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Stochastic derivation of conformal anomaly

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The conformal anomaly of the scalar and fermion fields in the background gravitational field is derived, in the framework of stochastic quantization, as a breaking of the naive Leibnitz formula. This breaking represents quantum effects. It is also shown that anomalous Ward-Takahashi identities are derived from a stationary property of physical observables.

Recently, we proposed a new formulation for understanding the quantum origin of anomalies in the framework of stochastic quantization.[1] This formulation has been applied to the chiral anomaly,[1,2] and it can be applied to any kind of anomaly. The basic idea of this formulation is that an anomaly comes from a breaking of the naive Leibnitz formula in stochastic processes. This breaking occurs because we cannot neglect the square or product of random quantities in the Langevin equation.[3] In this paper we apply this formulation to the conformal anomaly of the scalar and fermion fields in the background gravitational fields.

We consider the following conformal-invariant Euclidean actions of the scalar field ϕ and the fermion fields ψ, $\bar{\psi}$:

$$S^{(1)}=\int dx\sqrt{g}\left[\tfrac{1}{2}[g^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x)+\tfrac{1}{6}R\phi^2(x)]+\frac{\lambda}{4!}\phi^4(x)\right], \tag{1}$$

$$S^{(2)}=\int dx\sqrt{g}\left[\bar{\psi}(x)i\gamma^\mu(x)\left[\partial_\mu-\frac{i}{2}A_{\mu mn}(x)\sigma^{mn}\right]\psi(x)\right]. \tag{2}$$

Here $g^{\mu\nu}$ is the metric tensor, $g=\det g_{\mu\nu}$, R is the scalar curvature, $\{\gamma^\mu(x),\gamma^\nu(x)\}=2g^{\mu\nu}(x)$, $A_{\mu mn}(x)$ is the spin connection, and $\sigma^{mn}=(i/4)[\gamma^m,\gamma^n]$. The gravitational field is considered to be a background field. The actions $S^{(1)}$ and $S^{(2)}$ are invariant under conformal transformation:

$$g^{\mu\nu}(x)\rightarrow e^{2\alpha(x)}g^{\mu\nu}(x)\ , \tag{3a}$$

$$g_{\mu\nu}(x)\rightarrow e^{-2\alpha(x)}g_{\mu\nu}(x)\ , \tag{3b}$$

$$\phi(x)\rightarrow e^{\alpha(x)}\phi(x)\ , \tag{3c}$$

$$\psi(x)\rightarrow e^{3\alpha(x)/2}\psi(x)\ , \tag{3d}$$

$$\bar{\psi}(x)\rightarrow e^{3\alpha(x)/2}\bar{\psi}(x)\ , \tag{3e}$$

where $\alpha(x)$ is an arbitrary real function of x. The stress tensor $T^{(i)}_{\mu\nu}$ is defined by functional derivative of $S^{(1)}$ and $S^{(2)}$;

$$T^{(i)}_{\mu\nu}\equiv\frac{2}{\sqrt{g}}\frac{\delta S^{(i)}}{\delta g^{\mu\nu}}\quad(i=1,2)\ . \tag{4}$$

Then the conformal invariance of $S^{(1)}$ and $S^{(2)}$ gives the following identities, respectively:

$$T^{(1)\mu}{}_{\mu}=-\frac{1}{\sqrt{g}}\frac{\delta S^{(1)}}{\delta\phi}\phi\ , \tag{5}$$

$$T^{(2)\mu}{}_{\mu}=\frac{3}{2}\frac{1}{\sqrt{g}}\frac{\delta S^{(2)}}{\delta\psi}\psi-\frac{3}{2}\bar{\psi}\frac{1}{\sqrt{g}}\frac{\delta S^{(2)}}{\delta\bar{\psi}}\ , \tag{6}$$

where the spinor indices are suppressed, and the derivatives $\delta/\delta\psi,\delta/\delta\bar{\psi}$ are assumed to be left derivatives. In the classical field theory, these identities and the classical equations of motion $\delta S^{(1)}/\delta\phi=0$ and $\delta S^{(2)}/\delta\psi=\delta S^{(2)}/\delta\bar{\psi}=0$ give $T^{(i)\mu}{}_{\mu}=0$. Namely, classical stress tensors are traceless. On the other hand, it is known that quantum stress tensors diverge. The appearance of this divergence is called the conformal anomaly.[4,5] In the path-integral quantization scheme, this divergence comes from conformal noninvariance of the functional measure.[5]

