PHYSICAL REVIEW D VOLUME 38, NUMBER 5 1 SEPTEMBER 1988

Hadronic transition amplitudes under null-plane Bethe-Salpeter dynamics

N. Nimai Singh and A. N. Mitra
Department of Physics, University of Delhi, Delhi 110007, India
(Received 1 April 1988)

A two-tier form of the Bethe-Salpeter (BS) framework is employed for the evaluation of a few typical transition amplitudes involving strong and electromagnetic decays of $q\bar{q}$ hadrons with $L \geq 0$. The two-tier BS framework, whose rationale has been described in some recent publications, is characterized by (i) a three-dimensional reduction of the BS equation under the null-plane ansatz to make a first contact with mass spectral data and (ii) a reconstructed four-dimensional BS wave function to restore the virtual $q\bar{q}$ effects (higher Fock states) for the evaluation of hadronic transition amplitudes through the lowest-order Feynman (quark-triangle) diagrams. The processes explicitly considered are the pion form factor $F_\pi(k^2)$, $(\omega,a_2)\to\gamma\pi$, $(\rho,f_2)\to\pi\pi$, $(a_1,a_2)\to\rho\pi$, and $b_1\to\omega\pi$. These illustrate the different types of coupling structures that are generally encountered in such a composite-hadron model. The agreement with the data on almost all these amplitudes is excellent, using the same basic constants (ω_0, C_0, m_{ud}) as employed recently for a successful description of the mass spectra of $q\bar{q}$ *and* qqq hadrons, as well as of some pionic amplitudes.

I. INTRODUCTION

Any serious test of a dynamical theory, be it at the atomic (QED) or nuclear (QHD) or hadronic (QCD) levels of compositeness, must of neccessity include both "on-shell" (mass spectra) *and* "off-shell" (transition amplitudes) predictions. Of these, only QED is well tested on both fronts, while QHD (Ref. 1), by its very nature, remains at a more or less empirical level of theory. In contrast, QCD, which, by all consensus, is the candidate theory of strong interactions, has not yet found adequate observational confirmation in the nonperturbative regime of confinement, if only because of the inadequacy of the mathematical tools associated with its predictive powers precisely in this regime. Mainly for this reason effective models of confinement motivated by QCD ideas have over the years acquired a degree of (pseudo)credibility which, in all fairness, can endure only if they stand on the legs of solid experimental support on diverse fronts. In particular, any meaningful test of such models must come not only from "on-shell" (hadron mass spectra) but from a wide variety of successful "off-shell" (hadronic transition amplitudes) predictions as well. A good example of such approaches is provided by the QCD sum rules,[2] parametrized by the coefficients of certain higher-twist terms[3] (vacuum expectations of $q\bar{q}$ and gg operators), which have successfully predicted several transition amplitudes (off shell), but which have proved inadequate on the mass-spectral (on-shell) front. Bag models,[4] in contrast, exhibited just the opposite features (rich in on-shell, but poor in off-shell predictions). On the other hand, nonrelativistic models[5] of linear $q\bar{q}$ confinement have proved highly successful on heavy quarkonia, both with respect to their mass spectra as well as on a limited class of transition amplitudes involving their *own species* only. All these models contain the germs of QCD in varying degrees,[6] and so do other effective models, and the overall credibility of *any* of these must be judged by the depth and range of its predictive powers on both on-shell and off-shell counts, *and* the result of actual contact with the data. This will presumably be the scenario until such time as the results of all these models get subsumed within the predictions of lattice gauge theory[7] which is supposed to "represent" the exact content of QCD, but for (i) the lack of stability of its numerical predictions as a function of the lattice parameter and (ii) the slow pace of arrival of "key" results.

Yet another type of QCD-oriented effective confinement theory, somewhat different in spirit rather from the above, can be motivated through the Bethe-Salpeter (BS) equation[8] with an effective $q\bar{q}$ (or qq) kernel based on vector confinement,[9] as the input dynamics. It is characterized by a two-tier approach, viz., (i) a three-dimensional reduction in the language of the null-plane ansatz[10] (NPA) to provide contact with the mass spectral data[9] for $q\bar{q}$ and qqq hadrons (on-shell tests) in a unified fashion and (ii) a reconstructed four-dimensional BS wave function[10] to restore the virtual $q\bar{q}$ effects (higher Fock states), at least perturbatively. The hadron-quark vertex thus identified facilitates the evaluation of various hadronic transition amplitudes through the lowest-order Feynman diagrams for the off-shell tests of the theory bearing on the detailed structure of the wave function. This two-tier approach which was suggested some years ago[9,10] was subsequently refined on two accounts: (i) mathematically,[11] a Lorentz-invariant generalization of the (harmonic) kernel was designed to facilitate a three-dimensional reduction of the BS equation having the property of null-plane covariance; (ii) physically,[12] an ansatz of proportionality of the spring constant to α_s provided an explicit QCD motivation to the kernel. (For other details, see Refs. 11 and 12.)

In the refined form the theory has the following basic parameters: (i) a universal spring constant ω_0, common to all flavors; (ii) a second constant C_0 to simulate the correct zero-point energies (vacuum structure); and (iii) the quark mass m_q for the flavor sector under study. In addition a very small constant A_0 (with negligible effect for the *uds* sectors) interpolates smoothly from harmonic to linear confinement as one goes from the lightest to the heaviest quarkonia. With these inputs alone, the model seems already to have passed the "on-shell" tests through a successful set of predictions of the spectra of both the $q\bar{q}$ hadrons (five flavors)[12] *and* the qqq baryons (*ud* flavors),[13] in excellent agreement with experiment. In regard to off-shell tests, its predictions[14] of the $\pi \to l\bar{l}$, $\pi^0 \to \gamma\gamma$, and $\gamma \to 3\pi$ coupling amplitudes in good accord with the data,[15,16] constitute some preliminary off-shell tests at low energies, bearing on the structure of the pion's wave function predicted by the BS dynamics. At the other extreme on the energy scale, some applications at higher energies (structure and fragmentation functions of the pion) have also met with some success.[17,18]

Encouraged by these results, we have attempted in this paper to systematically extend these off-shell tests of the hadronic wave function to intermediate energies by considering two classes of hadronic transition amplitudes $h \to h' + h''$ and $h \to h' + \gamma$ ($h = q\bar{q}$) starting with the *ud* sector. Similar calculations had been performed earlier[10] on the basis of the BS wave functions then available within the instantaneous approximation, but whose noncovariant nature has the disadvantage of not being extensible to arbitrarily high energies. To overcome this problem, it had then been found necessary to introduce a "renormalization" of the hadron coupling[10] scale, thus greatly reducing the predictive powers of the model. The refined formalism[11-14] which is automatically a null-plane-covariant is now able to dispense with this theoretical assumption[10] and is thus able to make a cleaner set of predictions free from any extra parameter beyond the three basic constants (ω_0, C_0, m_q). Specifically we have considered the following amplitudes:

$$\text{(i) pion form factor } F_\pi(k^2) \text{ and } (\omega, a_2) \to \pi\gamma \;, \tag{1.1}$$

$$\text{(ii) } (\rho, f_2) \to \pi\pi, \;\; (a_1, a_2) \to \rho\pi, \;\; b_1 \to \omega\pi \;, \tag{1.2}$$

which among themselves cover fairly well the distinct types of couplings usually encountered in such hadronic transitions. All these are in good agreement with experiments, thus lending some confidence in the internal structure of the BS dynamics. Apart from this test, these processes have also provided us with an opportunity to bring out in some detail the algebraic structure of the matrix elements characteristic of the two-tier BS dynamics in the null-plane language. In Sec. II, we first summarize the main ingredients, viz., the structure of the hadron-quark vertex function as finally obtained after recent refinements[11,12] in the earlier formalism.[9,10] Some terminology is clarified regarding the different types of off-shell extensions of the three-dimensional wave function under the null-plane ansatz (NPA). The NPA normalizations of the corresponding hadron-quark vertex functions are collected in Appendixes A and B for $L = 0$ and $L \geq 1$ mesons, respectively. Section III outlines the applications of the NPA-BS dynamics to the electromagnetic (EM) transition amplitudes $h \to h' + \gamma$ to illustrate the structure of EM couplings (form factors and transition form factors) of $q\bar{q}$ hadrons through the explicit examples of (i) the pionic form factor $F_\pi(k^2)$ and (ii) the EM transition amplitudes $\omega \to \pi\gamma$ and $a_2 \to \pi\gamma$ for $L = 0$ and 1 states, respectively. Section IV is taken up with the corresponding calculations of the strong break-up processes $h \to h' + h''$ classified under three distinct heads in ascending order of algebraic complexity. Sections V and VI summarize our main conclusions especially the practical significance of these results in relation to other contemporary models.

II. HADRON-QUARK ($Hq\bar{q}$) VERTEX FUNCTION

In this section the main steps of the BS model, after the inclusion of certain refinements in the mathematical formulation and physical input parameters, are outlined. The BS kernel based on the vector confinement[19] provides a common confining framework for both $q\bar{q}$ and qqq systems;[13] it was first adapted to give a Lorentz-invariant generalization of a three-dimensional (3D) harmonic oscillator (HO) in the form[11]

$$V(q-k) = 3\pi\omega_{q\bar{q}}^2 \lim_{m \to 0} \left[-\frac{\partial^3}{\partial m^3}(m + q_\mu^2)^{-1} \right] . \tag{2.1}$$

[This differs from the more conventional 4D HO form $\Box^2\delta^4(q)$.] The complete BS equation for a $q\bar{q}$ system, viz.,[9]

$$(m_q + i\gamma^{(1)} \cdot p_1)(m_q - i\gamma^{(2)} \cdot p_2)\Psi(q) = \frac{(2\pi)^{-4}}{i}(\tfrac{1}{2}\lambda_1 \cdot \tfrac{1}{2}\lambda_2) \int \gamma_\mu^{(1)}\gamma_\mu^{(2)} V(q-k)\Psi(q') \;, \tag{2.2}$$

is further subjected to three-dimensional reduction in the NPA using null-plane variables $(q_\perp, q_+, q_-)$ by integrating out over the q_- variables as[10,11]

$$\phi(\mathbf{q}) = \int \tfrac{1}{2} dq_- \Phi(q) \tag{2.3}$$

after a prior Gordon reduction,[9] a step which is now better justified under NPA than under the earlier instantaneous approximation,[9] since the component p_{i-} of any four-momentum $p_{i\mu}$ is, in the light-cone formalism, defined as

$$p_{i-} = \omega_{i\perp}^2 / p_{i+}, \quad \omega_{i\perp}^2 = (m_q^2 + p_{i\perp}^2) \tag{2.4}$$

corresponding to the on-shell condition $p_{i\mu}^2 + m_q^2 = 0$. In this formalism, the third component A_3 of a three-vector $\mathbf{A}$ $(= \mathbf{A}_\perp, A_3)$ reads as[11]

$$A_3 = A_+ \frac{M}{P_+} \;, \tag{2.5}$$

where M is the composite hadron mass in the four-momentum P_μ. This form preserves "null-plane covariance" in the sense that the net number of $\pm$ indices in the numerator and denominator is "conserved" on both sides. This constitutes the necessary mathematical refinement[11] over the earlier formulation.[10] At the physical level, an explicit QCD motivation for the confining term was introduced through the α_s-proportionality ansatz:[12]

$$\omega_{q\bar{q}}^2=2m_q\tilde{\omega}^2,\quad \tilde{\omega}^2=\omega_0^2\alpha_s(4m_q^2)\ , \tag{2.6}$$

where

$$\alpha_s(Q^2)=\frac{12\pi}{33-2f}\left[\ln\frac{Q^2}{\Lambda^2}\right]^{-1},\quad \Lambda=250\ \text{MeV} \tag{2.7}$$

