Eq. (4.10) makes it apparent that in the vicinity of the singularity $\alpha=0$ ($u'^2+v'^2=1$ in the u,v coordinates), the asymptotic behavior of the metric is characterized by $\epsilon(\beta)\equiv 0$ [cf. Eqs. (3.17) and (3.18)].

Our third example is the colliding plane-wave spacetime which develops from the initial data

$$V_{1/2}(r,1)=\tfrac{1}{2}V_{\rm KP}(r,1)\ ,\quad V_{1/2}(1,s)=\tfrac{1}{2}V_{\rm KP}(1,s)\ , \tag{4.11}$$

or, equivalently

$$V_1(u)=\frac{1}{2}\ln\left[\frac{1+(u/a)}{1-(u/a)}\right] ,$$
$$V_2(v)=\frac{1}{2}\ln\left[\frac{1+(v/b)}{1-(v/b)}\right] . \tag{4.12}$$

Unlike with the Khan-Penrose solution, Eqs. (2.16) cannot be solved analytically with the initial data (4.12); and consequently, $U_1(u)$, $U_2(v)$, $\alpha(u,v)$ and $\beta(u,v)$ cannot be expressed in closed form for the solution (4.11). On the other hand, from the linear dependence [Eq. (3.13)] of $\epsilon(\beta)$ on the initial data $\{V(r,1),V(1,s)\}$, it is very easy to see that the asymptotic structure of the solution (4.11) near $\alpha=0$ is characterized by

$$\epsilon_{1/2}(\beta)\equiv -1\ ,\quad p_1(\beta)\equiv 0\ ,\quad p_2(\beta)\equiv 1\ ,$$
$$p_3(\beta)\equiv 0\quad \forall\beta\in(-1,1)\ . \tag{4.13}$$

Therefore (Sec. III C), the solution (4.11) possesses a nonsingular Killing-Cauchy horizon at $\alpha=0$ across which the spacetime can be smoothly extended. Although the metric for this solution cannot be expressed in closed form in the Rosen-type u,v coordinate system [since the transformation to (α,β) coordinates is not available in analytic form], it can be easily computed in the (α,β) coordinates: By the linearity of the field equation (2.44a), it is clear that $V_{1/2}(\alpha,\beta)=\frac{1}{2}V_{\rm KP}(\alpha,\beta)$; this implies, by Eq. (2.44b), that up to an additive constant, $Q_{1/2}(\alpha,\beta)=\frac{1}{4}Q_{\rm KP}(\alpha,\beta)$. Therefore, combining Eq. (2.43) with Eq. (4.5), we obtain

$$g_{1/2}=\frac{c_0\,ab}{\{[(1-\alpha)^2-\beta^2]^{1/2}[(1+\alpha)^2-\beta^2]^{1/2}[\sqrt{(1-\alpha)^2-\beta^2}+\sqrt{(1+\alpha)^2-\beta^2}]^2\}^{1/4}}(-d\alpha^2+d\beta^2)$$
$$+\alpha\left[\frac{\sqrt{1+\alpha-\beta}+\sqrt{1-\alpha-\beta}}{\sqrt{1+\alpha-\beta}-\sqrt{1-\alpha-\beta}}\right]^{1/2}\left[\frac{\sqrt{1+\alpha+\beta}+\sqrt{1-\alpha+\beta}}{\sqrt{1+\alpha+\beta}-\sqrt{1-\alpha+\beta}}\right]^{1/2}dx^2$$
$$+\alpha\left[\frac{\sqrt{1+\alpha-\beta}-\sqrt{1-\alpha-\beta}}{\sqrt{1+\alpha-\beta}+\sqrt{1-\alpha-\beta}}\right]^{1/2}\left[\frac{\sqrt{1+\alpha+\beta}-\sqrt{1-\alpha+\beta}}{\sqrt{1+\alpha+\beta}+\sqrt{1-\alpha+\beta}}\right]^{1/2}dy^2 , \tag{4.14}$$

where c_0 is a numerical constant. Although the solution (4.14) is the first example of an exact colliding *parallel-polarized* plane-wave solution producing Killing-Cauchy horizons at $\alpha=0$, it has one undesirable feature: The incoming plane waves described by the data (4.12) are not sandwich waves; that is, the focal plane[10,3] of each single incoming wave represents a curvature singularity of the single plane-wave spacetime instead of just a coordinate singularity. (Readers can convince themselves of this fact by inspecting the behavior of the curvature [Eqs. (2.19)] in the single plane-wave spacetimes defined by Eqs. (4.12), (2.16), and (2.8). For a more detailed discussion of these issues, see Sec. II of Ref. 10.) As a result, it seems exceedingly difficult to carry out and analyze a maximal extension (of which we know there are infinitely many) of the spacetime (4.14) beyond the Killing-Cauchy horizon $\{\alpha=0\}$. In our final example below, we will discuss another exact colliding plane-wave solution which similarly produces a Killing-Cauchy horizon at $\alpha=0$, and we will see that the above-mentioned difficulty with singular focal planes does not arise in this solution. In fact, the maximal analytic extension of this solution across the horizon is readily available and produces a maximal colliding plane-wave spacetime with a surprising global structure.

