New axionic instantons in quantum gravity

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New Euclidean solutions, describing the nucleation of a baby universe, are presented for an axion (described by an antisymmetric tensor field strength of rank $D-1$ in D dimensions) coupled to gravity including a cosmological constant.

It has long been speculated that the geometry of space-time should fluctuate at distances of the Planck scale.[1] This possibility that quantum gravity allow fluctuations in topology has recently been receiving serious consideration by many authors.[2–8] The work of Giddings and Strominger[5] is based on a definite axionic instanton. The four-dimensional Euclidean solution involves an axion (given as a three-form field strength) coupled to Einstein gravity, and describes the nucleation of a baby universe, which carries off a definite axion charge. In the present paper, we briefly report two extensions of this solution. The first is the inclusion of a cosmological constant. The resulting instanton may be used to describe tunneling between a large de Sitter space and a Planckian Robertson-Walker universe. The second extension is finding analogous solutions for space-time dimensions other than four. This requires replacing the rank-three antisymmetric tensor field strength by one of rank $D-1$ in D dimensions.[9] Such higher-rank tensor fields arise in higher-dimensional supergravity theories. It is hoped that the new solutions may prove useful in future studies involving baby universes.

Consider the Euclideanized action

$$S=-\frac{1}{16\pi G}\int_M d^Dx\sqrt{g}\,(R-2\Lambda)-\frac{1}{8\pi G}\int_{\partial M} d^{D-1}x\sqrt{h}\,(K-K^0)+\int_M d^Dx\sqrt{g}\,A^2 \qquad (1)$$

and the resulting equations of motion

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}=16\pi G[(D-1)A^2_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}A^2]\,,\qquad d^*A=0\,. \qquad (2)$$

Here G is Newton's constant with dimensions $(\text{length})^{D-2}$, and Λ is the cosmological constant. In the gravitational surface term, h_{ab} is the boundary metric and $K=h^{ab}K_{ab}$ is the trace of the second fundamental form. K^0 is the same for the boundary imbedded in flat space. This K^0 term is included in the case of an asymptotically flat space to render the surface contribution finite.[10,11] The boundary integral cancels surface terms arising in extremizing the gravitational action,[12,13] and is required for unitarity.[10,11] For the instantons considered below, the surface terms will vanish. The axion is described by a $(D-1)$-form $A=dB$ so that $dA=0$. In Eqs. (1) and (2), $A^2\equiv A_{\mu\nu\alpha\cdots}A^{\mu\nu\alpha\cdots}$, $A^2_{\mu\nu}\equiv A_{\mu\alpha\beta\cdots}A_\nu{}^{\alpha\beta\cdots}$, and an asterisk denotes the Hodge dual. The antisymmetric tensor field may be related to the more conventional pseudoscalar axion field a as follows. Define a conserved current $j={}^*A$. Since j is a closed one-form, $j=da$ at least locally. As noted in Ref. 5, this equivalence is only an on-shell relation, and in the following it is essential that in continuing to Euclidean space we use the tensor representation, and not the pseudoscalar. For simplicity, we have absorbed the Peccei-Quinn scale into A (Ref. 5).

We now construct an explicit solution to Eq. (2) as follows. Consider a spherically symmetric ansatz for the metric and the antisymmetric tensor:

$$ds^2=g^{-2}(r)dr^2+r^2d\Omega^2_{D-1},\quad A=f(r)\epsilon\,, \qquad (3)$$

where $d\Omega^2_{D-1}$ is a line element on the unit $(D-1)$-sphere, and ϵ is the corresponding volume form so that integrating over a constant r surface yields $\int\epsilon=A_{D-1}$. Here $A_{D-1}=2\pi^{D/2}/\Gamma(D/2)$ is the area of the unit $(D-1)$-sphere. Setting

$$f(r)=m \qquad (4)$$

where m is a constant, one finds that A is both closed and divergenceless (i.e., $dA=d^*A=0$) as desired. Defining $\chi\equiv 1-g^2$, $\hat\Lambda\equiv 2\Lambda/(D-1)(D-2)$, and $\hat m^2\equiv 16\pi G m^2(D-3)!$, the two nontrivial components of Einstein's equations may be written