Let us consider stochastic quantization of the scalar field in the background gravitational field described by the action $S^{(1)}$. We write the following Langevin equation:

$$d\phi(x,t)=-\frac{1}{\sqrt{g(x)}}\frac{\delta S^{(1)}(t)}{\delta\phi(x,t)}dt+dW(x,t)\ . \tag{7}$$

Here, t is the fictitious time variable, $S^{(1)}(t)$ is obtained by replacing the argument $\phi(x)$ with $\phi(x,t)$ in the classical action $S^{(1)}$, and $dW(x,t)$ is the Gaussian white noise satisfying the statistical properties

$$\langle dW(x,t)\rangle=0\ , \tag{8a}$$

$$\langle dW(x,t)dW(y,t)\rangle=2\hbar\frac{\delta(x-y)}{\sqrt{g(x)}}dt\ . \tag{8b}$$

Equations (7) and (8) are covariant under the general coordinate transformation, assuming that $dW(x,t)$ is a scalar variable.

Equation (8b) means that dW is of the order of $dt^{1/2}$ and so $(dW)^2$ is of the order of dt^1. This breaks the naive Leibnitz formula. In fact, for arbitrary functionals $F(t)\equiv F[\phi(\ ,t)]$ and $G(t)\equiv G[\phi(\ ,t)]$,

$$d[F(t)G(t)]=[\delta F(t)]G(t)+F(t)\delta G(t)+\tfrac{1}{2}\{[\delta^2F(t)]G(t)+2[\delta F(t)]\delta G(t)+F(t)\delta^2G(t)\}\ , \quad (9a)$$

$$\delta F(t)\equiv\int dx\left[\frac{\delta F(t)}{\delta\phi(x,t)}d\phi(x,t)\right]\ , \quad (9b)$$

$$\delta^2F(t)\equiv\int dx\int dy\left[\frac{\delta^2F(t)}{\delta\phi(x,t)\delta\phi(y,t)}d\phi(x,t)\times d\phi(y,t)\right]\ . \quad (9c)$$

Equation (9) is easily derived through the well-known prescription of the Ito calculus.[3] In particular, when $G(t)=1$, Eq. (9) can be written as

$$dF(t)=\int dx\frac{\delta F(t)}{\delta\phi(x,t)}\left[-\frac{1}{\sqrt{g(x)}}\frac{\delta S^{(1)}(t)}{\delta\phi(x,t)}dt+dW(x,t)\right]+\tfrac{1}{2}\int dx\,dy\frac{\delta^2F(t)}{\delta\phi(x,t)\delta\phi(y,t)}dW(x,t)dW(y,t)\ . \quad (9')$$

Equation (9′) is the Langevin equation for $F(t)$, and it is called the Ito formula.[3]

The Fokker-Planck distribution to give expectation values in our stochastic process is defined by

$$\langle F[\phi(\ ,t)]\rangle=\int\mathcal{D}\phi\,F[\phi]P[\phi,t]\ , \quad (10)$$

where the left-hand side means an ensemble average with respect to dW. Then, taking expectation values of both sides of Eq. (9′) and using the fact that F is arbitrary, we obtain the following Fokker-Planck equation:

$$\frac{d}{dt}P[\phi,t]=-H_{\rm FP}P[\phi,t]\ , \quad (11a)$$

$$H_{\rm FP}=-\int\frac{dx}{\sqrt{g(x)}}\left[\frac{\delta}{\delta\phi(x)}\frac{\delta S^{(1)}}{\delta\phi(x)}+\hbar\frac{\delta^2}{\delta\phi^2(x)}\right]\ . \quad (11b)$$

The positive semidefiniteness of the Fokker-Planck Hamiltonian $H_{\rm FP}$ is proved in the same way as in Ref. 6. This, with the assumption that $H_{\rm FP}$ has a nondegenerate eigenvalue zero, guarantees that $P[\phi,t]$ behaves as $\exp(-S/\hbar)$ at the $t\to\infty$ limit.