We now turn to this final example: a family of colliding *parallel-polarized* plane wave solutions producing nonsingular Killing-Cauchy horizons at $\alpha=0$. These solutions are derived by a procedure that is almost identical to the procedure by which we have constructed the infinite-parameter family of exact solutions discussed in Ref. 10. To follow the *details* of our presentation, the reader *must* refer to Ref. 10; however, the qualitative features of our example can be understood from the discussion here, and especially from the Figs. 3, 4, and 5. The equation numbers that refer to equations of Ref. 10 will be denoted by a prefix "10"; for example, Eq. (10.3.4) refers to Eq. (3.4) of Ref. 10.

Consider the colliding parallel-polarized plane-wave solution described by Eq. (10.2.4). In Ref. 10, this solution was constructed from the interior Schwarzschild metric in the following four steps (Fig. 3).

(i) The coordinate transformation (10.2.2) was carried out to define a new set of coordinates (u',v',x,y) in terms of the Schwarzschild coordinates (r,θ,ϕ,t), and the interior Schwarzschild metric was expressed [Eq. (10.2.3)] in terms of these new coordinates.

(ii) Two length scales a and b were introduced by rescaling the null coordinates u' and v' through the relations $u'=u/a$, $v'=v/b$, where $ab=4M^2$.

(iii) The resulting interaction-region metric (10.2.3) was then extended beyond the null surfaces $\{u=0\}$ and $\{v=0\}$ by the Penrose prescription:[5,9] $u/a \rightarrow (u/a)H(u/a)$, $v/b \rightarrow (v/b)H(v/b)$, where H is the Heaviside step function.

(iv) The global topology of the resulting spacetime was changed from $S^2 \times R^2$ to R^4 by means of the coordinate transformation (10.2.2) and the nonanalytic extension (iii) across the null surfaces $\{u=0\}$ and $\{v=0\}$. In Fig. 3, we

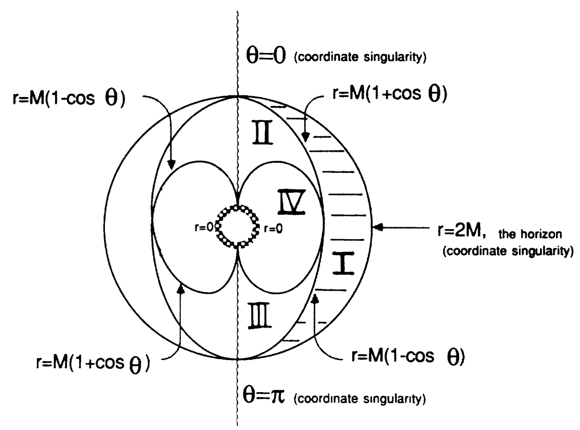

FIG. 3. The region I in Schwarzschild spacetime to which the interaction region of the colliding plane-wave solution (4.16) is locally isometric. This region I is shown shaded in this figure, which is drawn in a $\{t=\text{const}\}$, $\{\phi=0,\pi\}$ plane. As explained in the text, the geometry in region I is extended nonanalytically beyond the null surfaces $r=M(1+\cos\theta)$ and $r=M(1-\cos\theta)$, which correspond to the wave fronts $\{u=0\}$ and $\{v=0\}$, respectively. After this extension, the geometry in regions II and III represents incoming single plane sandwich waves [Eqs. (4.18)]; and region IV is flat. The interaction region I is bounded by a Killing-Cauchy horizon which corresponds to the event horizon of the Schwarzschild spacetime at $\{r=2M\}$. In Ref. 10, we have used the region IV of the interior Schwarzschild spacetime as the interaction region of the colliding plane-wave solution (10.2.4); the solution of Ref. 10 was obtained by exactly the same procedure as the solution (4.16) which we outline in Sec. IV here.

have indicated these null surfaces by their expressions in terms of the Schwarzschild coordinates; these expressions are $\{r=M(1+\cos\theta)\}$ and $\{r=M(1-\cos\theta)\}$ for $\{u=0\}$ and $\{v=0\}$, respectively.

