Brief Reports

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Strengths of shell-focusing singularities in marginally bound collapsing self-similar Tolman spacetimes

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Marginally bound self-similar collapsing Tolman spacetimes are examined, and the necessary conditions for formation of naked strong-curvature shell-focusing singularities are found.

Recently, Ori and Piran[1] have studied the self-similar spherical collapse of an adiabatic perfect fluid and have shown that with a soft enough equation of state the collapse can give rise to a naked shell-focusing singularity.[2] This is the first work which extends beyond dust,[2,3] known fluid collapse histories which have naked singular end states which are not instantaneous.

Newman[4] has found that for a wide class of Tolman spacetimes (spherical dust solutions) the shell-focusing singularities are not strong-curvature singularities as defined, for example, by Tipler, Clarke, and Ellis.[5] However, Lake[6] has shown that the shell-focusing singularities studied by Ori and Piran *are* strong-curvature singularities. This is not what one might expect; the addition of pressure in the spherical collapse of a perfect fluid might well lead to a *weaker* singularity than in the pressureless case.

We resolve this problem by considering the singularity structure of self-similar spacetimes—specifically, the marginally bound Tolman case here. (We have studied all self-similar spherically symmetric spacetimes. However, the case at hand provides a particularly clear and simple demonstration that this class of spacetimes gives examples of strong-curvature singularities.) We find that the form of the energy density used in Newman's work *excludes* self-similar spacetimes, and show that self-similar Tolman spacetimes admit *strong*-curvature singularities.

Consider the self-similar Tolman metric (in standard geometrical units) using comoving coordinates,[7]

$$ds^2=-dT^2+e^{2\omega}dR^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2) , \qquad (1)$$

where ω and $\tilde{r}\equiv r/T$ are functions of the self-similarity variable $y\equiv R/T$. Specifically, consider marginally bound, self-similar Tolman spacetimes,[2] where the equation for the areal radius is

$$r^3=\tfrac{9}{2}m\,[T-T_0(R)]^2 . \qquad (2)$$

Self-similarity demands that $m=\mu R$ and $T_0=KR$, where μ and K are strictly positive constants. The range of coordinates is $0\le R<\infty$ and $-\infty<T<KR$.

To compare our formulation to that of Newman,[4] consider the function ρ [defined as $\rho=\epsilon(R,0)$, where $\epsilon(R,T)$ describes the energy density at all times]. This is related to the mass by

$$m=4\pi\int_0^R \rho(s)s^2ds , \qquad (3)$$

where, following Newman, we have chosen to scale R so that $r(T=0,R)=R$. As a result

$$\rho=\frac{\mu}{4\pi R^2} , \qquad (4)$$

and so $\rho''>0$ (a prime denotes $\partial/\partial R$) for all R. Newman imposes the condition that for any T, ϵ is an even smooth function of R on the whole real line. This is not the case here, and so he necessarily *excludes* self-similar models.[8]

The metric coefficient g_{RR} is given by

$$e^{\omega}=r' . \qquad (5)$$

A shell-focusing singularity is the singular "point" at $R=T=0$ associated with radial null geodesics. The critical direction is the Cauchy horizon. We have shown elsewhere[9] that the Cauchy horizon of a spherically symmetric self-similar spacetime has $y=$const. Hence, along the Cauchy horizon

$$e^{\omega}=\frac{1}{y} . \qquad (6)$$

Then, with Eqs. (2), (5), and (6), with our choice of the scale for R, we have

$$\mu=\frac{6(1-Ky)}{y(1-3Ky)^3}=\frac{2}{9K^2} . \tag{7}$$

Thus, along the Cauchy horizon the singularity is globally naked if

$$K^3\geq\tfrac{2}{9}(\tfrac{26}{3}+5\sqrt{3}) . \tag{8}$$

This agrees with Eardley and Smarr.[2] [The inequality (8) is obtained here by calculating the maximum value of μ in the range $0<1/y<K$ (see below). Given K, for μ less than this maximum, two solutions $y=$const exist. The largest y gives the Cauchy horizon.]