and ω_0 is the new "universal spring constant," same for all flavor sectors. A second refinement was the replacement of the HO term as $r^2\to r^2-C_0/\omega_0^2$ to simulate the correct zero-point energies for the different flavor sectors at the cost of a second universal constant C_0. A third refinement was introduced for effecting a smooth transition from harmonic to linear confinement, during the passage from the lightest to the heaviest quarkonia at the cost of a small constant A_0 ($=0.0283$) (Ref. 12), but this aspect does not figure in this paper which is concerned only with the light ud sectors with equal quark masses ($m_1=m_2=m_q$). The NPA reduced three-dimensional BS equation for $\phi_L(\mathbf{q})$ has the form ($p_{1,2}^\mu=\frac{1}{2}P^\mu\pm q^\mu$) (Ref. 11)

$$D_+(\mathbf{q})\phi_L(\mathbf{q})=\omega_{q\bar{q}}^2\frac{P_+}{M}\tilde{D}(\mathbf{q})\phi_L(\mathbf{q})\ , \tag{2.8}$$

where

$$D_+(\mathbf{q})=2P_+\left[m_q^2+q_\perp^2+\frac{M^2q_+^2}{P_+^2}-\frac{M^2}{4}\right], \tag{2.9}$$

for a hadron with $\mathbf{p}_\perp=0$, is the NPA denominator function defined by

$$\frac{2\pi i}{D_+(\mathbf{q})}=\int\tfrac{1}{2}dq_-/\Delta_1\Delta_2,\quad \Delta_{1,2}=m_q^2+p_{1,2}^2 \tag{2.10}$$

and $\tilde{D}(\mathbf{q})$ is a quadratic differential operator defined in Ref. 11. The corresponding three-dimensional wave functions for $L\geq 0$ states have the standard HO forms[11]

$$\phi_0(\mathbf{q})=\exp\left[-\frac{1}{2\beta^2}\left[\mathbf{q}_\perp^2+\frac{M^2q_+^2}{P_+^2}\right]\right]\quad \text{for } L=0 \tag{2.11}$$

and[20]

$$\phi_L=n_L\beta^{-L}q_{i_1}\cdots q_{i_L}B_{i_1\cdots i_L}^L\phi_0(\mathbf{q})\quad \text{for } L>0,\quad n_L^2=\frac{2^L}{L!}\ . \tag{2.12}$$

The HO parameter β is given by[12]

$$\begin{aligned}\beta^2&=\omega_0(m_qM\alpha_s/\gamma^2)^{1/2}\ ,\\ \gamma^2&=1+\frac{2\omega_0^2}{Mm_q}-\frac{4C_0}{M}m_q\alpha_s\ .\end{aligned} \tag{2.13}$$

Using the input parameters of the three basic constant of the model:

$$\omega_0=158\ \text{MeV},\quad C_0=0.296,\quad m_q=270\ \text{MeV}\ , \tag{2.14}$$

the three-dimensional BS equation shows good mass spectral predictions for $q\bar{q}$, $q\bar{Q}$, $Q\bar{Q}$ mesons[12] and (qqq) baryons[13] under a common framework without further individual assumptions.

Having thus rooted the three-dimensional form of the BS dynamics in hadron individual spectroscopy (on-shell tests) the next step is to reconstruct the four-dimensional BS amplitude Ψ, essentially through an inversion of Eq. (2.3) (Refs. 10 and 11):

$$\Psi(p_1,p_2)=S_F(p_1)\Gamma(\mathbf{q})S_F(-p_2)\ , \tag{2.15}$$

$$\Gamma(\mathbf{q})=N_h^{(-)}\Gamma_iD_+(\mathbf{q})\phi_L(\mathbf{q})/(2\pi i)\ . \tag{2.16}$$

This has the consequence of incorporating the virtual $q\bar{q}$ effects (higher Fock states) on different types of hadronic transition amplitudes perturbatively through the lowest-order Feynman diagrams.[10] (For further arguments justifying the two-tier approach, see Refs. 14 and 18.) In Eq. (2.16) the constant Dirac matrix Γ_i corresponds to $q\bar{q}$ states in question (e.g., γ_5 for pseudoscalar, $i\gamma\cdot\hat{\epsilon}$ for vector mesons, etc.), and $N_h^{(-)}$ which should be proportional to P_+^{-1} in accordance with the demands of NPA covariance,[11] represents the standard BS normalizer (via current conservation). For an on-shell pseudoscalar meson P with the wave function ϕ_0, Eq. (2.11), the BS normalizer is given in Ref. 11, whereas the calculation of $N_h^{(-)}$ for $L=0$ V mesons, and all types of $L\geq 1$ mesons are outlined in Appendixes A and B, respectively, using the three-dimensional wave functions (2.11) and (2.12).

Further examination of the structure of the three-dimensional wave function, Eq. (2.11), indicates that the pseudoscalar particles ($\pi,k,\ldots$) do not conform to the conditions under which the general structure (2.11) was derived, viz., the condition $D_+\approx 0$, termed "on shell" for the hadron in question.[11] Such conditions are fairly well satisfied for the (vector) mesons ($\rho,\omega,\ldots$) as may be seen from the results of $V\to e^+e^-$ decays derived in Appendix A. The same presumably applies to L-excited hadrons, but not readily for π, k, etc., $q\bar{q}$ states of $L=0$, so that some relaxation of the "on-shell" condition for the pseudoscalar hadrons may be necessary in practice. This defect can be remedied in two ways.

(a) When the hadron (P) and one of the two quarks (p_2) are "free" (on shell),[17,18] the four-momentum (p_1) of the nonfree quark can be eliminated in favor of p_2 and P, so that

$$\frac{M^2q_+^2}{P_+^2}-q_+q_-=-(\tfrac{1}{2}P_+-p_{2+})(\tfrac{1}{2}P_--p_{2-}) \tag{2.17}$$

thus defining a "$\frac{1}{2}$-off-shell" wave function:[17,18]

$$\tilde{\tilde{\phi}}=\exp\left[-\frac{q_\perp^2}{4\beta^2}\left[\frac{P_+}{p_{2+}}\right]+\frac{1}{4\beta^2}\left[\frac{1}{2}-\frac{p_{2+}}{P_+}\right]\left[M^2-2m_q^2\frac{P_+}{p_{2+}}\right]\right] \tag{2.18}$$

and a similar function with $p_1\rightleftharpoons p_2$. The corresponding normalizer for π works out as[17,18]

$$\tilde{\tilde{N}}_{\pi}^{(-)}=(2\pi)^{-3/2}(\pi\beta^2)^{-1/2}\tilde{\tilde{N}}_{\pi}P_{+}^{-1} \tag{2.19}$$

in terms of a "reduced" normalizer $\tilde{\tilde{N}}_{\pi}$ given by

$$(2\tilde{\tilde{N}}_{\pi})^{-2}=(\pi\beta^2)^{-1}\int_{-\infty}^{+\infty}dq_{\perp}^2\int_0^1 x\,dx(m_q^2+xq_{\perp}^2+x^2M^2)\tilde{\tilde{\phi}}_0^2$$
$$=m_q^2(1.448)\ . \tag{2.20}$$

This type of wave function (and norm) has been found to be appropriate for high-energy processes involving structure[17] and fragmentation[18] functions. Still another kind of off-shell extension is provided by the replacement[14]

$$\frac{M^2q^2}{P_{+}^2}\to -q_{+}q_{-}=\frac{\omega_{\perp}^2q_{+}^2}{p_{1+}p_{2+}},\quad \omega_{\perp}^2=m_q^2+q_{\perp}^2 \tag{2.21}$$

as originally envisaged,[11] without invoking the on-shell hadron condition. The corresponding wave function, termed "off shell" is given by[14]

$$\tilde{\phi}=\exp[-\tfrac{1}{2}\beta^{-2}(q_{\perp}^2+x^2m_q^2)/(1-x^2)],\quad x=2q_{+}/P_{+}\ . \tag{2.22}$$

The associated normalizer $\tilde{N}^{(-)}$ works out as

$$\tilde{N}^{(-)}P_{+}=\left[\frac{1}{8\pi^4\beta^2}\right]^{1/2}\int_{-1}^{1}dx(1-x^2)[\beta^2(1-x^2)+m_q^2+\tfrac{1}{4}M^2(1+x^2)]\exp[-m_q^2x^2\beta^{-2}(1-x^2)^{-1}]\ . \tag{2.23}$$

This is appropriate for certain low-energy pionic amplitudes.[14] For the present calculation of certain hadronic decays, the "half-off-shell" form (2.18) for the pion seems to be the right choice on kinematical grounds (see Secs. III and IV).

III. EM TRANSITION AMPLITUDES

In this section we shall illustrate the structure of the EM couplings of $q\bar{q}$ hadrons through the explicit examples of the pionic form factor and the (ii) $(a_2,\omega)\to\pi\gamma$ transition amplitude. The calculational pattern follows the same formal procedure as described earlier[10] for the BS normalizations, form factors, and EM transition amplitudes such as $\omega\to\pi\gamma$. Thus, the general structure of an EM amplitude $\mathcal{F}_{\mu}$ which covers both situations (form factor and transition form factor) is of the form

$$\mathcal{F}_{\mu}(h\to h'\gamma)=(2\pi)^4N_h^{(-)}N_{h'}^{(-)}\frac{e}{2}\mathrm{Tr}\int d^4q[i\gamma_{\mu}\bar{\Psi}_{h'}(p_1',p_2')(m_q-i\gamma\cdot p_2)\Psi_h(p_1,p_2)+(1\rightleftharpoons 2)]\ , \tag{3.1}$$

where h and h' are the two hadrons involved in the initial and final states, and the second term corresponds to the photon (k_{μ}) interacting with the quark number 2 [Figs. 1(a) and 1(b)]. The necessary kinematics for Fig. 1(a) are

$$P=p_1+p_2,\quad P'=p_1'+p_2,\quad p_1-p_1'=k\ . \tag{3.2}$$

For the pionic form factor, $h=h'=\pi$ and k^2 will be considered in the spacelike region $(k^2>0)$ of the off-shell photon. For the transition amplitude $\omega\to\pi\gamma$, $h=\omega$, $h'=\pi$, and the photon is real $(k^2=0)$. Finally the limit $k_{\mu}\to 0$ (for $h=h'$) corresponds to the BS normalization of the hadron (h), whose calculations are summarized in Appendix A for the different types of h (except for $h=\pi$ which was given already in Ref. 11) that are involved in this paper. For all of these cases, the equal masses (m_q) of the ud quarks ensure that the second term $(1\rightleftharpoons 2)$ in (3.1) gives an overall multiplicative factor of 2 only.

A. Pion form factor

Inserting Eqs. (2.15) for Ψ_{π} in (3.1), the form factor of the pion may be expressed entirely in terms of Fig. 1(a) as

$$\mathcal{F}_{\mu}^{\pi}=(2\pi)^2\tilde{\tilde{N}}_{\pi}^{(-)}\tilde{\tilde{N}}_{\pi'}^{(-)}e\int d^2q_{\perp}dq_{+}\tfrac{1}{2}dq_{-}(D_{+}D'_{+}\tilde{\tilde{\phi}}_{\pi}\tilde{\tilde{\phi}}_{\pi'})(\Delta_1\Delta_1'\Delta_2)^{-1}\mathrm{Tr}_{\mu}(\pi\pi'\gamma)\ , \tag{3.3}$$

$$\Delta_{1,2}=p_{1,2}^2+m_q^2,\quad \Delta_1'=p_1'^2+m_q^2\ , \tag{3.4}$$

$$2q=p_1-p_2,\quad 2q'=p_1'-p_2\ , \tag{3.5}$$

$$\mathrm{Tr}_{\mu}(\pi\pi'\gamma)=\mathrm{Tr}[\gamma_5(m_q+i\gamma\cdot p_2)\gamma_5(m_q-i\gamma\cdot p_1)i\gamma_{\mu}(m_q-i\gamma\cdot p_1)]\ , \tag{3.6}$$

which simplifies to

$$4[\tfrac{1}{2}P_{\mu}'\Delta_1+\tfrac{1}{2}P_{\mu}\Delta_1'+2\bar{p}_{1\mu}+M_{\pi}^2\bar{P}_{\mu}-(M_{\pi}^2+\tfrac{1}{2}k^2)p_{2\mu}]\ , \tag{3.7}$$

$$\bar{p}_{1\mu}=\tfrac{1}{2}(p_{1\mu}+p_{1\mu}')=\bar{P}_{\mu}-p_{2\mu},\quad 2\bar{P}_{\mu}=P_{\mu}+P_{\mu}'\ . \tag{3.8}$$

The functions $\tilde{\tilde{\phi}}$ employed in (3.3) correspond to the half-off-shell structures (2.18), and the denominator functions D_{+}, D'_{+} are given by (2.9) for the momenta (p_1,p_2) and (p_1',p_2), respectively.