The interaction region of the resulting colliding plane-wave solution (10.2.4) is locally isometric to the region denoted by IV in Fig. 3. In particular, the Schwarzschild singularity at $r=0$ corresponds, under this isometry, to the singularity at $\alpha=0$ [at $(u/a)+(v/b)=\pi/2$ in the Rosen-type coordinates of Ref. 10] created by the colliding waves. In Ref. 10, the above steps (i)–(iv) were repeated almost identically for an infinite-parameter family of regular interior Weyl solutions generalizing the interior Schwarzschild solution. As a result, the infinite-parameter family (10.3.18)–(10.3.22) of colliding plane-wave solutions was obtained. Use of Eqs. (4.6) and inspection of the solution (10.2.4) reveal that for this solution (which corresponds to all parameters d_k being zero), and for all the other solutions (10.3.18)–(10.3.22) (as long as all but finitely many d_k are zero), the asymptotic behavior of the metric near the singularity is characterized by

$$\epsilon(\beta)\equiv-3\ ,\quad p_1(\beta)\equiv-\tfrac{1}{3}\ ,\quad p_2(\beta)\equiv\tfrac{2}{3}\ ,\quad p_3(\beta)\equiv\tfrac{2}{3}\quad \forall\beta\in(-1,1)\ . \tag{4.15}$$

Note that by combining Eqs. (4.15) with Eqs. (3.33)–(3.35), we can reproduce the results of Ref. 10 dealing with the asymptotic behavior of the curvature quantities near the singularity $(u/a)+(v/b)=\pi/2$.

Now, in order to obtain colliding plane-wave solutions which produce Killing-Cauchy horizons at $\alpha=0$, we simply reverse the roles of regions I and IV in Fig. 3; that is, we take region I to be our interaction region, and apply the steps (i)–(iv) above to this new interaction-region metric. This results in a new colliding plane-wave solution whose metric is easily seen to be given by Eq. (10.2.3), but this time for $u<0$, $v<0$ (which describe region I) instead of $u>0$, $v>0$ (which describe region IV). Therefore, redefining u and v as $-u$ and $-v$, respectively, the interaction-region metric of the new solution can be written in the form

$$g_{\rm I}=-\left[1+\sin\left[\frac{u}{a}+\frac{v}{b}\right]\right]^2 du\,dv+\left[\frac{1-\sin\left[\frac{u}{a}+\frac{v}{b}\right]}{1+\sin\left[\frac{u}{a}+\frac{v}{b}\right]}\right]dx^2+\left[1+\sin\left[\frac{u}{a}+\frac{v}{b}\right]\right]^2\cos^2\left[\frac{u}{a}-\frac{v}{b}\right]dy^2\ , \tag{4.16}$$

where the interaction region on which the metric (4.16) is defined is given by $\{u>0,v>0\}$. Note that Eq. (4.16) can be obtained by applying the simple transformations $u\rightarrow -u$, $v\rightarrow -v$ to Eq. (10.2.3). The metric on the rest of the solution (4.16) (i.e., in regions II, III, and IV) is obtained by extending the interaction-region metric (4.16) via the Penrose prescription;[5,9] $(u/a)\to(u/a)H(u/a)$, $(v/b)\to(v/b)H(v/b)$. It is clear from Eq. (4.16) that the solution thus obtained has an asymptotic structure near $\alpha=0$ characterized by

$$\epsilon(\beta)\equiv 1\ ,\quad p_1(\beta)\equiv 1\ ,\quad p_2(\beta)\equiv 0\ ,\quad p_3(\beta)\equiv 0\quad \forall\beta\in(-1,1)\ . \tag{4.17}$$

Therefore, $\alpha=0$ is a nonsingular Killing-Cauchy horizon produced by the colliding plane-wave solution (4.16). By applying exactly the same reasoning as above to the infinite-parameter family (10.3.18)–(10.3.22) of colliding plane-wave solutions, we obtain an infinite-parameter family of generalizations of the solution (4.16). These generalized solutions can be found by simply applying the transformations $u\to -u$, $v\to -v$ *throughout* Eqs. (10.3.18)–(10.3.22). As long as all but finitely many of the parameters d_k are nonzero, the generalized solutions all have the same asymptotic structure near $\alpha=0$ characterized by the exponents (4.17); i.e., all generalized solutions create Killing-Cauchy horizons at $\alpha=0$. The proof that the colliding plane-wave spacetime (4.16) and its generalizations described above are genuine solutions (in the sense of distributions) to the vacuum Einstein equations is provided by exactly the same arguments with which we showed the solutions (10.3.18)–(10.3.22) of Ref. 10 to be genuine vacuum solutions.