Following the work of Clarke and Królak[10] we consider the null geodesic along the Cauchy horizon, affinely parametrized by λ, with a four-tangent k^α, and terminating in the shell-focusing singularity at $R=T=\lambda=0$. The singularity is a strong-curvature singularity (as defined by Tipler, Clarke, and Ellis[5]) if

$$\lim_{\lambda\to 0}\lambda^2 R_{\alpha\beta}k^\alpha k^\beta\neq 0 . \tag{9}$$

For the present case the energy density at the singularity [Eq. (4)] and the equation for the general energy density

$$\epsilon=\frac{\rho R^2}{r^2 r'} \tag{10}$$

give

$$\epsilon=\frac{1}{6\pi(1-Ky)(1-3Ky)T^2}\equiv\frac{D(y)}{T^2} . \tag{11}$$

(Note that ϵ is singular at $T=KR$ and $T=3KR$.) It follows from (11) together with the Einstein equations and the comoving condition that

$$\lambda^2 R_{\alpha\beta}k^\alpha k^\beta=8\pi D\left[\frac{\lambda}{T}\frac{dT}{d\lambda}\right]^2 . \tag{12}$$

The null geodesic equations for the Cauchy horizon ($y=$const) integrate explicitly to give $T=\lambda^\delta$ where $\delta=1/(1+4\pi D)$. As a result

$$\lambda^2 R_{\alpha\beta}k^\alpha k^\beta=8\pi D\delta^2 . \tag{13}$$

That is, the Cauchy horizon ($y=$const) terminates at $R=T=0$ in a *strong*-curvature singularity.

The central singularity must not exist in the initial conditions of the spacetime. This will be shown here by considering the Kretschmann scalar:

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}=\frac{16}{27(T-KR)^2}\times\left[\frac{9}{(T-3KR)^3}-\frac{8}{(T-KR)(T-3KR)}+\frac{4}{(T-KR)^2}\right] . \tag{14}$$

It is apparent that the Kretschmann scalar does not diverge for $R=0$ unless $T=0$ also. Singularities also occur at the crunch ($T=KR$) and when $r'=0$ (i.e., a shell-crossing singularity) at $T=3KR$. Since $K>0$, the collapse is free of shell-crossing singularities down to the crunch. Since $T<KR$ along the Cauchy horizon it will arise *prior* to the crunch. Likewise, a simple calculation shows that the apparent horizon ($r=2\mu R$) always occurs *after* the Cauchy horizon. As a result the shell-focusing singularity at $R=T=0$ is a globally naked strong-curvature singularity. [The global nature can be emphasized, for example, by junction onto vacuum at some fixed $R>0$ (Ref. 2).]

Shell focusing is a uniquely relativistic phenomenon in the sense that it is characterized by a (coordinate) focusing of null geodesics. The fact that the gravitational collapse of dust can give rise to a naked strong-curvature shell-focusing singularity is, as regards the cosmic censorship hypothesis, at least a bit disturbing. Further, it is now known that this situation is not limited to dust.[1] The solution to this problem may lie in the "elastic" boundary condition at $R=0$ (Ref. 11), but at present the situation is not clear.

Professor Piran has kindly informed us that he and A. Ori have independently obtained the result on strength discussed here.[12] This work was supported by the Natural Sciences and Engineering Research Council of Canada.

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[4] R. P. A. C. Newman, Class. Quantum Gravit. **3**, 527 (1986).

[5] F. J. Tipler, C. J. S. Clarke, and G. F. R. Ellis, in *General Relativity and Gravitation*, edited by A. Held (Plenum, New York, 1980), Vol. 2, p. 181.

[6] K. Lake, Phys. Rev. Lett. **60**, 241 (1988). Note that our Eq. (13) is Lake's Eq. (4) with, of course, $\Gamma=1$.

[7] For example, M. E. Cahill and A. H. Taub, Commun. Math. Phys. **21**, 1 (1971); G. V. Bicknell and R. N. Henriksen, Astrophys. J. **225**, 237 (1978).

[8] It is worth noting that the condition $r(T=0,R)=R$ is a choice of scale for R, and not the choice of $T=0$ as the initial slice. The slice $T=0$ is singular at $R=0$ [see, e.g., Eq. (14)]. For a nonsingular initial slice any $T<0$ will do for the Tolman model considered here.

[9] B. Waugh and K. Lake (unpublished). We have found that all spherically symmetric self-similar spacetimes in comoving coordinates with real finite positive roots y_c to $y_c^2 g_{RR}/g_{TT}\,|_{y_c}=1$ admit globally naked strong shell-focusing singularities with Cauchy horizon $y=y_c$. Further, we have found that the Cauchy horizon is stable to the development of blue-shift instabilities for test electromagnetic fields as long as the weak energy condition holds.

[10] C. J. S. Clarke and A. Królak, J. Geom. Phys. **2**, 127 (1986).

[11] D. M. Eardley, in *Gravitation in Astrophysics, Cargèse, 1986,* edited by B. Carter and J. Hartle (Plenum, New York, 1987).

[12] A. Ori and T. Piran (unpublished).