$$(rr):\ \frac{\chi}{r^2}-\hat\Lambda=\frac{\hat m^2}{r^{2D-2}}\,,$$

$$(\theta\theta)=(\phi\phi)=\cdots:\ \frac{\chi'}{r}+(D-3)\frac{\chi}{r^2}-(D-1)\hat\Lambda=-(D-1)\frac{\hat m^2}{r^{2D-2}}\,,$$

where $\chi'\equiv\partial\chi/\partial r$. The (rr) component is simply an algebraic equation which yields

$$\chi=\hat\Lambda r^2+\frac{\hat m^2}{r^{2D-4}} \qquad (5)$$

or, in terms of the original metric function,

$$g^2 = 1 - \hat{\Lambda} r^2 - \frac{\hat{m}^2}{r^{2D-4}} , \tag{6}$$

where $D \geq 3$. The remaining equation is automatically satisfied as a result of the Bianchi identities, as may be checked in a straightforward manner. The solution given by Eqs. (3), (4), and (6) constitutes our main result.

Let us examine the solution in the limit of vanishing cosmological constant. In this case, Eq. (6) reduces to $g^2 = 1 - \hat{m}^2 / r^{2D-4}$. There is an apparent singularity at $r^{2D-4} = \hat{m}^2 \equiv r_{\min}^{2D-4}$. In fact, this is simply a coordinate singularity, as is revealed by examining the nonvanishing components of the Riemann tensor in an orthonormal frame:

$$R_{i\hat{r}j\hat{r}} = \frac{1}{2r} \delta_{ij} \mathcal{X}', \quad R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \frac{\mathcal{X}}{r^2} . \tag{7}$$

Above i, j, k, and l indicate frame indices corresponding to directions on the $(D-1)$-sphere, while $\hat{r}$ indicates that parallel to dr. Now as indicated in Eq. (5) with $\Lambda = 0$, $\mathcal{X} = 1$ and $\mathcal{X}' = -(2D-4)/r_{\min}$ at $r = r_{\min}$. Therefore there is no curvature singularity at this point.

A useful coordinate transformation, which provides a regular extension beyond $r_{\min}$, is

$$r^{D-2} = \frac{|\hat{m}|}{2} \left[\left[\frac{u}{\beta} \right]^{D-2} + \left[\frac{\beta}{u} \right]^{D-2} \right] ,$$

where β is some arbitrary scale. For this choice of coordinates, the line element is

$$ds^2 = \left\{ \frac{|\hat{m}|}{2} \left[\left[\frac{u}{\beta} \right]^{D-2} + \left[\frac{\beta}{u} \right]^{D-2} \right] \right\}^{2/(D-2)} \times \left[\frac{du^2}{u^2} + d\Omega_{D-1}^2 \right] .$$

Note that this form of the metric is invariant under the transformation $\tilde{u}/\beta = \beta/u$, since $d\tilde{u}/\tilde{u} = -du/u$. Hence for large u, $u \propto r$, and the instanton is asymptotically flat. As r approaches $r_{\min}$ or u approaches β, the instanton has a neck of minimum radius as shown in Fig. 1. Then in the range $u = \beta$ to 0, or $\tilde{u} = \beta$ to ∞, the geometry is repeated and the instanton expands out to a second asymptotically flat region. At the neck, the minimum proper volume is $A_{D-1} r_{\min}^{D-1} = A_{D-1} |\hat{m}|^{(D-1)/(D-2)}$. The noncontractible $(D-1)$-spheres all have an axion flux $\int_{S^{D-1}} A = m$ which is carried off through the throat.