The interaction region of the solution (4.16) is locally isometric to region I (Fig. 3) of the interior Schwarzschild solution. In particular, the Killing-Cauchy horizon at $\alpha=0$ [at $(u/a)+(v/b)=\pi/2$ in the Rosen-type coordinates] corresponds, under this isometry, to the horizon $\{r=2M\}$ of the Schwarzschild spacetime. This interaction region I of the solution (4.16) is formed by the collision of single plane waves whose forms in the precollision regions II and III (cf. Fig. 1) are

$$g_{\rm II}=-[1+\sin(u/a)]^2du\,dv+\left[\frac{1-\sin(u/a)}{1+\sin(u/a)}\right]dx^2+[1+\sin(u/a)]^2\cos^2(u/a)dy^2\ , \tag{4.18a}$$

$$g_{\rm III}=-[1+\sin(v/b)]^2du\,dv+\left[\frac{1-\sin(v/b)}{1+\sin(v/b)}\right]dx^2+[1+\sin(v/b)]^2\cos^2(v/b)dy^2\ . \tag{4.18b}$$

In contrast to Eq. (10.2.5), the incoming plane waves (4.18) are true sandwich waves; that is, the focal planes $u=\pi a/2$ and $v=\pi b/2$ of the incoming waves (4.18) represent nonsingular Killing-Cauchy horizons in the respective single plane-wave spacetimes. Similarly, it is easy to see (i) that the infinite-parameter family of generalizations of the solution (4.16) all have interaction regions locally isometric to an analogous region I in the interiors of the corresponding Weyl solutions, and (ii) that the Killing-Cauchy horizons created by these generalized solutions correspond to the horizons of the Weyl solutions from which they are derived. For each of these generalized solutions (as long as all but finitely many d_k are zero), the incoming single plane waves have a similar structure to the plane waves (4.18), and hence are also true sandwich waves.

In the remaining paragraphs of this section, we will concentrate on the colliding plane-wave solution (4.16) and discuss its properties in detail. In particular, we will show that the maximal analytic extension of (4.16) across the Killing-Cauchy horizon $\{\alpha=0\}$ is easy to find and yields a (weakly) asymptotically flat extended spacetime. The infinite-parameter family of generalizations of the solution (4.16) have qualitatively identical features with the solution (4.16), *provided* that all but finitely many parameters d_k are zero. In particular, each of these generalized solutions can be extended analytically across the horizon in a similar fashion; however, the extended spacetimes are not asymptotically flat since the generalized solutions are derived from Weyl solutions which violate asymptotic flatness.[10]

We first consider in some detail the structure of the colliding plane-wave spacetime (4.16) near the Killing-Cauchy horizon $\{\alpha=0\}$. Note that when we apply the coordinate transformation (10.2.2) to the interior Schwarzschild solution, and later extend the metric nonanalytically into the precollision regions as described in the steps (i)–(iv) above, we change the topology of the resulting spacetime from $S^2\times R^2$ to R^4 (cf. Ref. 10). Thus (Fig. 4), the topology of the Schwarzschild horizon is also changed from $S^2\times R^1$ to R^3 (when we regard the Schwarzschild horizon as the progenitor of the Killing-Cauchy horizon $\{\alpha=0\}$ which has topology R^3). The spacelike plane-symmetry-generating Killing vector $\partial/\partial x$ which becomes null on the horizon $\{\alpha=0\}$ corresponds, under the transformations (10.2.2), to the Killing vector $\partial/\partial t$ of the Schwarzschild spacetime which becomes null on the Schwarzschild horizon. The bifurcation two-sphere $\mathcal{S}^2$ of the Schwarzschild horizon, on which $\partial/\partial t$ vanishes, corresponds in our solution (4.16) to the crease singularity ($\equiv$bifurcation set[4]) of the Killing-Cauchy horizon $\{\alpha=0\}\equiv\{(u/a)+(v/b)=\pi/2\}$, on which the Killing vector $\partial/\partial x$ (which is tangent to the null generators of the horizon) vanishes (Fig. 4). Of course, when the solution (4.16) is *not* extended beyond the Killing-Cauchy horizon $\{\alpha=0\}$, the remaining Killing vector $\partial/\partial y$ is *not* cyclic, in contrast to the corresponding Killing vector $\partial/\partial\phi$ of the Schwarzschild spacetime which is cyclic. Therefore, the bifurcation set of the Killing-Cauchy horizon $\{\alpha=0\}$ has the topology R^2 in the unextended spacetime, in contrast to the bifurcation *sphere* $\mathcal{S}^2$ of the Schwarzschild horizon. [Note that, strictly speaking the Killing-Cauchy horizon $\{\alpha=0\}$ is *not* part of the spacetime manifold for the *unextended* colliding plane wave solution (4.16); in other words, the unextended solution (4.16) is represented by those points in Fig. 4 which lie *strictly* to the past of the horizon $\{\alpha=0\}$.] It is easy to see that the curvature quantities Ψ_0, Ψ_2, and Ψ_4 [Eqs. (2.19)] for the solution (4.16) are all finite and well behaved near and on the Killing-Cauchy horizon $\{(u/a)+(v/b)=\pi/2\}$. For example, the quantity Ψ_2 is given by