Before carrying out the integration in (3.3), it is first necessary to project out the four-momenta $p_{1\mu}$ and $p_{2\mu}$ in the (only surviving) direction of P_{μ}:

$$p_{2\mu}\Longrightarrow\frac{p_2\cdot\bar{P}}{\bar{P}^2}\bar{P}_{\mu},\quad \bar{p}_{1\mu}=\bar{P}_{\mu}-p_{2\mu}\ , \tag{3.9}$$

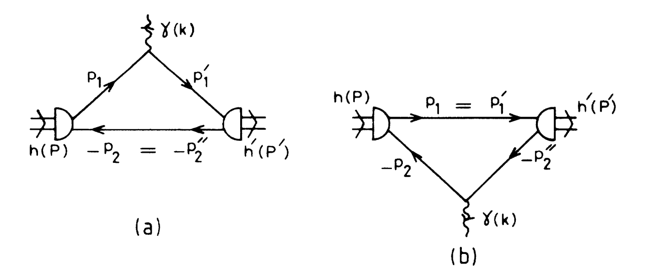

FIG. 1. Electromagnetic transition amplitudes for $h \to h' + \gamma$ via quark triangle. For notation see the text.

where

$$\bar{P}^2 = -M_\pi^2 - \tfrac{1}{4}k^2 . \tag{3.10}$$

Furthermore,

$$4p_2 \cdot \bar{P} = -2M_\pi^2 - \Delta_1 - \Delta_1' + 2\Delta_2 . \tag{3.11}$$

To carry out the integrations in (3.3), it is convenient to convert q_- to the p_{2-} variable[10] whose pole Δ_2^{-1} is on the *opposite* side to those of Δ_1^{-1} and $(\Delta_1')^{-1}$ (for small enough values of k_+). Some important results are[11]

$$\frac{D_+ D'_+}{2\pi i} \int \tfrac{1}{2} dp_{2-} \frac{1}{\Delta_2} \left[\frac{1}{\Delta_1 \Delta_1'} ; \frac{1}{\Delta_1} ; \frac{1}{\Delta_1'} \right] = [2p_{2+} ; D'_+ ; D_+] , \tag{3.12}$$

$$\int \tfrac{1}{2} dp_{2-} \left[\frac{1}{\Delta_1 \Delta_1'} ; \frac{1}{\Delta_1} ; \frac{1}{\Delta_1'} ; \frac{1}{\Delta_2} \right] = 0 . \tag{3.13}$$

Use of these results (3.9)–(3.13) in (3.3) via (3.7) gives

$$\mathcal{F}_\mu^\pi = 4(2\pi)^2 \tilde{\tilde{N}}_\pi^{(-)} \tilde{\tilde{N}}_{\pi'}^{(-)} e \int d^2 q_\perp dq_+ \tilde{\tilde{\phi}}_\pi \tilde{\tilde{\phi}}_{\pi'} [\cdots] \bar{P}_\mu \tag{3.14}$$

$$[\cdots] = (\tfrac{1}{2}D_+ + \tfrac{1}{2}D'_+ + 2p_{2+}M_\pi^2)[1 + \tfrac{1}{2}(M^2 + \tfrac{1}{2}k^2)/\bar{P}^2] .$$

Next, the integration over $d^2 q_\perp$ is carried out trivially, after the substitution from (2.18), so that only the integration over $dq_+ = dp_{2+}$ remains. Substituting $p_{2+} = xP_+$, so that $0 \le x \le 1$, and using (2.19), we have, after some simplifications,

$$\mathcal{F}_\mu^\pi = 2e\bar{P}_\mu F_\pi(k^2), \quad F_\pi(k^2) = f(k^2)/f(0) , \tag{3.15}$$

where

$$f(k^2) = \int_0^1 x\, dx\, \bar{D}(x,k^2) E(x,k^2) , \tag{3.16}$$

$$\bar{D}(x,k^2) = m_q^2 + 2x\beta^2 + x^2(M^2 + \tfrac{1}{4}k^2) , \tag{3.17}$$

$$E(x,k^2) = \exp\left\{ \frac{1}{\beta^2}[\tfrac{1}{4}M^2 + m_q^2 - \tfrac{1}{2}m_q^2 x^{-1} - \tfrac{1}{2}x(M^2 + \tfrac{1}{4}k^2)] \right\} . \tag{3.18}$$

In deriving Eqs. (3.15)–(3.18), use has been made of the following results which are valid in the Breit frame:

$$P_+, p'_+ = \bar{P}_+ \pm \tfrac{1}{2}k_+, \quad \hat{k} = k_+/P_+ , \tag{3.19}$$

$$\hat{k}^2 = k^2/(M^2 + \tfrac{1}{4}k^2) \quad (k^2 = k_\mu^2 > 0) . \tag{3.20}$$

It is easy to check that $f(0)$ is proportional to $\tilde{\tilde{N}}_\pi^{-2}$, as expected. The quantity $F_\pi(k^2)$ may be identified as the pion form factor, normalized to $F_\pi(0) = 1$.

Now the form factor at low k^2 is essentially governed by the "pion radius" which is *defined* through

$$\langle R_\pi^2 \rangle = -6 \left. \frac{\partial F_\pi(k^2)}{\partial k^2} \right|_{k^2=0} , \tag{3.21}$$

which works out as

$$\langle R_\pi^2 \rangle = \tfrac{3}{4}\beta^{-2} f^{-1}(0) \int_0^1 x^2 dx (m_q^2 + x^2 M_\pi^2) E(x,0) = (0.459 \text{ fm})^2 . \tag{3.22}$$

This value is rather small compared to the recent experimental value[21] of (0.6625±0.006), suggesting that our $q\bar{q}$ pion is a rather "tight" structure, perhaps not far removed from the Goldstone pion field of the modern chiral theories.[22] While this features has helped us in predicting[14] certain low-energy pionic amplitudes in conformity with the low-energy theorem[23] and with experiment,[15,16] our description of the pion as a $q\bar{q}$ composite has exacted a price through its prediction of a *small* radius which necessarily implies a more gentle fall in the form factor at small k^2 than observed,[21] even without a further need for an explicit display. We also record the prediction of the *closed form* (3.15)–(3.18) for large k^2, for which some data are also available.[21] The results of this comparison which are given in Fig. 2 over the extended range $0 \le k^2 \le 10$ GeV^2, reveals a much better

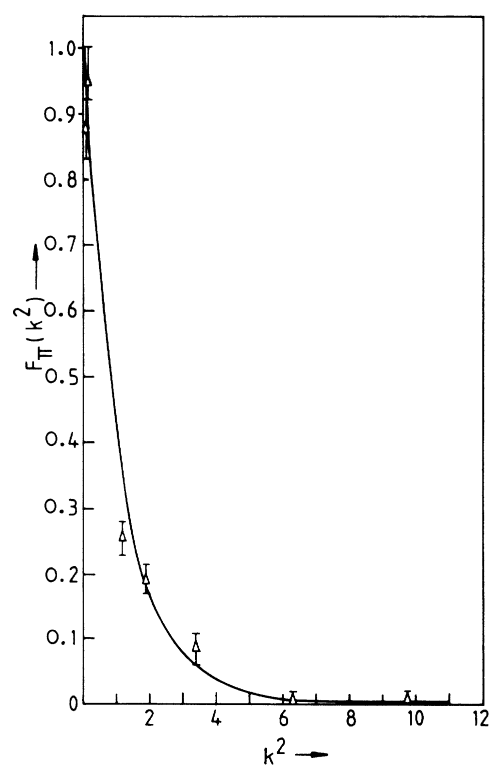

FIG. 2. Pion form factor F_π at large k^2: theory vs data (Ref. 21).

overlap with the corresponding data. For further discussion, see Sec. V.

B. $(\omega, a_2) \to \pi\gamma$ amplitudes

Our next example concerns a "transition form factor," $\omega \to \pi\gamma$ with $h=\omega$, $h'=\pi$, with the emission of a real photon ($k^2=0$), which implies either $k_+=0$ or $k_-=0$. The former ($k_+=0$) is more in conformity with the "form-factor" scenario (Breit-frame) of the last subsection and corresponds to the final hadron moving in the same direction as the initial one ($P'_+=P_+$). The corresponding transition amplitude is expressed by

$$\mathcal{F}(\omega \to \pi\gamma) = (2\pi)^2 N_\omega^{(-)} \tilde{\tilde{N}}_\pi^{(-)} e \int d^4q\, D_+ D'_+ \phi_\omega \tilde{\tilde{\phi}}_\pi (\Delta_1 \Delta'_1 \Delta_2)^{-1} \mathrm{Tr}(\omega\pi\gamma) , \tag{3.23}$$

$$\mathrm{Tr}(\omega\pi\gamma) = \mathrm{Tr}[i\gamma\cdot\omega(m_q + i\gamma\cdot p_2)\gamma_5(m_q - i\gamma\cdot p'_1) i\gamma(m_q - i\gamma\cdot p_1)] \equiv 4 m_q P_{5\gamma} , \tag{3.24}$$

$$P_{5\gamma} = \epsilon_{\mu\nu\lambda\rho}\omega_\mu \epsilon_\nu P_\lambda P'_\rho . \tag{3.25}$$

The integration over the internal momenta follows identical lines to (3.6)–(3.14), only the calculation is simpler now. The final result is

$$\mathcal{F}(\omega \to \pi\gamma) = 4 m_q e P_{5\gamma}(N_\omega \tilde{\tilde{N}}_\pi / \beta_\pi \beta_\omega) \int_0^1 2x\, dx\, B(x) \times \exp\left[-\frac{M_\omega^2}{2\beta_\omega^2}(\tfrac{1}{2} - x)^2 + \frac{1}{4\beta_\pi^2}\left[\frac{M_\pi^2}{2} + 2m_q^2\right] - \frac{1}{4\beta_\pi^2}(M_\pi^2 x + m_q^2/x)\right] , \tag{3.26}$$

where

$$B^{-1}(x) = \frac{1}{2\beta_\omega^2} + \frac{x^{-1}}{4\beta_\pi^2} , \tag{3.27}$$

$$x = p'_{2+}/P'_+ = p_{2+}/P_+ . \tag{3.28}$$

The integration yields

$$\mathcal{F}(\omega \to \pi\gamma) = e P_{5\gamma}(2.578) = \frac{2e}{M_\omega} P_{5\gamma}(0.999) . \tag{3.29}$$

The last equality shows a direct comparison with Schwinger's vector-meson-dominance (VMD) form[24] with $0.999 \to 1.00$. The width in turn is given by ($\hat{Q}=380$ MeV)

$$\Gamma_{\omega\to\pi\gamma} = \frac{e^2}{4\pi}\frac{\hat{Q}^3}{3}(2.578)^2 = 0.887 \text{ MeV} \quad (\text{cf. } 0.852 \text{ MeV, Ref. 16}) . \tag{3.30}$$

