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Exact method for nonlinear fermions on finite lattice: Two dimensions

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A new method is proposed to solve exactly the problem of a nonlinear fermion on a finite lattice. A lattice version of the two-dimensional chiral Gross-Neveu model with broken SU(2) is examined as a simple example. We found a phase transition through analyses of zeros of the partition function and of the specific heat.

In recent years various finite-lattice methods (typically Monte Carlo methods) have been used in the study of the properties of quantum field theory and statistical mechanics. Also it has become important to study the phase transition by evaluating the partition function exactly. Unfortunately, on an infinite lattice with more than one dimension only a few models of a limited class such as Ising models are exactly solved. From the physical point of view, however, a wider class of models should be discussed on the basis of the information on exact solutions. An exact partition function on a finite lattice is also certainly the case, but only some limited classes are solved: Z_N spin and gauge models.[1]

Exact evaluation of the finite partition functions beyond Z_N spin and gauge models, therefore, would be important to shed light on lattice field theory. In particular, it is very interesting to study the nonlinear lattice fermions by allowing such an exact evaluation. This is due to not only the field-theoretical interests in the context of the dynamical fermions, but also the long-lived interests in the distribution of zeros of the partition function[2] in statistical mechanics. Particularly, the latter will give us a signature of new phenomena.[3]

The purpose of this paper is to present a new method which exactly integrates out the nonlinear fermions on a finite lattice. For simplicity we shall explicitly discuss the two-dimensional four-Fermi model in which a fermion and antifermion are interacting with attractive or repulsive couplings. The attractive case is the lattice version of the chiral Gross-Neveu model[4,5] with broken SU(2), and the repulsive case is mainly related to the (extended) Hubbard model in condensed-matter physics.[6]

We take examples up to a 4×4 lattice. It is, however, possible to solve exactly for larger lattices and, furthermore, the general applicability of our methods to purely fermionic theory and the extension to higher dimensions are obvious.

To illustrate our method concretely, we consider a lattice version of a two-flavor four-Fermi interaction model in the fermion formulation of Kogut and Susskind[7] (KS). Since we adopt the KS fermion formulation, the model has a "chiral" symmetry even for finite lattice.[8,9] The action is given by

$$S=-\frac{a^2}{2}\left[\sum_{x,\mu}\eta_\mu(x)\frac{\bar\psi(x+a\hat\mu)\psi(x)-\bar\psi(x)\psi(x+a\hat\mu)}{a}\pm\frac{g^2}{2}\sum_{x,\mu}[\bar\psi(x)\psi(x)\bar\psi(x+a\hat\mu)\psi(x+a\hat\mu)]\right], \tag{1}$$

where the lattice points are denoted as $x=(x_1,x_2)=(n_1a,n_2a)$. $\eta_1(x)=1$ and $\eta_2(x)=(-1)^{n_1}$ are the standard representations.[8,9] This action has not only a discrete symmetry but also a continuous symmetry $U(1)_e\otimes U(1)_o$ which is a remnant of chiral symmetry. The interaction term in Eq. (1) can be rewritten by the standard flavor interpretation[10] as

$$S_I=-G_0[(\bar f f)^2-(\bar f\gamma_5T_3f)^2],$$

where $f=\binom{u}{d}$, and $U(1)_e\otimes U(1)_o$ symmetry implies the $e^{i\gamma_5T_3}$ invariance. Hence, Eq. (1) corresponds to a lattice version of the chiral Gross-Neveu model with broken SU(2). Therefore it is advantageous to use the action in studies of chiral symmetry. We impose periodic and antiperiodic boundary conditions along the spatial and the temporal directions, respectively.

The partition function is defined by the following Grassmann functional integration:

$$Z=\int\prod_{x}^{N_1N_2}d\psi(x)d\bar\psi(x)e^{-S}. \tag{2}$$

Here N_1 and N_2 are the total numbers (even number) of the lattice points along the spatial and the temporal directions, respectively. When N_μ are finite, the integral is just a polynomial with respect to the coupling constant g. Therefore, our task is to calculate exactly the unknown polynomial.