$$\Psi_2=-\frac{2}{ab}\{1+\sin[(u/a)+(v/b)]\}^{-3}\ ; \tag{4.19}$$

and the other curvature quantities also exhibit similar smooth behavior at $(u/a)+(v/b)=\pi/2$. Hence, clearly, the spacetime can be extended smoothly beyond the horizon $\{\alpha=0\}$ to obtain a maximal colliding plane-wave solution. In fact, since the metric in the interaction region I (Fig. 3) is everywhere locally isometric to interior Schwarzschild, and the maximal analytic extension of the Schwarzschild metric across the Schwarzschild horizon is well known, the maximal analytic extension of the colliding plane-wave solution (4.16) across the Killing-Cauchy horizon $\{\alpha=0\}$ is very easy to describe. In the following final paragraph we will discuss the local description and the global structure of this maximal analytic extension.

Before proceeding with the detailed description of the extension, note that the two boundary points of the unextended solution (4.16) denoted by P and Q in Fig. 4 are spacetime singularities. The reason is that the amplitudes of the delta-function contributions to the curvature quantities along the null surfaces $\{u=0\}$ and $\{v=0\}$ diverge as these two points are approached. However, aside from these two singular points P and Q located on the bifurcation set of the Killing-Cauchy horizon $\{\alpha=0\}$, the geometry at all the other boundary points of the unextended spacetime is perfectly smooth and well behaved. Now, the local description of the maximal analytic extension of (4.16) is particularly clear: Near the Killing-Cauchy horizon $\{(u/a)+(v/b)=\pi/2\}$, the metric is locally isometric to the Schwarzschild solution near the horizon $\{r=2M\}$. It is clear that, because of the specific time orientation that we are using on the unextended colliding plane-wave spacetime, this Schwarzschild horizon to which our Killing-Cauchy horizon $\{\alpha=0\}$ corresponds is the *past* horizon of the Schwarzschild spacetime, rather than the future one. Construct, then, the usual Kruskal-type regular coordinate system on the (past) horizon $\{\alpha=0\}$, and simply extend the solution as *the maximal analytic extension* of the metric in such a coordinate system. Clearly, this would give us *precisely* the usual Schwarzschild solution outside the past horizon. However, just as the maximal analytic extension of a metric like $dx^2+\sin^2 x\, dy^2$ *forces* on us the fact that the coordinate y is periodic, and forces on us the fact that the metric represents a two-sphere; so also here, the maximal analytic extension in the above-described manner *forces* the coordinate y to be 2π periodic; i.e., forces us to identify any two points $(u,v,x,y+2\pi n)$ and $(u,v,x,y+2\pi m)$ *throughout* the spacetime, including, of course, the regions I, II, III, and IV lying *before* the Killing-Cauchy horizon. Thus, the maximal analytic extension of (4.16) yields us an exact solution, which describes the collision of two plane-symmetric sandwich gravitational waves propagating in a *cylindrical universe* with topology $R^3\times S^1$. When these waves collide, they produce a Killing-Cauchy horizon, which, (i) when added to the unextended spacetime region $I^-[\{(u/a)+(v/b)=\pi/2\}]$ (with topology $R^3\times S^1$) causes the topology of the extended spacetime to become $R^2\times S^2$ because of the above identifications, and (ii) encloses a spacetime singularity (the future Schwarzschild singularity) that is spacelike. In fact, the collision produces a Schwarzschild black hole, complete with its future horizon and the two asymptotically flat regions. In Fig. 5, we have tried to depict symbolically the global structure of this maximal colliding plane-wave spacetime. Note that, after the maximal analytic extension is carried out, the singular points P and Q of the solution (4.16) are contained in the bifurcation sphere $\mathcal{S}^2$ of the extended Schwarzschild horizon. Also note that it might be helpful to visualize, as we have done in Fig. 5, the cylindrical spacetime with topology $R^3\times S^1$ (representing the history of the colliding waves "before" the Killing-Cauchy horizon forms) as the direct product of R^1 (representing the time direction) with a finite-sized but open-ended cylinder $R^2\times S^1$ (representing a slice of constant time). As we also explain in Fig. 5, the cylinder $R^1\times S^1$ is topologically equivalent (homeomorphic) to a twice-punctured two-sphere; therefore, the slices of constant time $R^2\times S^1$ are homeomorphic to the direct product of R^1 with a twice-punctured two-sphere. When the Killing-Cauchy horizon $\{\alpha=0\}$ forms and the spacetime is extended beyond it in the above-described manner, the missing pairs of points of these "twice-punctured" spheres are supplied by points from the horizon, and thereby the extended spacetime acquires the topological structure of $R^2\times S^2$, instead of the topology $R^3\times S^1$ of the original cylindrical background on which the colliding plane waves propagate.

