Four-dimensional parafermionic string

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It is suggested how to formulate a consistent string theory in four spacetime dimensions, based on parafermionic fields. A formula for the cosmological constant is given, together with some plausibility arguments for the existence of a corresponding conformal field theory and string.

There has recently been considerable interest in four-dimensional superstrings[1] which cannot be regarded as compactifications of ten-dimensional superstrings. For example, asymmetric orbifolds where the left movers and right movers of a closed string are compactified differently are of this type. Nevertheless, the underlying theory still possesses all the degrees of freedom which characterize a ten-dimensional superstring.

It is natural to ask whether there can exist a more truly four-dimensional string. Here we offer some comments and speculation on the possible properties of such a string which would possess a smaller number of degrees of freedom than any of those described in Ref. 1.

The guiding principles for a consistent (closed) string are those of Lorentz invariance, conformal invariance, and modular invariance. Although our arguments are incomplete, we shall attempt to satisfy all these necessary (and possible sufficient?) criteria. As we shall see, the construction will be related to the paraquantization and parafermions which have been discussed elsewhere.[2] Our remarks will show how to obtain the correct number of zero modes in such parafermionic theories.

Since the progenitor[3] of all strings seems to be the bosonic string we begin by considering the nonintegrally moded bosonic operators of Ref. 4. Taking $0 \leq \eta < 1$ and N_η oscillators with mode numbers equal to integers plus η one finds that, for absence of a Lorentz anomaly and presence of a massless ground state (we shall throughout count only the transverse modes in light-cone gauge),

$$\sum N_\eta = 24 , \tag{1a}$$

$$\sum N_\eta (6\eta^2 - 6\eta + 1) = 0 . \tag{1b}$$

The solutions of most interest are

$$N_0 = 24 , \tag{2a}$$

$$N_0 = 8 , \quad N_{1/2} = 16 , \tag{2b}$$

$$N_0 = N_{1/2} = 4 , \quad N_{1/4} = N_{3/4} = 8 , \tag{2c}$$

$$N_0 = N_{1/4} = N_{1/2} = N_{3/4} = 2 , \quad N_{1/8} = N_{3/8} = N_{5/8} = N_{7/8} = 4 . \tag{2d}$$

Since the number (N_0) of zero modes should equal the number of transverse dimensions, it is natural to identify $d = 26$, 10, 6, and 4 with (2a), (2b), (2c), and (2d), respectively. Bearing in mind the formula[2]

$$(d - 2) = 8p^{-1} \tag{3}$$

for parafermionic strings we may attempt to identify (2b), (2c), and (2d) with parafermions of order $p = 1$, 2, and 4, respectively.

If we compute the partition function for the nonintegrally moded bosonic oscillators we find straightforwardly the results

$$f_B(q)^{24} , \tag{4a}$$

$$f_B(q^{1/2})^8 f_F(q^{1/2})^8 , \tag{4b}$$

$$f_B(q^{1/4})^4 f_F(q^{1/4})^4 , \tag{4c}$$

$$f_B(q^{1/8})^2 f_F(q^{1/8})^2 , \tag{4d}$$

where

$$f_B(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-1} ,$$

$$f_F(q) = \prod_{n=1}^{\infty} (1 + q^n) , \quad q = e^{2\pi i \tau}$$

and where we have used the fact that

$$\prod_{n=1}^{\infty} (1 - q^{2n-1})^{-1} = \prod_{n=1}^{\infty} (1 + q^n) . \tag{5}$$

As pointed out in Ref. 4, the partition function (4b) is a familiar quantity: it is, within an overall factor, similar to the partition function of the Ramond sector of the ten-dimensional superstring.