From now on we calculate the integral by using the algebraic properties of the Grassmann numbers.[11] Let us take the minimal bases of the Grassmann algebra for each fermionic component as follows:

$$\eta^{i_\alpha}=(1,\psi^\alpha) \quad (i_\alpha=1,2, \quad \alpha=1,\ldots,N), \tag{3}$$

where index α specifies fermionic components, spinors ψ and $\bar{\psi}$ and flavor, color, etc. For each α, we can then write

$$\eta^{i_\alpha}\eta^{j_\alpha}=r_{k_\alpha}{}^{i_\alpha j_\alpha}\eta^{k_\alpha}, \tag{4}$$

where $r_{k_\alpha}^{i_\alpha j_\alpha}$ are the structure constants given by the elements of the following matrices:

$$(r_k)^{ij}=\delta_{k,i+j-1}$$

or (5)

$$r_1=\tfrac{1}{2}(1+\sigma_3), \quad r_2=\sigma_1 .$$

Furthermore, we numbered $\{\theta^j\}$, which is the set of the "direct products" of η^{i_α}:

$$\{\theta^j\}=\{\eta^{i_1}\eta^{i_2}\cdots\eta^{i_\alpha}\cdots\eta^{i_N}\}, \tag{6}$$

where

$$j-1=\sum_{\alpha=1}^{N}2^{\alpha-1}(i_\alpha-1)$$

is a binary system. The structure constants are also defined in terms of θ^j as

$$\theta^i\theta^j=R_k^{ij}\theta^k . \tag{7}$$

The structure constants R_k^{ij} are rewritten into the form by using Eqs. (4), (6), and (7):

$$(R_k)^{ij}=\prod_{\alpha=1}^{N}(-1)^{\sum_{\alpha'=\alpha+1}^{N}F_{i_{\alpha'}}F_{j_\alpha}}(r_{k_\alpha})^{i_\alpha j_\alpha}, \tag{8}$$

where F_{i_α} denotes the fermion number of η^{i_α}. Equations (5), (6), and (8) enable us to give the explicit form for $\{\theta^i\}$ and R_k as

$$\{\theta^i\}=(1,\psi,\bar{\psi},\psi\bar{\psi}) \quad (i=1,\ldots,4) \tag{9}$$

and

$$R_1=R_1^0=r_1\otimes r_1, \quad R_2=R_2^0=r_2\otimes r_1, \quad R_3=R_3^0=r_1\otimes r_2 ,$$

and

$$R_4=\mathcal{S}'R_4^0, \quad R_4^0=r_2\otimes r_2 , \tag{10}$$

with

$$\mathcal{S}'=\mathrm{diag}(1,1,-1,1) . \tag{11}$$

To include the parameter g, it is convenient to define a set explicitly for the one-component KS fermion:

$$\{\tilde{\theta}^i\}=(1,\psi,\bar{\psi},\tilde{g}\psi\bar{\psi}) \quad (i=1,\ldots,4) . \tag{12}$$

The structure of $\{\tilde{\theta}^i\}$ is similar to that of $\{\theta^i\}$, i.e.,

$$\tilde{\theta}^i\tilde{\theta}^j=\tilde{R}_k^{ij}\theta^k , \tag{13}$$

where $\tilde{R}_k^{ij}$ represent

$$\tilde{R}_k=R_k \ (k=1,2,3) \text{ and } \tilde{R}_4=\mathcal{S}'GR_4^0=GR_4 \tag{14}$$

with

$$G=\mathrm{diag}(\tilde{g},1,1,\tilde{g}) \tag{15}$$

and $\tilde{g}=(1\pm g^2)^{1/2}$.

We can rewrite the Boltzmann factor in Eq. (2) as (see also Appendix A)

$$e^{-S}=\prod_{\mu=1}^{2}\Omega_\mu , \tag{16}$$

where

$$\Omega_\mu=\prod_n\{1-\eta_\mu(n)[\psi(n)\bar{\psi}(n+\hat{\mu})+\bar{\psi}(n)\psi(n+\hat{\mu})]+(1\pm g^2)[\bar{\psi}(n)\psi(n)\bar{\psi}(n+\hat{\mu})\psi(n+\hat{\mu})]\} . \tag{17}$$