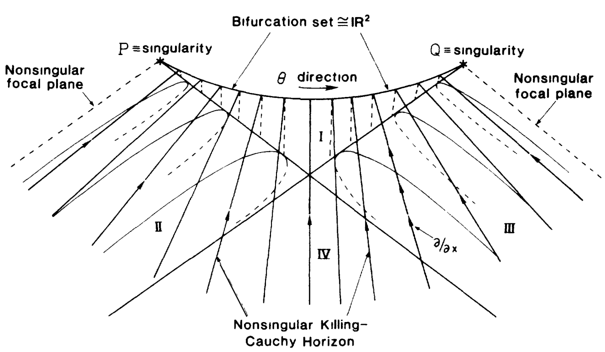

FIG. 4. The global structure of the colliding plane-wave solution (4.16). One of the spacelike Killing directions, namely, the y direction, is suppressed. The remaining spacelike Killing vector $\partial/\partial x$ becomes null on the Killing-Cauchy horizon as depicted; in fact, it is tangent to the null generators of the horizon. This Killing vector $\partial/\partial x$ corresponds to the Killing vector $\partial/\partial t$ of the Schwarzschild spacetime, which similarly becomes null on the event horizon. The suppressed Killing vector $\partial/\partial y$ corresponds, under the local isometry with the Schwarzschild spacetime, to the cyclic Killing vector $\partial/\partial\phi$. However, in constructing the solution (4.16), we have changed the topology of the spacetime from the topology $R^2\times S^2$ of Schwarzschild to R^4. Therefore, the Killing vector $\partial/\partial y$ is no longer cyclic. In particular, the bifurcation two-sphere $\mathcal{S}^2$ of the Schwarzschild horizon is "torn" open in our solution to a bifurcation set which has topology R^2. The Killing vector field $\partial/\partial x$, which becomes null on the horizon, vanishes on this bifurcation set. Although the focal planes for each of the incoming plane waves in the solution (4.16) are nonsingular, the points P and Q where these focal planes intersect the bifurcation set of the Killing-Cauchy horizon represent spacetime singularities (see the discussion in Sec. IV).

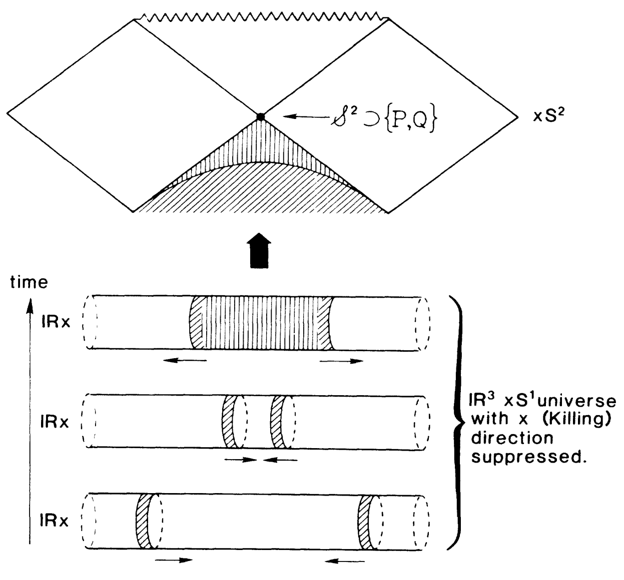

FIG. 5. The global structure of the maximal colliding plane-wave spacetime obtained by analytically extending the solution (4.16) across its Killing-Cauchy horizon using Kruskal-type global coordinates. The description in the figure is only symbolic, and is intended to help the reader in visualizing the true geometry of this maximal extension. As explained in the text (Sec. IV), the maximal analytic extension of the metric (4.16) (leading to the Schwarzschild spacetime outside the past horizon) causes the (Killing-) coordinate y to become cyclic, and thereby causes the topology of the extended spacetime to become $S^2 \times R^2$ instead of R^4 [see Fig. 4 for a description of the global structure of the unextended solution (4.16)]. Similarly, this change in the topological nature of the coordinate y implies that the region of the maximal spacetime which lies to the past of the Killing-Cauchy horizon has topology $R^3 \times S^1$ instead of R^4. Since this region describes the history of the colliding plane waves before they create the Killing-Cauchy horizon, it follows that the incoming waves propagate and collide in a cylindrical universe with topology $R^3 \times S^1$. Note that, after the maximal analytic extension is carried out, the singular points P and Q of the solution (4.16) are contained in the bifurcation sphere $\mathcal{S}^2$ of the extended Schwarzschild horizon. Also note that it might be helpful to visualize, as is done in the figure here, the cylindrical spacetime with topology $R^3 \times S^1$ (representing the history of the colliding waves "before" the Killing-Cauchy horizon forms) as the direct product of R^1 (representing the time direction) with a finite-sized but open-ended cylinder $R^2 \times S^1$ (representing a slice of constant time). The cylinder $R^1 \times S^1$ is topologically equivalent (homeomorphic) to a twice-punctured two-sphere; therefore, the slices of constant time $R^2 \times S^1$ are homeomorphic to the direct product of R^1 with a twice-punctured two-sphere. When the Killing-Cauchy horizon $\{\alpha = 0\}$ forms and the spacetime is extended beyond it, the missing pairs of points of these "twice-punctured" spheres are supplied by points from the horizon, and thereby the extended spacetime acquires the topological structure of $R^2 \times S^2$, instead of the topology $R^3 \times S^1$ of the original cylindrical background on which the colliding plane waves propagate. Thus, the analytic extension of the solution (4.16) gives a maximal spacetime, in which a Schwarzschild black hole is created out of the collision between two plane-symmetric sandwich waves propagating in a cylindrical universe.