The calculation of the $a_2 \to \pi\gamma$ amplitude follows identical lines, except for the replacement in (3.25):

$$\omega_\mu \to \hat{q}_{\mu_1} A_{\mu_1\mu}\frac{\sqrt{2}}{\beta_{a_2}}, \quad \hat{q}_\mu = q_\mu + \frac{q\cdot P}{M^2}P_\mu , \tag{3.31}$$

as may be seen from Eq. (2.12) with $L=1$. This factor must now be included in the q integration. This is readily achieved by noting that since only one $\hat{q}_\mu$ factor is involved, which is effectively a three-vector, only its q_+ component will contribute. [See Appendix B, Eq. (B4) and subsequent remarks for justification.] In addition, $\hat{q}_\mu$ can be decoupled from the external kinematics through the obvious replacement

$$\hat{q}_{\mu_1} A_{\mu_1\mu} \to \frac{\hat{Q}\cdot\hat{q}}{\hat{Q}^2}\hat{Q}_{\mu_1} A_{\mu_1\mu} , \tag{3.32}$$

$$\hat{Q}_\mu = \tfrac{1}{2}(P'_\mu - P''_\mu) + \tfrac{1}{2}\frac{P\cdot(P'-P'')}{M^2}P_\mu , \tag{3.33}$$

which ensures $\hat{Q}\cdot P=0$, for unequal-mass hadrons. Numerically, $|\hat{Q}|$ is the two-body decay momentum listed in the Particle Data Group (PDG) tables,[16] and $\hat{Q}_- = -\hat{Q}^2/\hat{Q}_+$, so that one finds

$$\hat{Q}\cdot\hat{q}\hat{Q}^{-2} = \frac{q_+}{P_+}\cdot\eta, \quad \eta = \left[1 + \frac{M^2}{4\hat{Q}^2}\right] . \tag{3.34}$$

The rest of the procedure is now identical to Sec. III B, with the extra factor q_+/P_+ included in x integration vide Eq. (3.26) and the invariant $P_{5\gamma}$ replaced by

$$P_{5\gamma} \to P'_{5\gamma} \equiv \hat{Q}_{\mu_1} A_{\mu_1\mu}\epsilon_{\mu\nu\lambda\rho}\epsilon_\nu P_\lambda P'_\rho . \tag{3.35}$$

The final result is

$$\mathcal{F}(a_2 \to \pi\gamma) = e P'_{5\gamma}(-2.3600) ,$$

which corresponds to

$$\Gamma(a_2 \to \pi\gamma) = \tfrac{1}{10}\alpha\,|\hat{Q}|^5(-2.3600)^2 = 0.48 \text{ MeV} \quad (\text{vs } 0.297 \pm 0.066, \text{ Ref. 16}) ,$$

which is in fair agreement with the data.

IV. HADRONIC TRANSITIONS ($h \to h' + h''$)

Our next task is to illustrate the analytic structure of a purely hadronic class of transition amplitudes ($h \to h' + h''$) as obtained from this formalism using lowest-order Feynman (quark-triangle) diagrams. The standard processes, which sample the main varieties of hadronic coupling structures as $q\bar{q}$ composites are similar to those considered at the initial stages[10] of this program but with two major refinements[11,12] (outlined in Sec. II) incorporated, thus saving a vital parameter representing the coupling-constant renormalization[10] (which now gets "determined" by the model). As in Sec. III, we shall present the results for three categories of $hh'h''$ couplings in ascending order of algebraic complexity, but it is first necessary to outline the main strategy for the integration over the internal q_- variable (common to all cases) which should ensure explicit Lorentz invariance of the resultant coupling structure, especially in view of the possibility of an unsymmetrical choice of the " + " components for the outgoing hadron momenta involving unequal masses.

A. General procedure

Figure 3(a) shows the quark triangle diagram for the hadronic transition $h \to h'+h''$, with masses (M,M',M'') and four-momenta (P,P',P''). We adopt the convention that the "quark" line of the parent hadron has four-momentum $p_{1\mu}$ and the "antiquark" line has four-momentum $p_{2\mu}$, each in the direction of its arrow. The corresponding duality diagram[25] is shown in Fig. 3(b) which describes direction of four-momentum flow, and justifies a relabeling of the quark internal momenta as

$$p_1=p_1', \quad -p_2'=p_1''=p_3, \quad -p_2=-p_2'' . \tag{4.1}$$

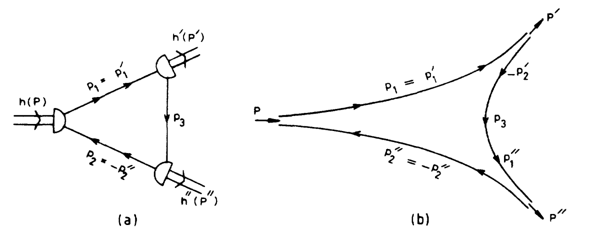

FIG. 3. Quark triangle diagram for (a) $h \to h'+h''$; corresponding duality diagram (b).

This ensures the overall conservation $P=P'+P''$ via the connections

$$P=p_1+p_2, \quad P'=p_1'+p_2', \quad P''=p_1''+p_2'' . \tag{4.2}$$

The internal four-momenta (q) are given by

$$2q=p_1-p_2, \quad 2q'=p_1'-p_2', \quad 2q''=p_1''-p_2'' . \tag{4.3}$$

The invariant matrix element for the transition is[10]

$$\mathcal{F}(h \to h'h'')=(2\pi)^8\sqrt{2/3}\int d^4q \, \mathrm{Tr}[\Gamma_h(q)S_F(p_1)\Gamma_{h'}(q')S_F(p_3)\Gamma_{h''}(q'')S_F(-p_2)] , \tag{4.4}$$

where the color and isospin factors for all the cases considered work out as $\sqrt{2/3}$ using the method of Ref. 10, and the vertex functions $\Gamma(q)$ are defined by Eq. (2.16).

The next step is the integration over $\frac{1}{2}dq_-$ which involves the poles of the three S_F propagators, viz., $\Delta_1^{-1}\Delta_2^{-1}\Delta_3^{-1}$, $\Delta_i=m_q^2+p_i^2$, at the positions

$$p_{i-}=\frac{\omega_\perp^2}{p_{i+}} \quad (i=1,3,3) , \tag{4.5}$$

where $\omega_\perp^2$ has the common value of $(m_q^2+q_\perp^2)$ by virtue of the collinear nature of the decay process $(q_\perp=q_\perp'=q_\perp'')$. Of these, the p_{1-} and p_{2-} poles are necessarily on opposite sides of the q_- plane, while the p_{3-} pole is on the same side as p_{1-} or p_{2-} accordingly, as the value of p_{1+}'' is >0 or <0 (i.e., p_{2+}' is <0 or >0), respectively.

Thus, for $p_{1+}''>0$, there is only the Δ_2 pole on one side which makes it convenient to close the contour on its side, taking p_{2-}'' as the integration variable. Similarly for $p_{2+}'>0$, there is only the Δ_1 pole on one side, whose integration is facilitated with p_{1-}' as the variable. There is a simple geometrical interpretation of these two integration zones which is intimately associated with the direction of the momentum flow in the null-plane language: In the rest frame of the parent hadron h, if the p_1 quark moves "forward," then $p_{1+} \gg p_{2+}>0$. Such an inequality results from the p_2 quark moving "backward," which implies that $p_{2+}=p_{2+}''$ has only a small positive value; thus it is reasonable to suppose that the joining quark p_{1+}'' adds a positive contribution to it (so as to ensure for the resulting hadron h'' a modest positive value $P_+''>0$). Since, on the other hand, $p_{1+}=p_{1+}' \gg p_{2+}''$, the hadron h' would still have $P_+'>p_+''$, despite some loss of its value due to the link momentum p_{2+}' being negative in this case (since $P_+'=p_{1+}'+p_{2+}'$). Thus, the momentum flow $p_{1+}''>0$ is associated with a "forward" h' and a "backward" h'' $(P_+'>P_+'')$. The opposite scenario holds for $p_{2+}'>0(P_+''>P_+')$. The two zones correspond precisely to the two integration zones noted above with only the Δ_2 pole or only the Δ_1 pole contributions, respectively. These two integration zones, which are complementary, are schematically shown in Fig. 4.

The *sum* of these two contributions ensures explicit null-plane covariance for the entire amplitude (much like *s*- and *u*-channel contributions to, say, electron-photon scattering together ensure explicit gauge invariance for the entire compton amplitude). Before discussing specific applications, it is useful to express (4.4) in a format more convenient for identifying individual decay modes. Thus substituting from (2.16), and writing $(\Gamma,\Gamma',\Gamma'')$ for the constant Dirac matrices associated with (h,h',h''), respectively, the transition amplitude may be written as

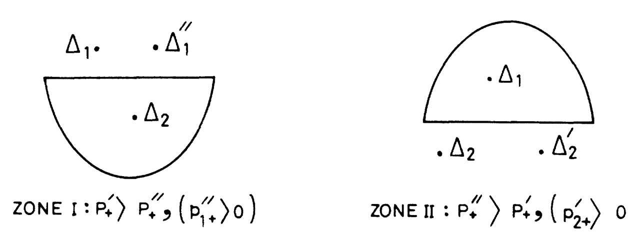

FIG. 4. Positions of poles in the integration zones I and II. For details, see text.

$$\mathcal{F}(h\to h'h'')=\frac{16}{\sqrt{3}}(\beta\beta'\beta'')^{-1}N_hN_{h'}N_{h''}\frac{1}{2\pi i}$$

$$\times\int d^4q[\mathrm{TR}]\frac{(\mathrm{DWF})}{\Delta_1\Delta_2\Delta_3}\;, \tag{4.6}$$

where, for any hadron the reduced normalizer N_h is defined as

$$N_h^{(-)}=(2\pi)^{-3/2}(\pi\beta^2)^{-1/2}P_+^{-1}N_h\;, \tag{4.7}$$

and the other symbols are

$$[\mathrm{TR}]=\tfrac{1}{4}\mathrm{Tr}[\Gamma(m_q-i\gamma\cdot p_1)\Gamma'(m_q-i\gamma\cdot p_3)$$

$$\times\Gamma''(m_q+i\gamma\cdot p_2)]\;, \tag{4.8}$$

$$(\mathrm{DWF})=\frac{D_+\phi}{P_+}\frac{D'_+\phi'}{P'_+}\frac{D''_+\phi''}{P''_+}\;, \tag{4.9}$$

$$\Delta_i^2=m_q^2+p_i^2\quad(i=1,2,3)\;. \tag{4.10}$$

Further, as explained for the pion form-factor calculation in Sec. III, and again in Appendix A, [TR] must first be expressed as much as possible in terms of the Δ_i functions, *before* the q_- integrations are performed over the two zones (I and II) noted above. We now consider the individual cases in the above format.

B. $\rho\to\pi\pi$, $f_2\to\pi\pi$

For the $\rho\to\pi\pi$ case, $\Gamma=i\gamma\cdot\hat{\epsilon}$, and $\Gamma'=\Gamma''=\gamma_5$, so that

$$[\mathrm{TR}]=\hat{\epsilon}\cdot q\left[\Delta_3+M_\pi^2-\frac{M^2}{2}\right]$$

$$+\tfrac{1}{2}\hat{\epsilon}\cdot\hat{Q}(\Delta_1+\Delta_2+M^2)\;, \tag{4.11}$$

where $\hat{Q}$, defined through Eq. (3.33), is now merely $(\frac{1}{2}P'-\frac{1}{2}P'')$. In this equal-mass case $(M'=M''=M_\pi)$ for two final hadrons, the two integration zones (I and II) will give identical contributions, so that it is sufficient to do only zone I and multiply the result by 2. Taking account of the kinematic conditions for zone I, viz., $P'_+\approx P_+\gg P''_+$ and $(0\le p''_{2+}\le P''_+)$ by virtue of the definitions

$$P'_+,P''_+=(M_\pi^2+\hat{Q}^{\,2})^{1/2}\pm|\hat{Q}|\quad(M_\pi^2/M^2\ll1)\;, \tag{4.12}$$

the factor $\hat{\epsilon}\cdot q$ in (4.11) is nearly equal to $\hat{\epsilon}\cdot\hat{Q}$ in this case, and the result of q_- integration is just

$$\frac{1}{2\pi i}\int\tfrac{1}{2}dp_{2-}\frac{[\mathrm{TR}]}{\Delta_1\Delta_1'\Delta_2}\approx\frac{\hat{\epsilon}\cdot\hat{Q}}{2D''_+}\;. \tag{4.13}$$