For most purposes, it is useful to divide the instanton in half by slicing the space through the minimal surface at the neck. This is the D-dimensional extension of the instanton given in Ref. 5. The present case describes tunneling between R^{D-1} and $R^{D-1} \oplus S^{D-1}$. The $(D-1)$-sphere baby universe may rejoin the parent universe through an instanton carrying the opposite axion flux, forming a handle. Alternatively by analytically continuing the boundaries back to a Lorentz signature, the instanton describes an interaction (annihilation or creation) with a baby Robertson-Walker universe. After analytic continuation, the fields and their conjugate momenta must be real on the boundaries. The r components of A all vanish, and the remaining components are independent of r. Therefore A trivially satisfies the reality condition upon analytic continuation. In the r coordinate system, the desired continuation is accomplished by $g^2 \to -g^2$. The reality condition is clearly satisfied on the flat R^{D-1} boundaries. The S^{D-1} boundary is a minimal surface and so has vanishing extrinsic curvature (as will be shown below). Therefore the metric also satisfies the reality condition on this boundary.

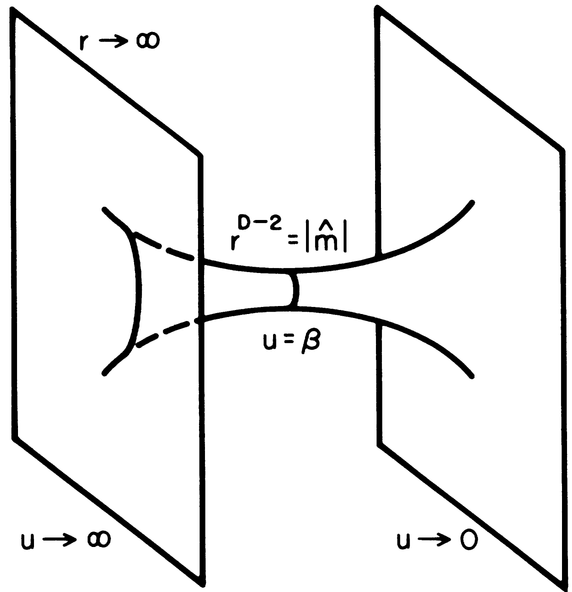

FIG. 1. The maximally extended solution with $\Lambda = 0$ has two asymptotically flat Euclidean regions. A single r coordinate patch only covers one-half of the space, while the u coordinate extends over the entire instanton. $D-2$ dimensions are suppressed so that each circle around the throat represents a $(D-1)$-sphere.

This new instanton may be used to generalize the remaining calculations of Ref. 5 to D dimensions. Of particular interest is the value of action. Using the trace of the equations of motion (2), $R = -16\pi G A^2$ with $\Lambda = 0$, the action (1) reduces to

$$\begin{aligned} S &= 2 \int d^D x \sqrt{g}\, A^2 \\ &= 2(D-1)! m^2 A_{D-1} \int_{r_{\min}}^{\infty} \frac{dr}{r (r^{2D-4} - \hat{m}^2)^{1/2}} \\ &= \frac{(D-1)(D-2)}{8\pi G} |\hat{m}| A_{D-1} \int_1^{\infty} \frac{du}{u (u^{2D-4} - 1)^{1/2}} \\ &= \frac{(D-1)}{16G} |\hat{m}| A_{D-1} = \frac{(D-1) A_{D-1}}{16} \left[\frac{r_{\min}}{l_{\rm Pl}} \right]^{D-2} . \end{aligned} \tag{8}$$

The last equality follows from $G \equiv l_{\rm Pl}^{D-2}$, where $l_{\rm Pl}$ denotes the Planck length. (Also recall $r_{\min}^{D-2} = |\hat{m}|$.) Therefore the interactions of baby universes large compared to the Planck length will be highly suppressed.

Now let us return to the solutions with a nonvanishing cosmological constant. We will concentrate on the de Sitter case (i.e., $\Lambda > 0$), and briefly comment on $\Lambda < 0$ at the end. Recall that the line element involved

$$g^2 = 1 - \hat{\Lambda} r^2 - \frac{\hat{m}^2}{r^{2D-4}} .$$

With positive Λ, g^2 is positive over a finite range of r so that a valid Euclidean solution exists if

$$\hat{m}^2 < \frac{1}{D-1} \left[\frac{D-2}{(D-1)\hat{\Lambda}} \right]^{D-2} .$$