In the case of the $d = 10$ closed type-II superstring the complete modular-invariant partition function is as follows. We note that

$$f_B(q)^2 f_F(q)^2 = \frac{\theta_2(0 \mid \tau)}{2\eta(\tau)^3} \tag{6}$$

and then modular invariance dictates that we must combine to form [from 4(b) and the well-known properties of $\theta_i(0 \mid \tau)$ $(i = 2,3,4)$ under SL(2,Z)], for the cosmological constant formula

$$\Lambda \sim \int \frac{d^2\tau}{(\mathrm{Im}\tau)^2} \frac{1}{(\mathrm{Im}\tau)^4} \left| \frac{\theta_2^4(0\,|\,\tau)-\theta_3^4(0\,|\,\tau)+\theta_4^4(0\,|\,\tau)}{\eta(\tau)^{12}} \right|^2 \quad (7)$$

(which happens to vanish), introducing the Neveu-Schwarz sectors in the θ_3 and θ_4 pieces in Eq. (7).

What if we begin with the partition function (4d) and naively parallel the step from (4b) to Eq. (7)? A modular invariant choice for Λ_4 is then

$$\Lambda_4 \sim \int \frac{d^2\tau}{(\mathrm{Im}\tau)^2} \frac{1}{\mathrm{Im}\tau} \times \frac{|\theta_2(0\,|\,\tau)|^2+|\theta_3(0\,|\,\tau)|^2+|\theta_4(0\,|\,\tau)|^2}{|\eta(\tau)|^6} \quad (8)$$

which does not vanish. This choice is almost unique; the only nonuniqueness is that in the numerator $|\theta_2|^2+|\theta_3|^2+|\theta_4|^2$ by some linear combination of itself with $(|\theta_2|+|\theta_3|+|\theta_4|)^2$. We may thus surmise that Λ_4 of Eq. (8) is the cosmological constant formula of some $d=4$ parafermionic string. This is our principal claim to which we now wish to add some supporting evidence.

Consider first the $d=10$ case with ordinary (non-para)fermions. Of the 24 modes ($N_0=8$, $N_{1/2}=16$) one clearly employs all but $N_{1/2}=8$ to reconstruct the first factor in (4b). For the remaining four complex $\eta=\frac{1}{2}$ modes one may expand each according to

$$X(Z)=i\sum_{n=1}^{\infty}\left[\frac{1}{n-\frac{1}{2}}\alpha_{n-1/2}Z^{-(n-1/2)} - \frac{1}{n-1/2}\beta^{\dagger}_{n-1/2}Z^{n-1/2}\right], \quad (9)$$

where we take X as complex and usual commutation relations for α and β. If we fermionize by

$$\Psi(Z)=\frac{1}{\sqrt{2}}:\exp[i\,\mathrm{Re}X(Z^2)]: \quad (10)$$

with $\mathrm{Re}X(Z)=[X(Z)+X^{\dagger}(Z)]/\sqrt{2}$ and expand

$$\Psi(Z)=\sum_{n=1}(b_nZ^{-n}+d_n^{\dagger}Z^n)+b_0 \quad (11)$$

then one can check that

$$\{b_m,b_n^{\dagger}\}=\delta_{mn}\ , \quad (12)$$

$$\{\Psi(Z),\Psi^{\dagger}(W)\}_{\tau_z=\tau_w}=2\pi\delta(\sigma_z-\sigma_w)\ . \quad (13)$$

Note that we must use Z^2 rather than Z on the right-hand side of Eq. (10). Alternatively if we start with Z, we must use $Z^{1/2}$ in the fermion, and hence use $q^{1/2}$ rather than q in the partition function.

In Eq. (10), when we add the Lorentz index on X and Ψ, it would appear at first sight that the fermions commute for different values of the index; however, this can be corrected[5] by appropriately weighting the exponentials of different linear combinations of the X's. This can be similarly achieved in the other examples given below. The point is that, as far as the two-dimensional world sheet is concerned, the Lorentz index may be regarded as an internal coordinate.