V. CONCLUSIONS

We can summarize the main results of this paper as follows.

(i) In a suitable coordinate system, the structure of the singularities produced by colliding parallel-polarized gravitational plane waves can be analyzed in full generality and detail. This analysis (a) reveals that the asymptotic structure of these singularities are of inhomogeneous Kasner type, and (b) provides explicit expressions for the asymptotic Kasner exponents in terms of the initial data posed by the incoming, colliding plane waves.

(ii) For specific choices of initial data for the colliding waves, the asymptotically Kasner form that the spacetime metric takes near the singularity can be that of a degenerate Kasner solution. In this case, the curvature singularities created by the colliding waves degenerate to coordinate singularities, and nonsingular Killing-Cauchy horizons are thereby obtained. The mathematical formalism that is built in this paper proves (a) that these horizons are unstable in the full nonlinear theory against small but generic perturbations of the initial data, and (b) that in a very precise sense, "generic" initial data always produce all-embracing, spacelike curvature singularities without Killing-Cauchy horizons.

(iii) An abundance of exact colliding parallel-polarized plane-wave solutions can be constructed, which exemplify some of the asymptotic singularity structures discussed in general terms in this paper. In particular, an infinite-parameter family of such solutions are found which create Killing-Cauchy horizons instead of curvature singularities. The analytic extension of one of these solutions across its Killing-Cauchy horizon results in a maximal spacetime, in which a Schwarzschild black hole is created out of the collision between two plane-symmetric sandwich waves propagating in a cylindrical universe.

There are a few specific directions for further research along the lines discussed in this paper that are worth listing. These are the following.

(i) A similar study of the more general problem of colliding plane waves with arbitrarily oriented polarizations. It will be interesting to find out whether the fundamental aspects of our results (i) and (ii) above remain intact after the new degree of freedom associated with a discrepancy in the incoming polarizations enters the problem. (In fact, recent work by the author[30] shows that this is indeed the case.)

(ii) A similar analysis of the problem of colliding plane waves coupled with matter fields.[9] Again the most interesting targets for such an inquiry will be understanding the validity of the results (i) and (ii) above under the presence of a nonzero stress-energy tensor.

(iii) Finally, an analysis of the structure of singularities produced by colliding *almost*-plane waves (see Refs. 3, 15, and 31 in this connection). Although such an analysis may well be beyond the capabilities of current analytical techniques, the question of whether the relaxation of strict plane symmetry in the initial data will cause the asymptotic singularity structure to deviate significantly from inhomogeneous Kasner (and, e.g., to become inhomogeneous mixmaster[12] or some more general structure) is an extremely interesting one.

Note added in proof. After this paper had been accepted for publication, the author learned that solutions similar to the solution derived from the Schwarzschild metric and studied in Sec. IV here have been discovered and studied from another viewpoint previously and independently by Ferrari, Ibanez, and Bruni.[32]