Next, for the remaining integrations, we must use the "half-off-shell" pion wave functions, Eq. (2.18), with corresponding normalizers $\tilde{\tilde{N}}_\pi^{(-)}$, Eq. (2.19), while the ρ wave function continues to be "on shell," Eq. (2.10), and its reduced normalizer N_ρ is given by Eq. (A9). Collecting all necessary factors, the total amplitude (including the effect of zone II) is expressible as[10]

$$\mathcal{F}(\rho\pi\pi)=2(\hat{\epsilon}\cdot\hat{Q})g_{\rho\pi\pi}\;, \tag{4.14}$$

$$g_{\rho\pi\pi}=\tfrac{32}{3}\pi\beta_\rho^{-1}\beta_\pi^{-2}N_\rho\tilde{\tilde{N}}_\pi^{\,2}\mathcal{G}_\rho\;, \tag{4.15}$$

$$\mathcal{G}_\rho=\int_0^1dx\,B_x[(m_q^2+B_x)^2+B_x^2]\exp[-F(x)]\;, \tag{4.16}$$

$$B_x^{-1}=\tfrac{1}{2}\beta_\rho^{-2}+\tfrac{1}{4}\beta_\pi^{-2}\left[1+\frac{1}{x}\right],\quad x=p''_{2+}/P''_+\;, \tag{4.17}$$

$$F(x)\approx\exp\left[-\left[\frac{M_\pi^2}{2}-m_q^2\right]\Big/4\beta_\pi^2\right.$$

$$\left.+(\tfrac{1}{2}-x)(M_\pi^2-2m_q^2x^{-1})/4\beta_\pi^2-\frac{M^2}{8\beta^2}\right]. \tag{4.18}$$

The value of $g_{\rho\pi\pi}$ works out as 6.234, leading to a width of

$$\Gamma_{\rho\pi\pi}=\frac{2}{3}\frac{\hat{Q}^{\,3}}{M^2}\frac{g_{\rho\pi\pi}^2}{4\pi}$$

$$=162\ \mathrm{MeV}\quad(\mathrm{cf.}\ 153\ \mathrm{MeV,\ Ref.\ 16})\;. \tag{4.19}$$

The case of the $\pi\pi$ decay of f_2, an $L=1$ tensor meson, is similar to the above, except for the replacement

$$\hat{\epsilon}_\mu\to\hat{q}_{\mu_1}T_{\mu_1\mu}\hat{Q}_{\mu_1}T_{\mu_1\mu}\quad(\mathrm{const}) \tag{4.20}$$

under the kinematical conditions (4.12) for zone I. The final form of the $f_2\to\pi\pi$ amplitude becomes

$$\mathcal{F}(f_2\pi\pi)-(\hat{Q}_\mu T_{\mu\nu}\hat{Q}_\nu)g_{f_2\pi\pi}\;, \tag{4.21}$$

where

$$g_{f_2\pi\pi}=\tfrac{32}{3}\pi\beta_{f_2}^{-1}\beta_\pi^{-2}N_{f_2}\tilde{\tilde{N}}_\pi^{\,2}\left[\frac{2\sqrt{2}}{\beta_{f_2}}\right]\mathcal{G}_{f_2} \tag{4.22}$$

and $\mathcal{G}_{f_2}$ has the same algebraic structure as $\mathcal{G}_\rho$ Eq. (4.16). The value of $g_{f_2\pi\pi}$ works out as 16.21 and the corresponding decay width is given by

$$\Gamma_{f_2\pi\pi}=\frac{\hat{Q}^{\,5}}{40\pi M^2}g_{f_2\pi\pi}^2$$

$$=119.7\ \mathrm{MeV}\ (148\pm17\ \mathrm{MeV,\ Ref.\ 16})\;. \tag{4.23}$$

C. $a_2\to\rho\pi$

This is a case of *unequal*-mass hadrons $(M'>M'')$ which requires separate treatments for zones I and II with appropriate changes in the definitions of P'_+, P''_+, viz.,

$$P'_+,P''_+=(M'^2+\hat{Q}^{\,2})^{1/2}\pm|\hat{Q}|\;,$$
$$(M''^2+\hat{Q}^{\,2})^{1/2}\mp|\hat{Q}|\;, \tag{4.24}$$

where the upper (lower) signs correspond to zones I (II), respectively. With the definitions $\Gamma=i\gamma\cdot\hat{\epsilon}$, $\Gamma'=i\gamma\cdot\hat{\epsilon}'$, $\Gamma''=\gamma_5$, we have simply

$$[\mathrm{TR}]=m_q\epsilon_{\mu\nu\lambda\sigma}\hat{\epsilon}_\mu\hat{\epsilon}'_\nu P_\lambda P'_\sigma\;, \tag{4.25}$$

where $\hat{\epsilon}_\mu$ and $\hat{\epsilon}'_\mu$ are the a_2 and ρ polarization vectors, respectively. In particular, $\hat{\epsilon}_\mu$ is given by Eq. (3.31), where the $\hat{q}_{\mu_1}$ vector can be factored out as in Eq. (3.32), with the same definition (3.33) for $\hat{Q}_\mu$, but now Eq. (3.34) is replaced by

$$\hat{Q}\cdot\hat{q}/\hat{Q}^2=\frac{1}{2}\frac{q_+}{P_+}\left[\sigma M^2\hat{Q}^{-2}+\frac{1}{\sigma}\right],\quad \sigma=\hat{Q}_+/P_+ . \tag{4.26}$$

The integration over $\frac{1}{2}dq_-$ is now much simpler, but one must now take separate account of zones I and II.

The final result, after all other integrations, is expressible as a *sum* over the two zones as

$$\mathcal{F}(a_2\rho\pi)=\frac{32\pi}{3}\frac{N_{a_2}N_\rho\tilde{\tilde{N}}_\pi}{\beta\beta'\beta''}m_q\frac{\sqrt{2}}{\beta}\times\left[\frac{P''_+}{P'_+}\mathcal{G}_{\rm I}+\frac{P'_+}{P''_+}\mathcal{G}_{\rm II}\right]P'_{5V} , \tag{4.27}$$

where P'_{5V} is given by (3.35) and

$$\mathcal{G}_{\rm I}=\int_0^1 dx\, x\frac{D'_+q_+}{P'_+P_+}B_x\exp[-F(x)] , \tag{4.28}$$

$$B_x^{-1}=\tfrac{1}{2}\beta_{a_2}^{-2}+\tfrac{1}{2}\beta_\rho^{-2}+\tfrac{1}{4}\beta_\pi^{-2}\frac{1}{x}\quad (x=p_{2+}''/P''_+) , \tag{4.29}$$

$$F(x)=\frac{M^2}{2\beta_{a_2}^2}\left[\frac{q_+^2}{P_+^2}\right]+\frac{M^2}{2\beta_\rho'^2}\left[\frac{q_+'^2}{P'^q_+}\right]-\frac{1}{4\beta_\pi''^2}(\tfrac{1}{2}-x)(M_\pi^2-2m_q^2x^{-1}) . \tag{4.30}$$

Similar formulas hold for $\mathcal{G}_{\rm II}$ (with the roles of P'_+ and P''_+, etc., interchanged). The total contribution, which comes almost entirely from zone II, works out as

$$\mathcal{F}(a_2\rho\pi)=(18.434)P'_{5V} \tag{4.31}$$

leading to the width estimate

$$\Gamma_{a_2\rho\pi}=\frac{Q^5}{20\pi}(18.434)^2 = 69.8 \text{ MeV (cf. 77 MeV, Ref. 16)} . \tag{4.32}$$

D. $b_1\to\omega\pi$

This is a process involving unequal-mass hadrons ($M'>M''$) on the one hand, and a matrix element of the VPP type (e.g., $\rho\pi\pi$) on the other hand. After substituting the respective Dirac matrices $\Gamma=\gamma_5\hat{q}\cdot B$, $\Gamma'=i\gamma\cdot\hat{\omega}$, $\Gamma''=\gamma_5$, the trace part may be displayed succinctly as

$$[\mathrm{TR}]=(B\cdot\hat{q})\left[p_1\cdot\hat{\omega}\left[\tfrac{1}{2}\Delta_2+\tfrac{1}{2}\Delta_3+\frac{M''^2}{2}\right]+p_2\cdot\hat{\omega}\left[\tfrac{1}{2}\Delta_1+\tfrac{1}{2}\Delta_3+\frac{M'^2}{2}\right]+p_3\cdot\hat{\omega}\left[\tfrac{1}{2}\Delta_1+\tfrac{1}{2}\Delta_2+\frac{M^2}{2}\right]\right] . \tag{4.33}$$

From the definitions of p_1,p_2,p_3 in Eq. (4.3), the following results hold:

$$p_2\cdot\hat{\omega}=(P''-p_1)\cdot\hat{\omega},\quad p_3\cdot\hat{\omega}=p_1\cdot\hat{\omega}=q'\cdot\hat{\omega} , \tag{4.34}$$

Using the obvious result $P'\cdot\hat{\omega}=0$, so that $\hat{\omega}\cdot P=\hat{\omega}\cdot P''$. As in the $a_2\to\rho\pi$ case, the unequal masses of the final hadrons require separate pole integrations for zones I and II, in accordance with the pattern of Fig. 4. The results are

$$\frac{1}{2\pi i}\int dq_-\frac{[\mathrm{TR}]}{\Delta_1\Delta_2\Delta_3}=(B\cdot\hat{q})\tfrac{1}{2}\left[(\hat{\omega}\cdot P'')\left[\frac{1}{D_+}+\frac{1}{D''_+}+\frac{2p_{2+}M'^2}{D_+D''_+}\right]+(\hat{\omega}\cdot q')\frac{2p_{2+}t}{D_+D''_+}\right]\quad\text{(zone I)} , \tag{4.35}$$

$$=(B\cdot\hat{q})\left[(\hat{\omega}\cdot P'')\tfrac{1}{2}\left[\frac{1}{D_+}+\frac{2p_{1+}M^2}{D_+D'_+}\right]+(\hat{\omega}\cdot q')\left[\frac{1}{D'_+}+\frac{p_{1+}t}{D_+D'_+}\right]\right]\quad\text{(zone II)} , \tag{4.36}$$

where

$$t=M^2+M''^2-M'^2 . \tag{4.37}$$

The factor $(B\cdot\hat{q})$ and $(\hat{\omega}\cdot q')$ above may be broken up for subsequent $(\mathbf{q}_\perp,q_+)$ integrations as

$$(B\cdot\hat{q})=(B_\perp\cdot\mathbf{q}_\perp)+\frac{q_+}{P_+}\frac{B_+}{P_+}M^2 ,\qquad \hat{\omega}\cdot q'=\hat{\omega}_\perp\cdot\mathbf{q}'_\perp+\frac{q'_+}{p'_+}\frac{\hat{\omega}_+}{P'_+}M'^2 . \tag{4.38}$$

The values of B_+ and $\hat{\omega}_+$ may be determined on the lines of Eq. (4.26) from the respective relations

$$B\cdot\hat{Q}=-B\cdot P'=\frac{B_+}{P_+}\xi ,\qquad \xi=\tfrac{1}{2}(\hat{Q}^2P_+/\hat{Q}_++M^2\hat{Q}_+/P_+) , \tag{4.39}$$

$$(\hat{\omega}\cdot P'')=\frac{\hat{\omega}_+}{P_+}\cdot\xi' ,\qquad \xi'=\tfrac{1}{2}(M'^2P_+P''_+/P_+^2-M''^2P_+/P''_+) , \tag{4.40}$$

$$\hat{\omega}\cdot\hat{Q}=(\hat{\omega}\cdot P'')(M^2+M'^2-M''^2)/(2M^2) = (\hat{\omega}\cdot P'')\xi'' , \tag{4.41}$$

where $\hat{Q}$ is given by Eq. (3.33). These results may be used to obtain the following connection between the transverse components of B and ω polarizations and the Lorentz-invariant quantity $(B\cdot\hat{\omega})$:

$$(B_\perp\cdot\hat{\omega}_L)=(B\cdot\hat{\omega})+\frac{(\hat{\omega}\cdot P'')(B\cdot P')\cdot\xi''}{\hat{Q}^2} . \tag{4.42}$$