Next, we derive the conformal anomaly for the scalar field. Equation (9′) gives an easy prescription to derive it without solving Eq. (7). Classically, the identify (5) and the equation of motion give $T^{(1)\mu}{}_{\mu}=0$. In our quantized system, however, we have to use the Langevin equation (7) instead of the classical equation of motion. Then we obtain

$$T^{(1)\mu}{}_{\mu}(x,t)dt=\phi(x,t)d\phi(x,t)-\phi(x,t)dW(x,t)\ . \quad (12)$$

On the other hand, choosing $F=\phi^2$ in Eq. (9′),

$$d[\phi^2(x,t)]=2\phi(x,t)d\phi(x,t)+dW(x,t)dW(x,t)\ . \quad (13)$$

Then Eqs. (12) and (13) give

$$\langle T^{(1)\mu}{}_{\mu}(x,t)\rangle=\langle\phi(x,t)d\phi(x,t)\rangle\frac{1}{dt}=\frac{1}{2}\frac{d\langle\phi^2(x,t)\rangle}{dt}-\hbar\frac{\delta(x-x)}{\sqrt{g(x)}}\ . \quad (14)$$

In the first equality, we used a formula of the Ito calculus, $\langle\phi(x,t)dW(x,t)\rangle=0$ (see Ref. 3). In the equilibrium state at the $t\to\infty$ limit, Eq. (14) reduces to the following unregularized anomalous Ward-Takahashi identity:

$$\langle T^{(1)\mu}{}_{\mu}(x)\rangle=-\hbar\frac{\delta(x-x)}{\sqrt{g(x)}}\ . \quad (15)$$

The above argument shows that the anomalous Ward-Takahashi identity (15) can be derived by rewriting the stationary property of ϕ^2 at $t\to\infty$, that is, $d\langle\phi^2(d,t)\rangle/dt=0$. In this rewriting we use Eqs. (12) and (13). This is comparable to the fact that the anomalous Ward-Takahashi identity for the chiral U(1) anomaly can be derived from the stationary property of the pseudoscalar density $\bar\psi\gamma_5\psi$ (see Ref. 1).

Note that the anomaly term $-\hbar\delta(x-x)/\sqrt{g(x)}$ comes from the last term of Eq. (9′) which includes the square of the random noise $(dW)^2$. This term breaks the naive Leibnitz formula, and this breaking causes the anomaly.

The regularization of the anomaly term can be performed in the usual way.[4] As a result, when the coupling constant λ is equal to zero, we obtain the following regularized form of the anomalous Ward-Takahashi identity:

$$\langle T^{(1)\mu}{}_{\mu}(x)\rangle=\frac{-\hbar}{(4\pi)^2}\lim_{s\to\infty}\frac{1}{s^2}+\frac{\hbar}{2880\pi^2}(\partial_\mu\partial^\mu R+R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-R_{\mu\nu}R^{\mu\nu})\ . \quad (16)$$

The conformal anomaly for the fermion fields ψ, $\bar\psi$ described by the action $S^{(2)}$ can be derived in a similar way. The Langevin equations are[1,7]

$$d\psi^a(x,t)=-\frac{1}{\sqrt{g(x)}}\frac{\delta S^{(2)}(t)}{\delta\bar\psi^a(x,t)}dt+d\theta^a(x,t)\ ,$$
$$d\bar\psi^a(x,t)=\frac{1}{\sqrt{g(x)}}\frac{\delta S^{(2)}(t)}{\delta\psi^a(x,t)}dt+d\bar\theta^a(x,t)\ , \quad (17)$$

where $d\theta$ and $d\bar\theta$ are Grassmann Gaussian white noises satisfying

$$\langle d\theta^a\rangle=\langle d\bar\theta^a\rangle=0\ , \quad (18a)$$

$$\langle d\theta^a d\theta^b\rangle=\langle d\bar\theta^a d\bar\theta^b\rangle=0\ , \quad (18b)$$

$$\langle d\theta^a(x,t)d\bar\theta^b(y,t)\rangle=-\langle d\bar\theta^b(y,t)d\theta^a(x,t)\rangle=2\hbar\delta^{ab}\frac{\delta(x-y)}{\sqrt{g(x)}}dt\ , \quad (18c)$$

with a,b spinor indices. The Langevin equation for an arbitrary functional of ψ and $\bar\psi$, $f(t)\equiv f[\psi(\ ,t),\bar\psi(\ ,t)]$ is