Here we use the nilpotency property $[\bar{\psi}(n)\psi(n)]^2=0$. Applying Eqs. (12) and (13) to Eq. (17), we have

$$\Omega_\mu=\prod_{n_\nu=1}^{N_\nu}\left[\prod_{n_\mu=1}^{N_\mu}\sum_{i_\mu,i_{\mu+\hat{\mu}}=1}^{4}\tilde{\theta}^{i_\mu}(n)(\mathcal{S}^{n_\mu^{(2)}+1}K)_{i_\mu i_{\mu+\hat{\mu}}}\tilde{\theta}^{i_{\mu+\hat{\mu}}}(n+\hat{\mu})\right] \quad (\mu\neq\nu) \tag{18}$$

$$=\prod_{n_\nu=1}^{N_\nu}\left[\prod_{n_\mu=1}^{N_\mu}\sum_{i_\mu,i_{\mu+\hat{\mu}}=1}^{4}\sum_{p_\mu=1}^{4}[\mathcal{S}^{n_\mu^{(2)}+1}K\tilde{R}_{p_\mu}(n)]^{i_\mu i_{\mu+\hat{\mu}}}\theta^{p_\mu}(n)\right] \quad (\mu\neq\nu) . \tag{19}$$

Here $\tilde{R}_{p_\mu}(n)=\tilde{R}_p$ except at the spatial boundary, and $\tilde{R}_{p_1}(N_1,n_2)=\tilde{R}_p\mathcal{S}$ at the boundary due to the periodic boundary condition (see Appendix B for details). $\mathcal{S}$, K, and $n_\mu^{(2)}$ are, respectively,

$$\mathcal{S}=\sigma_3\otimes\sigma_3 , \tag{20}$$

$$K=\tfrac{1}{2}\sum_{i=1}^{4}\sigma_i\otimes\sigma_i \quad (\sigma_4=1) , \tag{21}$$

and $n_\mu^{(2)}=n_1\delta_{\mu 2}$. $\mathcal{S}$ gives the minus sign to the fermonic components of θ^i and $n_\mu^{(2)}$ is an exponent of $\eta_\mu(n)$ in Eq. (1).

To perform the functional integration over ψ and $\bar{\psi}$, it is convenient to bring $\theta^{p_1}(x)$ and $\theta^{p_2}(x)$ associated with the same lattice point x into one place. The above operations require the nonlocal phase factors which are given by $(-1)^{\tilde{F}_{p_1}(x)}$, where

$$\tilde{F}_{p_1}(n_1,n_2)=F_{p_1}(n_1,n_2)\left[\sum_{m_1=n_1}^{N_1}\sum_{m_2=n_2}^{N_2}F_{p_1}(m_1,m_2)\right]_{\text{mod}2} \tag{22}$$

for each site (n_1,n_2). Here we use $\theta^{p_1}(x)\theta^{p_2}(y)=(-1)^{F_{p_1}F_{p_2}}\theta^{p_2}(y)\theta^{p_1}(x)$ and $F_{p_1}(n_1,n_2)+F_{p_2}(n_1,n_2)=0$, because only terms proportional to $\prod_x^{\text{all}}\bar{\psi}(x)\psi(x)$ do not vanish in performing the Grassmann integration (see Appendix C for details). Since the above expression is rather formal, we can further simplify $\tilde{F}_{p_1}(x)$ by $\sum_{n_\mu=1}^{N_\mu}F_{p_\mu}(n_\mu)=0$ being easily derived from Eqs. (1), (17), (18), and (19). After $\theta^{p_1}(x)$ and $\theta^{p_2}(x)$ are set on the same place over all lattice points, we apply Eq. (7) to the product $\theta^{p_1}(x)\theta^{p_2}(x)$, i.e., $\theta^{p_1}(x)\theta^{p_2}(x)=R_r^{p_1p_2}\theta^r(x)$. Then, the Grassmann functional integration can be performed by applying $\int d\bar{\psi}(x)d\psi(x)\theta^r(x)=\delta_{r4}$ because of a separate integral at each lattice point. Finally we can write the partition function as

$$Z=\text{SpTr}\prod_{N_1=1}^{N_1}\prod_{n_2=1}^{N_2}L(n_1,n_2)\ , \tag{23}$$

where

$$L(n_1,n_2)=\sum_{p_1,p_2=1}^{4}(-1)^{\tilde{F}_{p_1}(n)}\{[\mathcal{R}_{p_1}(n)]^{i_1i_1+\hat{1}}[\mathcal{S}^{n_1}\mathcal{R}_{p_2}(n)]^{i_2i_2+\hat{2}}R_4^{p_1p_2}\} \tag{24}$$

and

$$\mathcal{R}_p=\mathcal{S}K\tilde{R}_p\ . \tag{25}$$

Here Sp and Tr are traces with indices of transfer specified by i_1 and i_2 along the spatial and the temporal axes, respectively.