The rest of the procedure is a straightforward but lengthy application of the methods already outlined in the preceding subsections. The final result after adding the contributions of zones I and II (the latter being dominant as in the $a_2\rho\pi$ case) may be expressed in the invariant form

$$(b_1\to\omega\pi)=a(B\cdot\hat{\omega})+b\frac{(\hat{\omega}\cdot P'')(B\cdot P')}{\hat{Q}^2} , \tag{4.43}$$

where

$$a=1.681\,168, \quad b=5.173\,77 . \tag{4.44}$$

The unpolarized decay width is then given by

$$\Gamma(b_1\to\omega\pi)=\tfrac{1}{3}\sum_{\text{pols}} |\mathcal{F}(b_1\to\omega\pi)|^2\frac{|\hat{Q}|}{8\pi M^2}$$
$$=129.0 \text{ MeV} \quad (\text{vs } 136 \text{ MeV, Ref. 16}) . \tag{4.45}$$

E. $a_1\to\rho\pi$

Our last example of hadron couplings is the case of $a_1\rho\pi$ with $\Gamma=i\gamma\cdot\hat{\epsilon}\gamma_5$, $\Gamma'=i\gamma\cdot\hat{\epsilon}'$, and $\Gamma''=\gamma_5$, which yield

$$[\mathrm{TR}]=m_q[-(\hat{\epsilon}\cdot\hat{\epsilon}')(\Delta_3+S)+(\hat{\epsilon}\cdot P')(\hat{\epsilon}'\cdot P'')$$
$$-2(\hat{\epsilon}\cdot P')(\hat{\epsilon}\cdot q')] , \tag{4.46}$$

$$S=(M'^2+M''^2-M^2)/2 . \tag{4.47}$$

Following an identical procedure to the $b_1\to\omega\pi$ case for the pole integrations in zones I and II, and using Eq. (4.38) to simplify $(\hat{\epsilon}'\cdot q')$ together with Eqs. (4.39)–(4.41), the final result for $\mathcal{F}(a_1\to\rho\pi)$ (with zone II $\gg$ zone I) is again expressible in the form (4.43) with

$$a\to a'=3.223\,348, \quad b\to b'=0.677\,436\,6 \tag{4.48}$$

leading to the decay width

$$\Gamma(a_1\to\rho\pi)=\tfrac{1}{3}\sum_{\text{pols}} |\mathcal{F}(a_1\to\rho\pi)|^2\frac{|\hat{Q}|}{8\pi M^2}2$$
$$=173.6 \text{ MeV} , \tag{4.49}$$

which is about 54.94% of the total a_1 width of $\sim$316 MeV (Ref. 16). For ease of reading, the results are collected in Table I which also includes the masses predicted in this model.[12]

V. DISCUSSION, SUMMARY, AND CONCLUSIONS

In retrospect, we have evaluated some hadronic transition amplitudes of the types $h\to h'+\gamma$ and $h=h'+h''$ designed as a first (intermediate energy) test of the *second* stage of a two-tier BS formalism whose first stage, involving a three-dimensional (NPA) form of the BS equation, was meant as a "low-energy" check (mass spectra of hadrons). This second stage has been characterized by a reconstructed four-dimensional BS wave function Ψ, Eq. (2.15), which provides the normalized $q\bar{q}H$ vertex function $\Gamma(\mathbf{q})$, Eq. (2.16), in terms of the three-dimensional wave function ϕ and the denominator function D_+. The four-dimensional wave function Ψ carries with it the signature of the remaining BS degree of freedom characterized by virtual $q\bar{q}$ effects (higher Fock states). The basic philosophy of this approach[10] has been one of gradual unfolding of these effects so as to warrant a perturbative treatment through lowest-order Feynman diagrams. Now the good agreement of the spectral data[16] with the three-dimensional BS-equation predictions for both $q\bar{q}$ (Ref. 11) and qqq (Ref. 13) systems, unlike the predictions of certain O(4)-type theories,[26] had implied a relative insensitivity of the mass spectra to the (virtual $q\bar{q}$) degree of freedom contained in the full four-dimensional BS framework. Now these (perturbative) calculations of hadronic transition amplitudes through quark triangle diagrams should be regarded as a first systematic test of this hy-

TABLE I. Prediction of the decay widths (Γ), using the experimental masses (Ref. 16) (in MeV), except for the pion (for which the value 163 is used).

Particle (Ref. 16)	Predicted mass (Ref. 12)	Decay mode	Partial decay width (Γ) Theor.	Expt. (Ref. 16)
$\pi(140)$				
$\rho(770)$	915	$\rho_0\to e^+e^-$	6.025 keV	(6.9 ± 0.3) keV
		$\to\pi\pi$	162 MeV	153 MeV
$\omega(783)$	915	$\omega\to e^+e^-$	0.69 keV	(0.66 ± 0.4) keV
		$\to\pi\gamma$	0.887 MeV	0.852 MeV
$\phi(1020)$	1051	$\phi\to e^+e^-$	1.11 keV	(1.31 ± 0.06) keV
$a_2(1320)$	1352	$a_2\to\pi\gamma$	0.48 MeV	0.297 MeV
		$\to\rho\pi$	69.8 MeV	77 MeV
$f_2(1270)$	1352	$f_2\to\pi\pi$	119.7 MeV	(148 ± 17) MeV
$b_1(1235)$	1182	$b_1\to\omega\pi$	129.0 MeV	136 MeV
$a_1(1270)$	$\cdots$	$a_1\to\rho\pi$	173.6 MeV	dom. % of 316 MeV

pothesis of gradual unfolding of the virtual $q\bar{q}$ effects (termed "off-shell" effects in Sec. I), thus providing a stepwise extension of this two-tier BS formalism, up the energy scale, after the agreement with the mass-spectral data had served as a "low-energy calibration" of the formalism.

Viewed in this overall perspective, the good agreement of the results presented in Secs. III and IV with the data seems, by and large, to bear out our two-tier BS strategy. These processes have also provided the opportunity to describe in some detail the integration procedure involving an interplay of the four-dimensional structure of the matrix elements with the language of null-plane variables, leading to explicit Lorentz-invariant structures at the end. Further, the mathematical refinements affected[11] over the initial formalism[9] have dispensed with the need for a "coupling-constant renormalization" which had to be resorted to during the early applications to such amplitudes[10] in the instantaneous approximation. In this respect the present calculations are not only free from any other parameters beyond the three basic constants (ω_0, C_0, m_q) of the model,[12] but conceptually more satisfactory insofar as no "formal" restrictions on the energy scale are involved.

As to the actual results, one of the items, viz., the tight shape of the pion (small radius) is a weak feature of the model as applied to the pion which we do not yet fully understand (see Ref. 12, the last paragraph), since the unusually low mass of the pion makes it rather sensitive to the approximations employed. Nevertheless, the small value obtained for its radius, puts our $q\bar{q}$ pion somewhat nearer to its description as an "elementary" field, characteristic of the more familiar chiral theories.[22,23] Some recent results on certain "low-energy" EM couplings of the pion as obtained in this model,[14] in agreement with data as well as chiral theory predictions,[23] seem to support this view. On the other hand, the present $q\bar{q}$ picture of the pion at least allows *in principle* the calculation of its form factor which has a natural conceptual basis only in a composite description, while its dynamical significance for an (effectively elementary) pion field, characteristic of chiral models[22,23] needs further clarification. As to the actual fits to the data, the rather small radius found for the pion has already implied (even without a formal plot) disagreement with the observed form factor at low k^2, a result for which we do not as yet have a remedy without further assumptions. On the other hand, the sharp improvement in the fits to the data at higher k^2 (Fig. 2) seems to suggest that, our $q\bar{q}$ pion still works fairly well on the global scale of a wider range of k^2, while lacking precision of details (such as small-k^2 regions). Similar remarks apply to pion's structure[17] and fragmentation[18] functions which were also calculated in this model with a fair amount of success.

The results on the other EM amplitudes $(\omega, a_2 \rightarrow \pi\gamma)$ in Sec. III, as well as several purely hadronic amplitudes found in Sec. IV, must be regarded more from the viewpoint of this two-tier formalism at the level of essential physics, then from more specific motivations. The range of agreement with the data on a fairly representative list of hadronic transition amplitudes, especially those involving *L-excited* hadrons, do warrant a general degree of confidence in the conceptual framework of this model on one hand, and the methodology of the calculational procedures (including normalization techniques) on the other. In other words, these diverse hadronic amplitudes may be regarded as very useful (intermediate-energy) checkpoints (having earlier satisfied the "low-energy" spectroscopic data) which must first be confirmed before the wider ramifications of the model can be subjected to further tests. These ramifications lie in the capacity of this model to make unambiguous predictions on a truly wide range of transition amplitudes of which the applications given in this paper form perhaps a small sample. These include, among other things, unambiguous predictions[27] on such theoretical quantities as the leptonic decay constants of $Q\bar{q}$ mesons, involving very unequal-mass kinematics (f_D, f_B) whose values can only be inferred very indirectly through their *nonleptonic* modes whose dynamical mechanisms such as final-state interaction[28] or W annihilation[29] have not yet reached a consensus. On the other hand, such nonleptonic modes are *in principle* accessible to the present model through its formal capacity to "test" standard mechanisms such as penguin diagrams[30] for kaon decay[31] or the W-exchange diagram in D_0, D_s decays[29] by considerably limiting the "uncertainties" in the (strong) $q\bar{q}H$ vertex. Some of these applications are under way.[32]

VI. COMPARISON WITH OTHER MODELS

It should be of some interest to compare the scope of this model which seeks to make up for its semiempirical basis through an integrated view of a wide range of hadronic phenomena from the lowest to the highest energies with those of some contemporary approaches in the literature. First, while there have been several approaches[33] based on the null-plane language, most of them seem to start at the "wave-function" level without trying to connect this vital quantity to an underlying dynamical equation (which holds the key to, among other things, the prediction of mass spectra). In this respect, the approach to which ours has the closest resemblance is that of Brodsky and Lepage[34] (BL) who had developed the BS equation in the null-plane langauge through a chain of equations connecting to higher Fock states (much like the Tamm-Dancoff method[35] of the 1950s). However, their underlying dynamics represented by the "QCD evolution" equation had suffered from the disadvantage of overemphasis (?) on the "hard-QCD" component of strong interaction, with less attention to the "soft-QCD" aspects which presumably play a crucial role in the form-factor determinations.[36] (A similar disagreement with data on their $\pi^0 \rightarrow 2\gamma$ amplitude determination[34] was also ascribed[37] to the same cause, as could be traced to the structure of their null-plane wave function.) The present approach escapes this problem, albeit at the cost of theoretical depth. Indeed, the BL approach, had it been experimentally successful, would have been a serious QCD-oriented candidate for a strong-interaction theory. The present model, in contrast, seeks to bridge the "transition region" between soft and hard QCD, through a "guess" for the confining BS kernel, consistent with the disciplines of Lorentz and gauge invariance to the extent to which these can be incorporated in an effective $(q\bar{q})$ interaction. As for other contemporary approaches, we have already given in Sec. I a comparison with QCD sum-rule techniques[2] which represents a powerful method of extrapolation (based on operator product expansions) from the "high-energy" to the "low-energy" end. The price paid for such extrapolation seems to lie in the loss of information incurred on the mass *spectral data* as well as the wave functions for L-excited states inherent in any QCD sum-rule program.[38] Such techniques, while providing valuable information on several amplitudes involving $L=0$ mesons,[39] have not proved equally resilient for L-excited meson amplitudes (involving b_1, a_2, etc.) which we have now predicted with considerable success in this (less ambitious but more oriented to applications) model.

Among other models proposed in recent times (apart from the older "bag"-type models[4]) an interesting approach is the "flux-tube" model[40] with a high degree of predictive power (including L-excited mesonic amplitudes), but such models presumably stop short of dynamical predictions of hadron mass spectra.