$$df(t)=\int dx\left[-\frac{1}{\sqrt{g(x)}}\frac{\delta S^{(2)}(t)}{\delta\bar{\psi}^a(x,t)}dt+d\theta^a(x,t)\right]\frac{\delta f(t)}{\delta\psi^a(x,t)}+\int dx\left[\frac{1}{\sqrt{g(x)}}\frac{\delta S^{(2)}(t)}{\delta\psi^a(x,t)}dt+d\bar{\theta}^a(x,t)\right]\frac{\delta f(t)}{\delta\bar{\psi}^a(x,t)}$$
$$+\int dx\,dy\left[d\theta^a(x,t)d\bar{\theta}^b(y,t)\frac{\delta^2 f(t)}{\delta\bar{\psi}^b(y,t)\delta\psi^a(x,t)}\right]+\tfrac{1}{2}\int dx\,dy\left[d\theta^a(x,t)d\theta^b(y,t)\frac{\delta^2 f(t)}{\delta\psi^b(y,t)\delta\psi^a(x,t)}\right]$$
$$+\tfrac{1}{2}\int dx\,dy\left[d\bar{\theta}^a(x,t)d\bar{\theta}^b(y,t)\frac{\delta^2 f(t)}{\delta\bar{\psi}^b(y,t)\delta\bar{\psi}^a(x,t)}\right], \tag{19}$$

corresponding to Eq. (9′). Applying Eq. (19) to the case $f=\bar{\psi}\psi$, and performing the similar calculation as before, we obtain

$$\langle T^{(2)\mu}{}_{\mu}(x,t)\rangle=\frac{3}{2}\frac{d\langle\bar{\psi}(x,t)\psi(x,t)\rangle}{dt}+3\hbar\delta^{ab}\delta^{ab}\frac{\delta(x-x)}{\sqrt{g(x)}}\ . \tag{20}$$

By the stationary property of the scalar density $\bar{\psi}\psi$, Eq. (20) reduces to

$$\langle T^{(2)\mu}{}_{\mu}(x)\rangle=3\hbar\delta^{ab}\delta^{ab}\frac{\delta(x-x)}{\sqrt{g(x)}}$$
$$=\frac{3\hbar}{(4\pi)^2}\left[\lim_{s\to\infty}\left[\frac{4}{s^2}-\frac{R}{s}\right]+\left(\tfrac{1}{30}\partial_\mu\partial^\mu R+\tfrac{1}{72}R^2-\tfrac{1}{45}R_{\mu\nu}R^{\mu\nu}-\tfrac{7}{360}R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}\right)\right], \tag{21}$$

in the equilibrium state at the $t\to\infty$ limit, corresponding to Eqs. (15) and (16). The Ward-Takahashi identity (21) exhibits the appearance of the conformal anomaly.[4,5]

We have derived the conformal anomaly for the scalar and fermion fields in the background gravitational fields. And we have shown (i) the conformal anomaly appears by applying the Langevin equations to the identities which reflect conformal invariance of the action functionals, (ii) the origin of the anomaly is the breaking of the naive Leibnitz formula in stochastic processes, and (iii) the anomalous Ward-Takahashi identities are obtained from the stationary property of ϕ^2 or $\bar{\psi}\psi$.

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[1]M. Namiki, I. Ohba, S. Tanaka, and D. M. Yanga, Phys. Lett. B **194**, 530 (1987).

[2]M. Reuter, Phys. Rev. D **37**, 567 (1988).

[3]M. Namiki, I. Ohba, and S. Tanaka, Phys. Lett. B **182**, 66 (1986); S. Caracciolo, H.-C. Ren, and Y.-S. Wu, Nucl. Phys. **B260**, 381 (1985). See also textbooks about the Ito calculus, e.g., N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes* (North-Holland mathematical library) (North-Holland, Amsterdam, 1981).

[4]L. Parker, in *Recent Developments in Gravitation*, edited by S. Deser and M. Levy (Plenum, New York, 1979); A. Z. Petrov, *Einstein Spaces* (Pergamon, Oxford, 1969); N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).

[5]K. Fujikawa, Phys. Rev. Lett. **44**, 1733 (1980).

[6]M. Namiki and Y. Yamanaka, Prog. Theor. Phys. **69**, 1764 (1983).

[7]T. Fukai, H. Nakazato, I. Ohba, K. Okano, and Y. Yamanaka, Progr. Theor Phys. **69**, 1600 (1983).