We have introduced $L(n_1,n_2)$, Eq. (24) , on the analogy of the elementary vertex weights for the Baxter's eight-vertex model.[12] However, a remarkable difference with it is the existence of the nonlocal factor $(-1)^{\tilde{F}_{p_1}(x)}$, $\tilde{F}_{p_1}(x)$ being given by Eq. (22). Since this factor does not explicitly depend on the indices of transfer specified by i_1 and i_2, this enables us advantageously to factor each term in Eq. (23) into (N_1+N_2) traces along each axis:

$$Z=\sum_{\{p\}=1}^{4}\prod_{n_2=1}^{N_2}\text{Sp}\left[\prod_{n_1=1}^{N_1}(-1)^{\tilde{F}_p(n)}\mathcal{R}_p(n)\right]\prod_{n_1=1}^{N_1}\text{Tr}\left[\prod_{n_2=1}^{N_2}(-1)^{\delta_{3p}}\mathcal{S}^{n_1}\mathcal{R}_{5-p}(n)\right]\ . \tag{26}$$

We should have recourse to a computer in order to calculate Eq. (26) on lattices as large as possible. The following idea enables us to calculate analytically even a FORTRAN program. Each term of Eq. (26) can be expressed by the integer matrix $(p_{n_1n_2})$. For example, the column $(p_{n_1N_0})$ (N_0 fixed) is able to correlate uniquely with $\text{Sp}(\prod_{n_1=1}^{N_1}\mathcal{R}_p(n_1,N_0))$, which can be easily evaluated. Although Eq. (26) has $4^{N_1N_2}$ terms, almost all of them will vanish. Therefore, it is desirable to generate efficiently a set of the nonvanishing terms $(\tilde{p}_{n_1n_2})$.

Fortunately, the following method does not generate the vanishing terms of $(p_{n_1n_2})$ at all. Here, we note $\mathcal{S}K\tilde{R}_p\mathcal{S}K=(-1)^{F_p}K_{pp'}\tilde{R}_{p'}^T$ and $\tilde{R}_p$ forms $(r_{k^{(1)}}\otimes r_{k^{(2)}})$ (diagonal matrix). First, the entire set $\{(\tilde{k}_{n_1n_2})\}$, being a nonvanishing term of

$$z_0=\sum_{\{k_{n_1n_2}\}=1}^{2}\prod_{n_2=1}^{N_2}\text{Sp}\left[\prod_{n_1=1}^{N_1}r_{k_{n_1n_2}}\right]\prod_{n_1=1}^{N_1}\text{Tr}\left[\prod_{n_2=1}^{N_2}r_{3-k_{n_1n_2}}\right]\ , \tag{27}$$

can be found by applying the properties of Eqs. (5) and (28) (see below). Second, we generate the entire set $(\hat{p}_{n_1n_2}-1)=(\tilde{k}^{(1)}_{n_1n_2}-1,\tilde{k}^{(2)}_{n_1n_2}-1)_{\text{binary}}$. Here, the $(k^{(i)}_{n_1n_2})$ $(i=1,2)$ are two arbitrary elements of the set $\{(\tilde{k}_{n_1n_2})\}$. An element of the set $\{(\hat{p}_{n_1n_2})\}$ is not always identified with $(\tilde{p}_{n_1n_2})$ precisely because of the existence of the matrix K (note $K\tilde{R}_pK=K_{pp'}\tilde{R}_{p'}{}^T$). However, the set $\{(\hat{p}_{n_1n_2})\}$ is uniquely transformed into $\{(\tilde{p}_{n_1n_2})\}$ by a permutation of the values of $\hat{p}$'s satisfying $n_1+n_2=$odd (or equivalently even), such as $(2\leftrightarrow3)$ (see Appendix D for details). As a result, only 90^2 nonvanishing terms[13] are picked up and evaluated in a 4×4 lattice size, even if symmetries such as translation invariance are not considered (see Appendix E for details). The above principle which drastically reduces terms is essentially based on the relation

$$\mathrm{tr}\left[\prod_{i=1}^{N} r_1^{k_i} r_2^{m_i}\right] = \prod_{i=1}^{N} \mathrm{tr}(r_1 r_2^{m_i}) = 0 \quad (\text{if all } m_i \neq \text{even}) \; . \tag{28}$$