At the end points of the above range, g^2 vanishes and hence g_{rr} diverges. These are merely coordinate singularities as is displayed by the Riemann tensor in an orthonormal frame. The expressions in Eq. (7) remain valid with $\Lambda \neq 0$, and so the nonvanishing components remain finite since when $g^2 = 0$, $(g^2)'$ and $\mathcal{X}'$ are finite, and $\mathcal{X} = 1$. Finding the zeros of g^2 involves solving for the roots of a $(D-1)$-order polynomial in r^2. Hence analytic solutions only exist for $D = 3$, 4, and 5 by Galois's theorem, but given Λ and m solving numerically is straightforward for higher dimensions. Given the roots, one may proceed perturbatively to construct a regular extension of the metric beyond the maximum and minimum radii (which we will denote $r = r_m$). Setting $r = r_m + \delta$, $g^2 = P\delta + O(\delta^2)$ where $P = (g^2)' \mid_{r=r_m}$, and hence to leading order the line element involves $d\delta^2/(P\delta)$. Making the coordinate transformation $\rho = 2(P\delta)^{1/2}$, yields $d\delta^2/(P\delta) = d\rho^2/(P^2)$ removing the singularity at $\delta = 0$. Such a perturbative approach is unappealing since it might break down at higher orders, and also since it provides no insight into the geometry of the extended space.

Finding an exact coordinate transformation for arbitrary dimension, which yields a nonsingular metric at $r = r_m$, is extremely difficult. Instead we proceed with the following observation. The extrinsic curvature of surfaces of constant radius may be written (see, for example, Ref. 14) $K_{ij} = -n_\mu \Gamma^\mu_{ij} = r g \tilde{g}_{ij}$, where $n_\mu dx^\mu = dr/g$ is the unit normal to these surfaces, and $\tilde{g}_{ij}$ is a metric on the unit $(D-1)$ sphere. Therefore the extrinsic curvature vanishes on precisely the spheres $r = r_m$ where $g^2 = 0$, and hence these are extremal surfaces, as illustrated in Fig. 2. One may analytically continue back to a Lorentzian signature since these are extremal surfaces, as discussed above for the asymptotically flat case. In this case the instanton describes tunneling between a de Sitter space at the maximum radius where Λ is the main source of curvature, and a baby Robertson-Walker universe at the minimum radius where the axion is the dominant source. The instanton may also represent the birth (demise) of a de Sitter and a baby universe from (into) nothing. The instanton may be extended beyond $r = r_m$ by sewing together a number of the Euclidean manifolds as shown in Fig. 3. The sign of the axion charge m would be reversed on the neighboring spaces. Instantons can be constructed in this way to describe tunneling between two baby universes or between two de Sitter spaces. Of course, solutions with no boundaries may also be constructed by

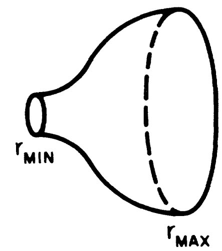

FIG. 2. The instanton with $\Lambda > 0$ covered by a single r coordinate patch. Near r_{max}, the cosmological constant dominates the curvature, while near r_{min}, the axion is the chief source of curvature. Again, $D-2$ dimensions are suppressed.

identifying the end surfaces in either of the latter two cases. For any of these possible tunneling processes, there are an infinite number of different instantons, which oscillate between the maximum and minimum radii. If in the semiclassical approximation these processes are weighted by the classical action, the resulting amplitude for de Sitter–baby-universe tunneling is

$$e^{-S_0} + e^{-3S_0} + e^{-5S_0} + \cdots = \frac{e^{-S_0}}{1 - e^{-2S_0}} , \tag{9}$$

where S_0 is the action for the instanton shown in Fig. 2. One should be aware that this resummation is a formal manipulation since S_0 is negative in many cases of interest. [In fact, S_0 is negative in all the cases that we have examined (including some numerical investigations). We conjecture that this is always true, but have been unable to prove it in general.] A related issue is whether the contribution of these instantons to the semiclassical am-