ACKNOWLEDGMENT

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[1]J. Ehlers and W. Kundt, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).
[2]R. Penrose, Rev. Mod. Phys. **37**, 215 (1965).
[3]U. Yurtsever, Phys. Rev. D. **37**, 2803 (1988).
[4]U. Yurtsever, Phys. Rev. D **36**, 1662 (1987).
[5]K. Khan and R. Penrose, Nature (London) **229**, 185 (1971).
[6]P. Szekeres, J. Math. Phys. **13**, 286 (1972).
[7]Y. Nutku and M. Halil, Phys. Rev. Lett. **39**, 1379 (1977).
[8]R. A. Matzner and F. J. Tipler, Phys. Rev. D **29**, 1575 (1984).
[9]S. Chandrasekhar and B. C. Xanthopoulos, Proc. R. Soc. London **A398**, 223 (1985); **A402**, 37 (1985); **A402**, 205 (1985); **A403**, 189 (1986).
[10]U. Yurtsever, Phys. Rev. D. **37**, 2790 (1988).
[11]S. Chandrasekhar and B. C. Xanthopoulos, Proc. R. Soc. London A (to be published).
[12]E. Kasner, Am. J. Math. **43**, 217 (1921); E. M. Lifshitz and I. M. Khalatnikov, Usp. Fiz. Nauk **80**, 391 (1964) [Sov. Phys. Usp. **6**, 495 (1963)]; C. W. Misner, Phys. Rev. Lett. **22**, 1071 (1969); V. A. Belinsky and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **57**, 2163 (1969) [Sov. Phys. JETP **30**, 1174 (1970)]; V. A. Belinsky, I. M. Khalatnikov, and E. M. Lifshitz, Adv. Phys. **19**, 525 (1970).
[13]E. Newman and R. Penrose, J. Math. Phys. **3**, 566 (1962).
[14]R. Price and K. S. Thorne, Phys. Rev. D **33**, 915 (1986) (Appendix A).
[15]U. Yurtsever (in preparation).
[16]U. Yurtsever (in preparation).
[17]S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, Cambridge, England, 1973).
[18]P. R. Garabedian, *Partial Differential Equations* (Wiley, New York, 1967).
[19]I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1980).
[20]R. Penrose, in *General Relativity: An Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).
[21]F. J. Tipler, C. J. S. Clarke, and G. F. R. Ellis, in *Einstein Centenary Volume*, edited by A. Held and P. Bergmann (Plenum, New York, 1979).
[22]M. Simpson and R. Penrose, Int. J. Theor. Phys. **7**, 183 (1973); J. M. McNamara, Proc. R. Soc. London **A358**, 499 (1978); **A364**, 121 (1978); Y. Gursel, I. D. Novikov, V. D. Sandberg, and A. A. Starobinsky, Phys. Rev. D **19**, 413 (1979); **20**, 1260 (1979); R. A. Matzner, N. Zamorano, and V. D. Sandberg, *ibid.* **19**, 2821 (1979); S. Chandrasekhar and J. B. Hartle, Proc. R. Soc. London **A384**, 301 (1982).
[23]M. Morris and K. S. Thorne, Am. J. Phys. (to be published).
[24]S. Chandrasekhar and B. C. Xanthopoulos, Proc. R. Soc. London A (to be published).
[25]V. Moncrief and J. Isenberg, Commun. Math. Phys. **89**, 387 (1983); J. Math. Phys. **26**, 1024 (1985).
[26]F. J. Tipler, Phys. Rev. D **16**, 3359 (1977).
[27]K. Yosida, *Functional Analysis* (Springer, Berlin, 1985).
[28]I am indebted to R. Geroch for pointing this out to me.
[29]Y. Choquet-Bruhat, C. DeWitt-Morette, and M. Dillard-Bleick, *Analysis, Manifolds and Physics* (North-Holland, Amsterdam, 1982).
[30]U. Yurtsever (in preparation).
[31]U. Yurtsever, following paper, Phys. Rev. D **38**, 1731 (1988).
[32]V. Ferrari and J. Ibanez, Gen. Relativ. Gravit. **19**, 405 (1987); V. Ferrari, J. Ibanez, and M. Bruni, Phys. Rev. D **36**, 1053 (1987); V. Ferrari and J. Ibanez, Proc. R. Soc. London A (to be published). I am grateful to V. Ferrari for bringing these references to my attention.

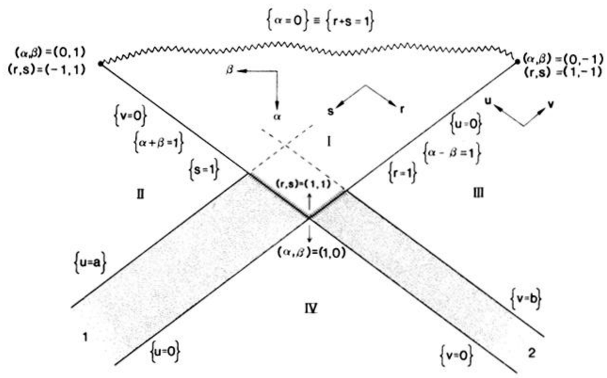

FIG. 1. The two-dimensional geometry of the characteristic initial-value problem for colliding plane waves. The null surfaces $\{u=0\}$ and $\{v=0\}$ are the past wave fronts of the incoming plane waves 1 and 2. Initial data corresponding to waves 1 and 2 are posed, respectively, on the upper portions of the surfaces $\{v=0\}$ and $\{u=0\}$ that are adjacent to the interaction region I. The geometry in the region IV is flat, and the geometry in regions II and III is given by the metric describing the incoming waves 1 and 2, respectively. The geometry of the interaction region I is uniquely determined by the solution of the above initial-value problem. The directions in which the various lines of constant coordinates u, v, α, β, r, and s run are also indicated, along with the descriptions of the initial null surfaces in these different coordinate systems.