Consequently, the analytic calculation turns out to sort the integer matrices, and the sum of nonzero terms gives the exact partition function on a finite lattice:

$$Z(2\times2)=8(g^4 \pm 2g^2+2)/2^4 \; , \tag{29}$$

$$Z(4\times4)=(17g^{16}\pm136g^{14}+540g^{12}\pm1384g^{10} +2530g^8\pm3472g^6 +3712g^4\pm2880g^2+1296)/2^{16} \; . \tag{30}$$

In the attractive-coupling case, the partition functions are polynomials whose terms all have positive signs. Thus in the repulsive-coupling case, the partition functions are polynomials whose terms have alternating signs which is easily seen from Eqs. (1) and (2). It is also easily understood that the maximal powers mean the total number of lattice points if one considers the strong-coupling limit in Eqs. (1) and (2).

In this method, it is very easy to calculate the partition function of a 2×2 lattice by hand. Now it will also be possible to calculate the partition function by hand, even for a 4×4 lattice, although we have done it with the FORTRAN program (see Appendix E for a rough estimate of the amount of calculations for a 4×4 lattice by hand).

Since we regard g^2 as temperature T (see Appendix F), we can define the following "specific heat" as a response function for the free energy:

$$C=\frac{1}{N_1N_2}\frac{\partial}{\partial T}T^2\frac{\partial}{\partial T}\ln Z \quad (T=g^2) \; . \tag{31}$$

However, the above specific heat does not guarantee its positivity due to the nilpotency property of Grassmann variables.[14]

In Fig. 1 we show the specific heat for both attractive and repulsive couplings. In the attractive case, the specific heat increases monotonically and has little size dependence. Therefore, there is no indication of a phase transition. In the repulsive case, on the other hand, the specific heat has a clear peak at $g^2 \simeq 2$, whose height increases with the lattice size. This means the occurrence of a phase transition.

The distribution of zeros in the complex g^2 plane is shown in Fig. 2. The negative $\mathrm{Re}g^2$ region has richer zeros than that of positive $\mathrm{Re}g^2$. The distribution is correlated with the structure of the specific heat. We can see the closest zero to the negative $\mathrm{Re}g^2$ axis at $\mathrm{Re}(-g^2)\simeq 2$, which will approach the axis as the lattice size increases. This also indicates that a phase transition occurs around $\mathrm{Re}(-g^2)\simeq 2$ on an infinite lattice.

A clear physical interpretation for the phase transition in the repulsive coupling has not been obtained. There needs to be a calculation of other physical quantities such as the correlation length and proper order parameter[15] in this formulation on larger lattices to give reliable information on the phase transition.

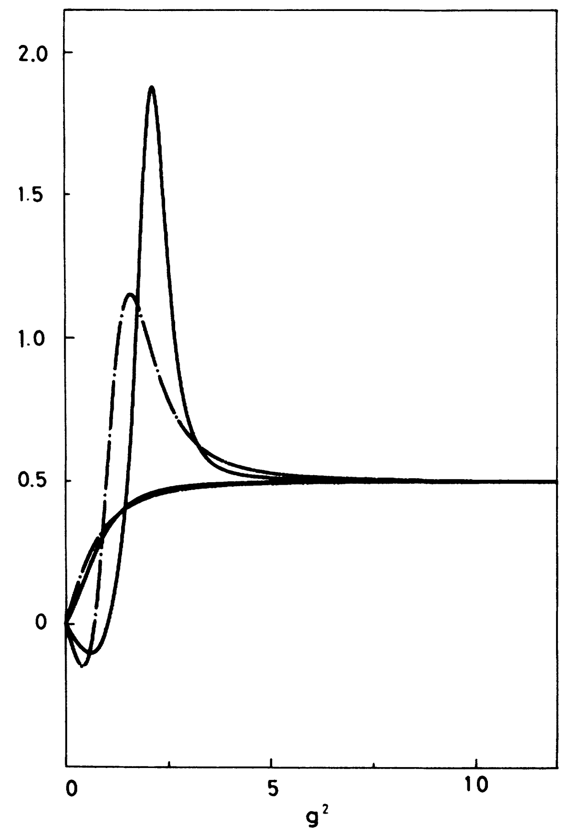

FIG. 1. The upper and lower solid (dashed-dotted) lines denote the specific heat for 4×4 (2×2) lattices for repulsive and attractive couplings, respectively.