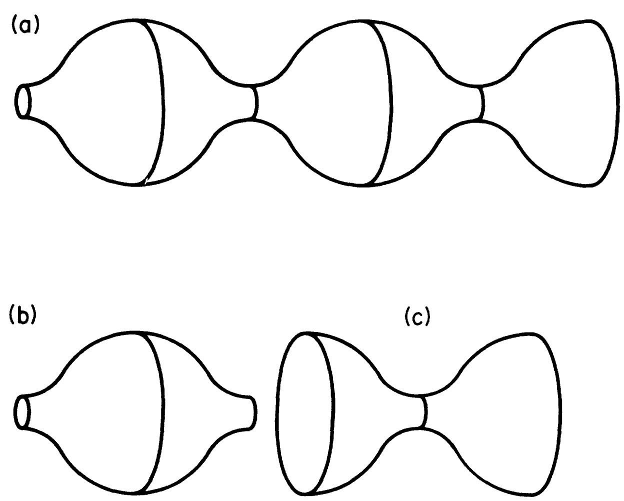

FIG. 3. Examples of instantons constructed by sewing together a number of copies of the solution illustrated in Fig. 2. Such instantons may describe tunneling between (a) de Sitter space and a baby universe, (b) two baby universes, or (c) two de Sitter spaces.

plitude is e^{-S} as above, or e^{+S} as suggested by some authors.[15,16] Note that the result in Eq. (9) is singular where S_0 vanishes. We have only found this to occur in the limit

$$\hat{m}^2 \to \frac{1}{D-1}\left[\frac{D-2}{(D-1)\hat{\Lambda}}\right]^{D-2},$$

where the zeros of g^2 coincide, and the instanton actually ceases to exist. A similar result occurs for the other tunneling processes described.

It should be possible (although very tedious) to calculate the action S_0 in closed form for $D=3,4,5$. We have only done the calculation for three dimensions, for which the result is

$$S_0(D=3)=\frac{\pi}{2}\frac{|\hat{m}|}{G}-\frac{\pi}{4}\frac{1}{G\hat{\Lambda}^{1/2}}<0 \ . \qquad (10)$$

In higher dimensions, numerical calculations for specific values of Λ and m would be required to determine S_0. In more general tunneling calculations, one would assume that the baby universes are small compared to the parent de Sitter space (i.e., $\hat{m}^2 \ll [1/(D-1)][(D-2)/(D-1)\hat{\Lambda}]^{D-2}$), and then the action can be approximated as the sum of that with the cosmological constant alone

$$S(\Lambda,m=0)=-\frac{\pi^{(D-1)/2}}{4\Gamma\left[\frac{D-1}{2}\right]}\frac{1}{G\hat{\Lambda}^{(D-2)/2}},$$

and the asymptotically flat result given by Eq. (8) for any number of baby universes.[5] It is curious that for a single instanton in three dimensions, this approximation in fact agrees with the exact result given in Eq. (10). In any event, the existence of an exact axionic instanton solution including a cosmological constant should be regarded as a nontrivial result.

Consider the case with negative cosmological constant. When $\hat{\Lambda}<0$, g^2 as given by Eq. (6) has a single zero at *small* radius. As before, this is only a coordinate singularity and actually indicates the presence of an extremal surface with vanishing extrinsic curvature. Asymptotically at large radius, the solution is approximately a constant curvature surface with $R=2D\Lambda/(D-2)$. It is relatively straightforward to find surfaces for which the extrinsic curvature vanishes as $(m/r^{D-1})^2$ in this region. These would be analogous to the R^{D-1} boundaries in the asymptotically flat instantons. Continuing to a Lorentzian signature on these boundaries yields anti–de Sitter spaces, and an axion-dominated Robertson-Walker space on the extremal surface at $g^2=0$. This would then describe an interaction between a baby universe and some number of anti–de Sitter spaces. Unfortunately this candidate instanton suffers from the fatal flaw that the action is not finite. The volume of the solution is infinite and hence the cosmological constant contribution to the action, which is $-[(D-1)\hat{\Lambda}/8\pi G]V$, diverges. This defect reoccurs in any Euclidean gravity path integrals with a negative cosmological constant, and so perhaps such theories may require a more subtle analysis. In conclusion, we have presented and briefly discussed a number of new axionic instantons, which we hope may prove useful in future studies of quantum gravity.

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