Finally, a word about lattice gauge theories[7] (LGT's) may be in order. Since, in principle, LGT uses the fullest physical context of QCD, none of the models discussed above (including of course the present one) can dare to rank with it in terms of an absolute theoretical status, were it not for the severe computational problems which have so far failed to lend a fair degree of stability[41] to its unchallenged predictive powers. Nevertheless, recent spectacular advances in computer technology have put LGT's on the verge of a breakthrough (?) in terms of concrete QCD results for $Q\bar{Q}$ mass spectra[42] as well as certain pionic amplitudes.[43] Once LGT gets a firm foothold at the observational level it will automatically "put in place" other QCD-oriented efforts of varying degrees of theoretical sophistication. Even so, the latter would probably retain their value as effective "bridges" between a formal theory and various *ad hoc* data-based models. The BS type approach described here, by virtue of its integrated emphasis, and considerable predictive power, could still serve as a useful practical tool for a very economical analysis (with very few parameters) of *unexplored* data, once its detailed working has been checked against "known" data. These predictions lend a sort of practical status to such models which may survive comparison with more fundamental theories, when it is remembered that even modern methods of data analysis entail several *method-dependent* assumptions, especially where strong interactions are concerned. With such a perspective in view, several more applications of the model are in progress.

ACKNOWLEDGMENTS

A. Mittal was actively associated with the early stages of this investigation when the development of the null-plane dynamics, as presented here, was in progress. We are grateful to Dr. Mittal for crucial discussions and to Krishna K. Gupta for help with some computations.

APPENDIX A: BS NORMS FOR $L=0$ STATES

The normalization constant $N_h^{(-)}$ for a $q\bar{q}$ hadron (h) is defined by the following standard relation (via current conservation) for equal quark masses:[10]

$$2iP_\mu=(2\pi)^4\mathrm{Tr}\int d^4q[\bar{\Psi}i\gamma_\mu\Psi(m_q-i\gamma\cdot p_2)\tfrac{1}{2}+(-\tfrac{1}{2})\bar{\Psi}(m_q+i\gamma\cdot p_1)\Psi i\gamma_\mu] , \tag{A1}$$

where the four-dimensional BS amplitude Ψ and its associated quantities $D_+, \phi_L, \Delta_{1,2}$, etc., are explained in Sec. II. In the light of the discussion of the general case of EM transition amplitudes in Sec. III, the normalization calculation corresponds to the specific case $h=h'$, $k_\mu=0$, implying $p_{1_\mu}=p'_{1_\mu}$, $P_\mu=P'_\mu$. For the pseudoscalar $q\bar{q}$ meson, the corresponding result was derived in Ref. 11.

1. V mesons $(\Gamma=i\gamma\cdot\hat{\epsilon})$

For V mesons (ρ,ω,ϕ) Eq. (A1) reduces to

$$2iP_\mu=(2\pi)^3(N_V^{(-)})^2\int d^2q_\perp dq_+\tfrac{1}{2}dq_-[D_+^2\phi_V^2(\Delta_1^2\Delta_2)^{-1}\times\mathrm{Tr}_\mu(VV\gamma)] , \tag{A2}$$

where

$$\mathrm{Tr}_\mu(VV\gamma)=\mathrm{Tr}[i\gamma\cdot\hat{\epsilon}(m_q+i\gamma\cdot p_2)i\gamma\cdot\hat{\epsilon}(m_q-i\gamma\cdot p_1)\times i\gamma_\mu(m_q-i\gamma\cdot p_1)] . \tag{A3}$$

After routine simplification, Eq. (A1) reduces to the form

$$\tfrac{1}{4}\mathrm{Tr}_\mu(VV\gamma)=\{-(\hat{\epsilon}\cdot\hat{\epsilon})[p_{2\mu}\Delta_1+p_{1\mu}(\Delta_1+\Delta_2+M^2)]+4p_{1\mu}\hat{\epsilon}\cdot p_1\hat{\epsilon}\cdot p_2+2\Delta_1 p_2\cdot\hat{\epsilon}\hat{\epsilon}_\mu\} , \tag{A4}$$

where the Δ's are defined in Eq. (3.4) of text. Further simplification is achieved by projecting out (A4) in the (only surviving) direction P_μ as in Eq. (3.9), noting that $\hat{\epsilon}\cdot P=0$, and doing the replacement

$$\hat{\epsilon}_\mu\hat{\epsilon}_\nu\Longrightarrow\langle\hat{\epsilon}_\mu\hat{\epsilon}_\nu\rangle=\frac{1}{3}\left[\delta_{\mu\nu}+\frac{P_\mu P_\nu}{M^2}\right]=\tfrac{1}{3}Q_{\mu\nu}(P) \tag{A5}$$

to eliminate the four-vectors $\hat{\epsilon}_\mu$ from the other terms. At this stage, we encounter terms like $p_1\cdot P, p_2\cdot P$ which can be further manipulated as

$$p_1\cdot P=\tfrac{1}{2}\Delta_1-\tfrac{1}{2}\Delta_2-\tfrac{1}{2}M^2,\ \text{etc} , \tag{A6}$$

by using Eq. (3.11) of text. Substitution of these results in (A4) will make it proportional to P_μ times a scalar containing terms up to $O(\Delta^3)$. Of these only the terms independent of Δ's and those proportional to Δ_1 only will survive, vide Eqs. (3.12) and (3.13) of text. The Δ_2-pole integration with surviving terms with the help of Eq. (3.13) gives

$$2iP_\mu=(2\pi)^3 4i(N_V^{(-)})^2 P_\mu \int d^2q_\perp dq_+ P_+ \frac{2}{3}\left[\left[\omega_\perp^2+\frac{p_{2+}^2}{P_+^2}M^2\right]\left[1-\frac{2m_q^2}{M^2}\right]+\frac{4p_{2+}}{P_+}m_q^2\right]\phi_0^2 , \quad (A7)$$

where ϕ_0 is the "on-shell" wave function, Eq. (2.10), for $L=0$, assumed valid for all $q\bar{q}$ hadrons except the $L=0$ pseudoscalars (see Sec. II).

The integration is Gaussian for $\mathbf{q}_\perp$ but a truncated one for $x=q_+/P_+$, since $-\frac{1}{2}P_+ \le q_+ \le \frac{1}{2}P_+$. However, the latter integration is seen to correspond to the error function $\mathrm{erf}(x_0)$ where $x_0=M/\beta \gg 1$, which ensures that even the x integration is almost fully Gaussian for all V mesons. The final result may be expressed, analogously to Eq. (2.18), in terms of a "reduced" normalizer N_V through

$$N_V^{(-)}=(2\pi)^{-3/2}\frac{1}{P_+}\frac{1}{(\pi\beta^2)^{1/2}}N_V , \quad (A8)$$

where

$$N_V^{-2}=\frac{2}{M}(\pi\beta^2)^{1/2}\frac{2}{3}\left[\left[1-\frac{2m_q^2}{M^2}\right]\left[m_q^2+\tfrac{3}{2}\beta^2+\frac{M^2}{4}\right]+2m_q^2\right] . \quad (A9)$$

As a simple check on our choice of the "on-shell" wave function ϕ for V mesons, we record the decay coupling constant g_V for the process $V\to e^+e^-$ which is defined through[10]

$$e\frac{M_V^2}{g_V}\hat{\epsilon}_\mu=\sqrt{3}ee_Q \mathrm{Tr}\int d^4d\, \Psi_V(P,q)i\gamma_\mu , \quad (A10)$$

where $e_Q^2=(\frac{1}{2},\frac{1}{9},\frac{1}{18})$ for $V=\rho,\ \phi,\ \omega$, respectively. After necessary substitutions from Sec. II corresponding to the "on-shell" function ϕ_0, Eq. (2.11), the right-hand side works out as

$$4\sqrt{3}ee_Q\int d^4q\frac{1}{\Delta_1\Delta_2}\hat{\epsilon}_\mu[(-m_q^2+p_1\cdot p_2) -2p_1\cdot\hat{\epsilon}p_2\cdot\hat{\epsilon}]N_V^{(-)}\phi_0 . \quad (A11)$$

Further reduction is made with the help of Eqs. (A5) and (A6) and after similar integrations to those encountered for the V meson normalization, the final result is

$$\frac{M_V^2}{g_V}=4\sqrt{3}e_Q\cdot\tfrac{1}{3}(M^2+2m_q^2)N_V\frac{\beta^2}{\sqrt{\pi}}\mathrm{erf}\left[\frac{M}{\sqrt{2}\beta}\right] \quad (A12)$$

giving

$$g_\rho^2/4\pi=2.282 \quad (A13)$$

and

$$\Gamma_{\rho\to e^+e^-}=\frac{4\pi}{3}\alpha^2 M_\rho g_\rho^{-2} =6.052\ \mathrm{keV}\quad (6.9\pm 0.3) \quad (A14)$$

(Ref. 16). The corresponding results again for "on-shell" ω and ϕ mesons are

$$\Gamma_{\omega\to e^+e^-}=0.69\ \mathrm{keV}\quad (0.66\pm 0.04) ,$$

$$\Gamma_{\phi\to e^+e^-}=1.11\ \mathrm{keV}\quad (1.31\pm 0.06)$$

(Ref. 16), all in good agreement with data. This provides a sort of *a fortiori* justification for the use of "on-shell" wave functions.

2. BS norm for a_1 meson

The calculational technique and general procedure is almost identical to the V-meson case, except for the Dirac matrix $\Gamma_i=i\gamma\cdot\hat{\epsilon}\gamma_5$ appropriate to the structure of a_1 with polarization vector $\hat{\epsilon}_\mu$. This amounts to the replacement of Eq. (A3) by

$$\mathrm{Tr}_\mu(a_1a_1\gamma)=\mathrm{Tr}[i\gamma\cdot\hat{\epsilon}\gamma_5(m_q+i\gamma\cdot p_2)i\gamma_5\gamma\cdot\hat{\epsilon} \times(m_q-i\gamma\cdot p_1)i\gamma_\mu(m_q-i\gamma\cdot p_1)] , \quad (A15)$$

which now works out on identical lines to the V case, and in the same notation, as

$$4P_\mu\left[\frac{\Delta_1}{2}+\left[1-\frac{\Delta_1}{M^2}\right]\left[\frac{M^2}{3}+\frac{\Delta_1}{6}+\frac{\Delta_2}{6}-\frac{4m_q^2}{3}\right]\right] . \quad (A16)$$

After carrying out the integration over the Δ_2 pole, exactly as before, the corresponding form of Eq. (A7) is

$$(N_{a_1}^{(-)})^2=2(2\pi)^3\int d^2q_\perp dq_+\frac{1}{3}\left[D_+\left[1+\frac{4m_q^2}{M^2}\right]+2p_{2+}(M^2-4m_q^2)\right]\phi_0^2 . \quad (A17)$$

The reduced normalizer, defined through Eq. (A8), finally works out as

$$N_{a_1}^{-2}=\frac{2}{M}(\pi\beta^2)^{1/2}\left[\frac{2}{3}\left[1+\frac{4m_q^2}{M^2}\right](m_q^2+\tfrac{3}{2}\beta^2+\tfrac{1}{6}M^2)-2m_q^2\right] . \quad (A18)$$

APPENDIX B: BS NORMS FOR $L\geq 1$ STATES

The L-excited $q\bar{q}$ states belong to four broad categories, three for $S=1$ (ρ-like) and one for $S=0$ (π-like). We shall designate the ρ states as

$$T^J(J=L+1),\quad A^J(J=L),\quad S^J(J=L-1) , \quad (B1)$$

and the π-like ones as $B^J(J=L)$. Their three-dimensional orbital wave functions which are given by Eqs. (2.11) and (2.12) may be relativistically expressed through the formal replacement[20]

$$q_{i_1}\cdots q_{i_L}B^L_{i_1\cdots i_L}\Longrightarrow \hat{q}_{\mu_1}\cdots\hat{q}_{\mu_L}B^L_{\mu_L\cdots\mu_L}(P)\equiv\hat{q}^{\,L}\cdot B^L\ , \tag{B2}$$

where

$$\hat{q}_\mu=q_\mu+\frac{q\cdot P}{M^2}P_\mu\ . \tag{B3}$$

Thus the $\hat{q}_\mu$ vectors in (B2) are effectively three-dimensional in content, but expressed in four-dimensional form. In particular, the null-plane component $\hat{q}_-$ must be read as[11]

$$\hat{q}_-=-\hat{q}_+M^2/P_+^2=-\hat{q}_3M/P_+ \tag{B4}$$

in accordance with Eq. (2.5), and does *not* have an independent "minus" component status unlike, e.g., the other q_- variables which appear in the two S_F functions of Eq. (2.15), via $p_{1,2}=\frac{1}{2}P\pm q$. Equation (B2) is now already adequate for the B^J mesons of $S=0$ (π-like) for purposes of insertion in the four-dimensional BS amplitude, Eqs. (2.15) and (2.16), through the identification

$$\Gamma_i=\gamma_5(\hat{q})^L\cdot B^L\ . \tag{B5}$$

For the $S=1$ (ρ-like) meson types in (B1), we follow the broad procedure outlined in Ref. 20. The highest J state T^J may be easily identified as the leading term (coefficient unity) in the Clebsch-Gordan reduction:[20]

$$\hat{\epsilon}_\mu\otimes B^L_{\mu_1\mu_2\cdots\mu_L}=T^J_{\mu\mu_1\cdots\mu_L}+(A^J,S^J\text{ terms})\ . \tag{B6}$$