For the $d=6$ case, the fermionization becomes parafermionization (of order $p=2$), and is hence more interesting. For the quarter-integer modes $N_{1/4}=N_{3/4}=4$ one defines

$$X(Z)=i\sum_{n=1}\left[\frac{1}{n-\frac{1}{4}}\alpha_{n-1/4}Z^{-(n-1/4)} - \frac{1}{n-\frac{3}{4}}\beta^{\dagger}_{n-3/4}Z^{n-3/4}\right] + i\sum_{n=1}\left[\frac{1}{n-\frac{3}{4}}\gamma_{n-3/4}Z^{-(n-3/4)} - \frac{1}{n-\frac{1}{4}}\delta^{\dagger}_{n-1/4}Z^{n-1/4}\right]. \quad (14)$$

From this, we now construct

$$b_{2n}^{(1)}=B_{2n}(-1)^{N_1+N_2}\ , \quad (15a)$$

$$b_{2n+1}^{(2)}=B_{2n+1}(-1)^{N_2}\ , \quad (15b)$$

in which

$$B_{2n}=\int\frac{dZ}{2\pi i}\frac{1}{\sqrt{2}}:\exp[i\,\mathrm{Re}X(Z^4)]:Z^{2n-1}\ , \quad (16a)$$

$$B_{2n+1}=\int\frac{dZ}{2\pi i}\frac{1}{\sqrt{2}}:\exp[i\,\mathrm{Re}X(Z^4)]:Z^{2n}\ , \quad (16b)$$

and

$$N_i=\sum_{n=1}^{\infty}B^{\dagger}_{2n+1-i}B_{2n+1-i}$$

in the Klein operators[6] of Eq. (15). The parafermions are then

$$\Psi^{(1)}(Z)=\sqrt{2}\sum_{n=1}(b_{2n}^{(1)}Z^{-2n}+d_{2n}^{+(1)}Z^{2n})+b_0\ , \quad (17a)$$

$$\Psi^{(2)}(Z)=\sqrt{2}\sum_{n=1}(b_{2n-1}^{(2)}Z^{-(2n-1)}+d_{2n-1}^{+(2)}Z^{2n-1})\ . \quad (17b)$$

The operators in Eq. (15) satisfy the algebra

$$[b^{(\alpha)}_{2n+1-\alpha},b^{(\beta)}_{2m+1-\beta}]=0\quad(\alpha\neq\beta)\ , \quad (18a)$$

$$\{b^{(\alpha)}_{2n+1-\alpha},b^{(\alpha)\dagger}_{2m+1-\alpha}\}=\delta_{m,n}\ , \quad (18b)$$

$$[d^{(\alpha)}_{2n+1-\alpha},d^{(\beta)}_{2m+1-\alpha}]=0\quad(\alpha\neq\beta)\ , \quad (18c)$$

$$\{d^{(\alpha)}_{2n+1-\alpha},d^{(\alpha)\dagger}_{2m+1-\alpha}\}=\delta_{m,n}\ , \quad (18d)$$

$$[d^{(\alpha)}_{2n+1-\alpha},b^{(\beta)}_{2m+1-\alpha}]=0\ ,\quad \text{etc.}\ , \quad (18e)$$

and similarly

$$\{\Psi^{(\alpha)}(Z),\Psi^{(\alpha)\dagger}(W)\}_{\tau_z=\tau_w}=2\pi\delta(\sigma_z-\sigma_w)\ , \quad (19a)$$

$$[\Psi^{(\alpha)}(Z),\Psi^{(\beta)\dagger}(W)]=0\quad(\alpha\neq\beta)\ . \quad (19b)$$

Note that the parafermions can only be of order $p=2$ in $d=6$. Again the Z^4 in Eq. (14), correlates with the appearance of $q^{1/4}$ in Eq. (4c).

