Self-dual fields and the Thirring model

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A recently proposed Lagrangian for self-dual fields is used to analyze the Thirring model as a superposition of right- and left-moving fields. We show that the systematic construction of the Thirring field, based on the current-algebra approach, follows from this Lagrangian and Dirac's theory for constrained systems.

The quantization of a self-dual field ($\dot{\chi}=\chi'$) was recently examined by Floreanini and Jackiw,[1] who interpreted it as a charge-density soliton. They proposed a singular (nonlocal) Lagrangian from which all the properties of the χ field can be derived. The whole approach can be thought of as a Lagrangian counterpart of a Hamiltonian program of formulating fermion fields in terms of currents, previously proposed by Sugawara.[2] In this paper we will make this point clearer by applying the aforementioned construction to the Thirring model. We will also clarify other points such as the relation between the "bosonization rules" and the "Mandelstam representation" of the fermion fields. In the past, the discovery of the Lagrangian of the nonlinear σ models and the explicit construction of the energy-momentum tensor for free fermion theories, realizing Sugawara's ideas, were some of the achievements of this program.[3]

Consider two boson fields j_+ and j_- whose dynamics are defined by the Lagrangian (the common time argument of all fields has been suppressed)

$$\mathcal{L}=\frac{1}{4}\int dx\, dy[j_+(x)\epsilon(x-y)\partial_t j_+(y) - j_-(x)\epsilon(x-y)\partial_t j_-(y)] - \frac{1}{2}\int dx[j_+^2(x)+j_-^2(x)] . \tag{1}$$

The singular nature of this Lagrangian follows from the expressions for the canonical conjugate momenta:

$$\pi_+(x)=\frac{1}{4}\int dy\, j_+(y)\epsilon(y-x) ,\qquad \pi_-(x)=-\frac{1}{4}\int dy\, j_-(y)\epsilon(y-x) , \tag{2}$$

which are constraints, since they do not contain the velocities. They are second-class constraints, so that the dynamics in the Hamiltonian formulation is best described by means of Dirac's procedure[4] (for details see the works of Refs. 5 and 6).

The canonical Hamiltonian and momentum corresponding to the Lagrangian (1) are

$$H=\frac{1}{2}\int dx\,(j_+^2+j_-^2) ,\quad P=\frac{1}{2}\int dx\,(j_-^2-j_+^2) , \tag{3}$$

and together with the Lorentz generator $\int dx\,[x\mathcal{H}(x)-t\mathcal{P}(x)]$ form the Poincaré algebra.

Using (2), Dirac's brackets can be computed in a straightforward way. In particular, we have

$$\{j_\pm(x),j_\pm(0)\}_D=\pm\delta'(x) ,\quad \{\pi_\pm(x),\pi_\pm(0)\}_D=\mp\frac{1}{8}\epsilon(x) ,\quad \{\pi_+(x),\pi_-(0)\}_D=0 ,\quad \{j_+(x),j_-(0)\}_D=0 . \tag{4}$$

The Euler-Lagrange equations for $j_\pm$ are

$$2\dot{\pi}_\pm(x)=-j_\pm(x) . \tag{5}$$

Thus, using the identity

$$2\pi'_\pm(x)=\mp j_\pm(x) \tag{6}$$

which follows from (2), we can rewrite (5) as ($u=x+t$, $v=t-x$)

$$\partial_u j_-=0 ,\quad \partial_u\pi_-=0 ,\quad \partial_v j_+=0 ,\quad \partial_v\pi_+=0 . \tag{7}$$

Equations (3), (4), and (6) are the classical form of the assumptions contained in the current-algebra approach of Ref. 6. Indeed, canonical quantization is consistently done by replacing Dirac's brackets by commutators and, taking into account (7), introducing a Wick ordering in the expressions (3) for H and P. It is remarkable that all this structure, from which the Thirring field can be systematically constructed, is already embodied in the Lagrangian (1) and Dirac's recipe to quantize constrained systems. The Thirring field is a charge-creating field ψ satisfying the Dirac's brackets[6]

$$\{j_+^{\pm}(x),\psi(x')\}_D=i(a+\bar{a}\gamma_5)\psi(x')\delta^{(\pm)}(x-x') ,\quad \{j_-^{\pm}(x),\psi(x')\}_D=i(a-\bar{a}\gamma_5)\psi(x')\delta^{(\pm)}(x-x') , \tag{8}$$

and having its time evolution governed by (3). Looking at the equations of motion which follow (3) and (8),

$$\partial_u\psi=-i(a+\bar{a}\gamma_5)j_+(u)\psi(u,v) ,\quad \partial_v\psi=-i(a-\bar{a}\gamma_5)j_-(v)\psi(u,v) , \tag{9}$$

we see that two of them coincide with the ones usually written for the Thirring model. The other two correspond, in the quantized theory, to additional information which must be given to construct the composite field

$j \approx \psi^\dagger \psi$ by the point-splitting method.[7] In fact, as we have shown in Ref. 8, these two extra equations are, at both classical and quantum level, just the integration conditions of the problem.

By using (6), a solution to (9) can be written as

$$\psi = \exp[2i(a+\bar{a}\gamma_5)\pi_+(u) + 2i(a-\bar{a}\gamma_5)\pi_-(v)] \begin{bmatrix} 1 \\ 1 \end{bmatrix} . \tag{10}$$

After quantization, the renormalized (i.e., Wick-ordered) form of this relation will have both a dimension and a Lorentz spin. It is the well-known bosonization rule.[9]

Another useful representation for ψ is obtained by noting that, due to (7), it is possible to write $j_\mu = \partial_\mu \phi / \sqrt{2}$ with ϕ satisfying $\partial^2 \phi = 0$. Therefore, replacing this on (2) and (9), we get

$$\psi = \exp \left[-ia\phi(x,t) + i\bar{a}\gamma_5 \int dy \frac{\epsilon(x-y)}{2} \dot{\phi}(y) \right] \begin{bmatrix} 1 \\ 1 \end{bmatrix} . \tag{11}$$

The quantum version of this equation, which differs from (11) by just a Wick-ordering prescription to treat the exponentiated field, is the well-known Mandelstam bosonization form of the Thirring model.

The equivalence of the Thirring model to a free field theory can be made still more transparent if we make a change of variables

$$j_\pm = \frac{p \pm \phi'}{\sqrt{2}} . \tag{12}$$

In terms of these new variables the Hamiltonian and Dirac's brackets become

$$H = \frac{1}{2} \int dx (p^2 + \phi'^2) \tag{13}$$

and

$$\begin{aligned} \{\phi(x), \phi(0)\}_D &= 0 , \\ \{\phi(x), \phi(0)\}_D &= i\delta(x) , \\ \{p(x), p(0)\}_D &= 0 , \end{aligned} \tag{14}$$

which is precisely the Hamiltonian description of an unconstrained free field. At this point the Dirac brackets can, of course, be replaced by ordinary Poisson brackets. However, we must stress that the fermionic character (the Lorentz spin) and a nonzero scaling dimension of the Thirring field are purely quantum effects. They come as a consequence of the multiplicative renormalization corresponding to the Wick-ordering prescription, needed to well define the composite field ψ.

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