1. T^J mesons (a_2, f_2, etc.)

The factor Γ_i of (2.16) in this case corresponds to the replacement

$$i\gamma\cdot\hat{\epsilon}\Longrightarrow i\gamma_\mu\hat{q}_{\mu_1}\cdots\hat{q}_{\mu_L}T^J_{\mu\mu_1\cdots\mu_L}\equiv i\gamma_\mu\hat{\epsilon}^J_\mu\ . \tag{B7}$$

We are now in a position to work out the BS normalizers for T^J states, on lines identical to those of Appendix A, except for the replacement $\hat{\epsilon}_\mu\rightarrow\hat{\epsilon}^J_\mu$. This last has the effect of replacing Eq. (A5) by[20]

$$\langle\hat{\epsilon}^J_\mu\hat{\epsilon}^J_\nu\rangle=\frac{(\hat{q}^{\,2})^L}{2J+1}\left[a_L\theta_{\mu\nu}(P)+b_L\frac{\hat{q}_\mu\hat{q}_\nu}{\hat{q}^{\,2}}\right]\ , \tag{B8}$$

$$(a_L;b_L)=\frac{1}{2}\frac{L!}{(2L+1)!!}[L+2;L]\ , \tag{B9}$$

so that in Eq. (A4), one now finds, e.g.,

$$\langle 4p_1\cdot\hat{\epsilon}^Jp_2\cdot\hat{\epsilon}^J\rangle=\frac{(\hat{q}^{\,2})^L}{2J+1}[a_L(2\Delta_1+4m_q^2-M^2)+b_L(-2\Delta_1+4m_q^2-M^2)] \tag{B10}$$

after dropping certain Δ_i factors which will not contribute. The integrations may now be carried out exactly as in Appendix A remembering that $\hat{q}_\mu$ is effectively a three-dimensional vector, vide Eq. (B4). The final result for the reduced normalizer N_J, Eq. (A9) is

$$N_{TJ}^{-2}=\frac{2}{M}(\pi\beta^2)^{1/2}\left[\left[m_q^2+\frac{M^2}{4}+\frac{2J+1}{2}\beta^2\right]\left[\frac{J+1}{2J+1}-\frac{4Jm_q^2}{2J+1}\frac{1}{M^2}\right]+\frac{4m_q^2J}{2J+1}\right]\ . \tag{B11}$$

2. B^J mesons (π-like)

In this case, Γ_i is given by (B5), and the result is derivable either, from Ref. 11, or simply from Appendix A, with the replacement

$$i\gamma\cdot\hat{\epsilon}\rightarrow\gamma_5(\hat{q})^L\cdot B^L\ , \tag{B12}$$

so that the only additional factor to be handled, over and above the pion case,[11] is

$$\frac{2^L\beta^{-2L}}{L!}\langle[(\hat{q})^L\cdot B^L]^2\rangle=\frac{2^L}{(2L+1)!!}(q^2/\beta^2)^L\ . \tag{B13}$$

The final result (after necessary integration) for the reduced B^J normalizer (N_L):

$$N_{BJ}^{-2}=\frac{2}{M}(\pi\beta^2)^{1/2}[m_q^2+\tfrac{1}{4}M^2+(L+\tfrac{3}{2})\beta^2]\quad(J=L)\ . \tag{B14}$$

3. A^J and S^J mesons

Finally, for the sake of completeness we shall merely record the BS norms for the lower-rank L-excited mesons (ρ like) $A^J(J=L)$ and $S^J(J=L-1)$. For these cases, the Γ_i vectors are, respectively,[20]

$$\Gamma_i(A^J)=\gamma_5\sigma_{\mu_1\beta}\hat{q}_\beta\hat{q}_{\mu_2}\cdots\hat{q}_{\mu_L}A^J_{\mu_1\cdots\mu_L}\ , \tag{B15}$$

$$\Gamma_i(S^J)=i\gamma\cdot\hat{q}\hat{q}_{\mu_2}\cdots\hat{q}_{\mu_L}S^J_{\mu_2\cdots\mu_L}\ . \tag{B16}$$

The calculational procedure is identical to the other cases and the final results for the "reduced" BS normalizers, Eq. (A9), are

$$(J=L):\quad N_{AJ}^{-2}=\frac{2}{M}(\pi\beta^2)^{1/2}\frac{J+1}{J}\frac{2m_q^2}{M^2}\times[\tfrac{3}{2}M^2-2m_q^2-(2J+3)\beta^2]\ , \tag{B17}$$

$$(J=L-1):\quad N_{SJ}^{-2}=\frac{2}{M}(\pi\beta^2)^{1/2}\frac{2J+3}{J+1}\frac{2m_q^2}{M^2}\times[\tfrac{3}{2}M^2-2m_q^2-(2J+5)\beta^2]\ . \tag{B18}$$

The S^J states which have $J=0$ for their lowest-order realization ($L=1$) correspond to the "scalar" mesons (f_0,a_0, etc.) (Ref. 16), while the A^J states, may be considered as an alternative description for the a_1 meson. However, as in Ref. 10, we have preferred in this paper to consider the simpler choice $i\gamma_5\gamma\cdot\hat{\epsilon}$ for its Dirac Γ factor.

[1]See, e.g., B. D. Serot and J. D. Walecka, in *Advances in Nuclear Physics,* edited by J. W. Nigel and E. Vogt (Plenum, New York, 1986), Vol. 26.
[2]M. Shifman, A. Vainshtein, and V. Zakharov, Nucl. Phys. **B147**, 448 (1979).
[3]L. Reinders, H. Rubinstein, and S. Yazaki, Nucl. Phys. **B186**, 109 (1981).
[4]See, e.g., A. Chodos, Phys. Rev. D **9**, 3471 (1974); **10**, 2594 (1974).
[5]E. Eichten *et al.*, Phys. Rev. Lett. **34**, 369 (1975); C. Quigg and J. Rosner, Phys. Rep. **56**, 167 (1979).
[6]See, e.g., D. B. Lichtenberg, Int. J. Mod. Phys. A **2** (1987).
[7]K. G. Wilson, Phys. Rev. D **10**, 2445 (1974); Phys. Rep. **23C**, 331 (1976).
[8]E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951); M. Gell-Mann and F. E. Low, *ibid.* **84**, 350 (1951).
[9]A. N. Mitra and I. Santhanam, Z. Phys. C **8**, 23 (1981); **8**, 25 (1981); Phys. Lett. **104B**, 62 (1981).
[10]A. N. Mitra and D. S. Kulshreshtha, Phys. Rev. D **26**, 3123 (1982); **28**, 588 (1983).
[11]N. Nimai Singh, Y. K. Mathur, and A. N. Mitra, Few-Body Syst. **1**, 47 (1986).
[12]A. Mittal and A. N. Mitra, Phys. Rev. Lett. **57**, 290 (1986).
[13]D. S. Kulshrestha and A. N. Mitra, Phys. Rev. D **37**, 1268 (1988).
[14]A. N. Mitra, A. Pagnamenta, and N. N. Singh, Phys. Rev. Lett. **59**, 2408 (1987).
[15]Yu. M. Antipov *et al.*, Phys. Rev. Lett. **56**, 797 (1986).
[16]Particle Data Group, M. Aguilar-Benitez *et al.*, Phys. Lett. **170B**, 1 (1986).
[17]A. Dhaul, A. Pagnamenta, and A. N. Mitra, Z. Phys. C **36**, 115 (1987).
[18]A. N. Mitra, A. Pagnamenta, and N. N. Singh, Phys. Lett. B **196**, 212 (1987).
[19]See, e.g., A. Faessler *et al.*, Nucl. Phys. **A466**, 445 (1987).
[20]A. N. Mitra and S. Sood, Fortschr. Phys. **25**, 649 (1977).
[21]S. R. Amendolia *et al.*, Nucl. Phys. **B277**, 168 (1986); C. J. Bebek *et al.*, Phys. Rev. D **17**, 1693 (1978).
[22]J. Wess and B. Zumino, Phys. Lett. **37B**, 95 (1971); E. Witten, Nucl. Phys. **B223**, 422 (1981).
[23]M. V. Terentev, Yad. Fiz. **14**, 1199 (1972) [Sov. J. Nucl. Phys. **15**, 665 (1972)].
[24]J. Schwinger, Phys. Rev. Lett. **18**, 923 (1967).
[25]H. Harari, Phys. Rev. Lett. **22**, 262 (1969); J. Rosner, *ibid.* **22**, 689 (1969).
[26]M. Bohm *et al.*, Acta Phys. Austriaca Suppl. XI, **3** (1973); R. F. Meyer, Nucl. Phys. **B71**, 226 (1974).
[27]K. K. Gupta, A. N. Mitra, and A. Pagnamenta (unpublished).
[28]J. F. Donoghue, Phys. Rev. D **33**, 1516 (1986).
[29]I. I. Y. Bigi and M. Fukugita, Phys. Lett. **91B**, 121 (1980); see also, A. N. Kamal, Phys. Rev. D **33**, 1344 (1986).
[30]M. Shifman, Nucl. Phys. **B120**, 316 (1977).
[31]R. S. Chivukla, J. M. Flynn, and H. Georgi, Phys. Lett. B **171**, 453 (1986); W. A. Bardeen *et al.*, *ibid.* **180**, 133 (1986); M. B. Gavella *et al.*, Phys. Lett. **148B**, 225 (1984); J. F. Donoghue, Phys. Rev. D **30**, 1499 (1984).
[32]K. K. Gupta, A. N. Mitra, N. Sinha, and R. Sinha (unpublished).
[33]M. V. Terentev, Yad. Fiz. **24**, 207 (1976) [Sov. J. Nucl. Phys. **24**, 106 (1976)]; J. M. Namyslowski, Prog. Part. Nucl. Phys. **14**, 49 (1988); N. H. Fuchs and M. D. Scadron, Nuovo Cimento **A93**, 205 (1986).
[34]S. Brodsky and G. P. Lepage, in *Particle and Fields—2,* edited by A. Capri and A. N. Kamal (Plenum, New York, 1983).
[35]I. E. Tamm, J. Phys. (Moscow) **1**, 449 (1945); S. M. Dancoff, Phys. Rev. **78** 382 (1950).
[36]N. Isgur and C. H. Llewellyn Smith, Phys. Rev. Lett. **52**, 1080 (1984).
[37]N. H. Fuchs and M. D. Scadron, J. Phys. G **11**, 299 (1985).
[38]V. L. Chernyak and A. R. Zitnitsky, Phys. Rep. **112**, 173 (1984).
[39]B. L. Ioffe, Z. Phys. C **18**, 67 (1983).
[40]S. Godfrey and N. Isgur, Phys. Rev. D **32**, 189 (1985); R. Kokoski and N. Isgur, *ibid.* **35**, 907 (1987).
[41]J. B. Kogut, Rev. Mod. Phys. **55**, 775 (1983).
[42]See, e.g., M. Campostrini, K. Moriarty, and C. Rebbi, Phys. Rev. D **36**, 3450 (1987).
[43]G. Martinelli and C. T. Sachrajda, Phys. Lett. B **196**, 184 (1987).