The most interesting case is obviously $d=4$ where one expands the transverse complex coordinate ($N_{1/8}=N_{3/8}=N_{5/8}=N_{7/8}=2$) as

$$\begin{aligned}X(Z)=&i\sum_{n=1}^{\infty}\left[\frac{1}{n-\frac{1}{8}}\alpha_{n-1/8}Z^{-(n-1/8)}-\frac{1}{n-\frac{7}{8}}\beta^{\dagger}_{n-7/8}Z^{(n-7/8)}\right]\\&+i\sum_{n=1}^{\infty}\left[\frac{1}{n-\frac{7}{8}}\gamma_{n-7/8}Z^{-(n-7/8)}-\frac{1}{n-\frac{1}{8}}\delta^{\dagger}_{n-1/8}Z^{(n-1/8)}\right]\\&+i\sum_{n=1}^{\infty}\left[\frac{1}{n-\frac{3}{8}}A_{n-3/8}Z^{-(n-3/8)}-\frac{1}{n-\frac{5}{8}}B^{\dagger}_{n-5/8}Z^{(n-5/8)}\right]\\&+i\sum_{n=1}^{\infty}\left[\frac{1}{n-\frac{5}{8}}C_{n-5/8}Z^{-(n-5/8)}-\frac{1}{n-\frac{3}{8}}D^{\dagger}_{n-3/8}Z^{(n-3/8)}\right].\end{aligned} \tag{20}$$

We may then construct parafermions of order 4 as ($\alpha=1-4$)

$$\Psi^{(\alpha)}(Z)=2\sum_{n=1}^{\infty}(b^{(\alpha)}_{4n-\alpha+1}Z^{-(4n-\alpha+1)}+d^{(\alpha)\dagger}_{4n-\alpha+1}Z^{4n-\alpha+1})+b_0\delta_{\alpha,1} \tag{21}$$

in which

$$b^{(\alpha)}_{4n+1-\alpha}=B_{4n+1-\alpha}(-1)^{\sum_{i=\alpha}^{4}\sum_{n=1}^{\infty}B^{\dagger}_{4n+1-i}B_{4n+1-i}} \tag{22}$$

and

$$B_{4n+1-\alpha}=\int\frac{dZ}{2\pi i}\frac{1}{\sqrt{2}}:\exp\left[i\frac{1}{\sqrt{2}}\mathrm{Re}X(Z^8)\right]:Z^{4n-\alpha}. \tag{23}$$

One easily checks from Eq. (22) and the corresponding correlation function that

$$\{b^{(\alpha)}_{4n+1-\alpha},b^{\dagger(\alpha)}_{4m+1-\alpha}\}=\delta_{n,m}, \tag{24a}$$

$$[b^{(\alpha)}_{4n+1-\alpha},b^{(\beta)\dagger}_{4m+1-\beta}]=0\quad(\alpha\neq\beta), \tag{24b}$$

$$\{\psi^{(\alpha)}(Z),\psi^{\dagger(\alpha)}(W)\}_{\tau_z=\tau_w}=2\pi\delta(\sigma_z-\sigma_w), \tag{25a}$$

$$[\psi^{(\alpha)}(Z),\psi^{(\beta)}(W)]=0\quad(\alpha\neq\beta). \tag{25b}$$

Indeed, the degrees of freedom in our four-dimensional model may thus be regarded as parafermions of order 4, because of the above algebra. In the partition functions of Eqs. (4) they appear as in the normal (nonpara)fermionic case, because the different commuting parafermions satisfy different boundary conditions on the worldsheet.

Although the fields we have discussed pertain only to the first of the three terms in Eq. (8), the fields corresponding to the second and third terms can be deduced completely from the modular transformations which permute the three terms; this will be discussed further elsewhere.

To show that a four-dimensional string of the type we are discussing is equally as consistent as the well-known superstrings will probably require construction of the two-dimensional field theory. Since we have taken account of the principal constraints for consistency, such as the conformal invariance (and even modular invariance), there is a good chance that such a conformal field theory (and the corresponding string) exists and is consistent.

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