The analogy between chiral-symmetry breaking and magnetization leads us to the idea of the distribution of zeros, as suggested by the theorem[2] of Lee and Yang (circle) in the complex mass plane. We find, however, that

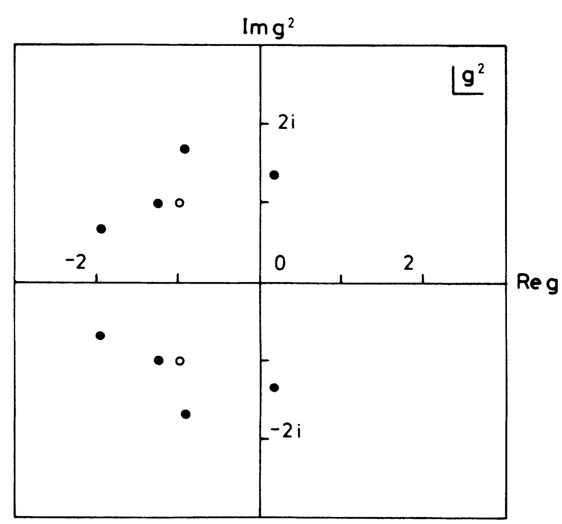

FIG. 2. The zeros of the 4×4 (solid circles) and 2×2 (open circles) partition functions in the g^2 plane.

the positivity of the partition function does not hold unless $m \lesssim 1/a$ in the repulsive-coupling case (the stable ground state disappears owing to the correlation length being too short). Therefore, careful analyses are required in the complex mass plane.[16]

It is clear that our method can be generalized in three- and four-dimensional nonlinear fermion theories. They are effective actions of QCD,[17] gauge-Higgs theories with fermions,[18] and the new high-T_c superconductors.[19] In particular, (reduced) KS fermion versions at a finite temperature will hopefully be attacked along the same line of thought developed above.

The present method is also successful in solving exactly two-dimensional gauge-fermion systems (including QCD_2) on a finite lattice. Actually, we have obtained exact partition functions and Wilson loops in the lattice Schwinger model with the KS fermion version up to a 6×4 lattice.[20]

Such remaining but interesting models as the γ_5-invariant Gross-Neveu model[5] and a generalized nonlinear interaction term such as

$$S_I = g\sum_{x,\mu}\exp[\bar\psi(x)\psi(x+a\hat\mu)+\text{H.c.}]$$

will be discussed in a separate paper, because these are also solvable on finite lattices by minor extensions.

Finally, we comment on the improvement of our method. The problem is whether it can directly give a (simple) formula to express the terms in analytic forms when the integer matrices $(\tilde p_{n_1 n_2})$ are given. The formula will be useful for more efficient calculations in our method, because it will clarify the symmetries of each nonvanishing term and also among terms. Furthermore, from a more general standpoint, the formula will be useful to clarify how the original action's symmetries reflect the expression of the partition function.

We think now that it is not so difficult, and an attempt is in progress. The formula will have a general structure, even though it will, of course, have a part, which depends on the details of the models. The main reason for it is that the basic structure of the kinetic term is common. It will be expected that the basic ideas of the formula will be applicable also to a wider class of nonlinear fermion models.

In conclusion, we presented a new method to solve exactly the problem of nonlinear fermions on finite lattices. We explained our method with a concrete example, which is a lattice version of the two-dimensional chiral Gross-Neveu model with broken SU(2). We found a phase transition through analyses of zeros of the partition function and of the specific heat in the case of repulsive coupling. The basic ideas will be applicable to a general class of purely fermionic theory and also the exact calculations of fermionic integrals in fermion-gauge (or boson) systems on finite lattices.

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APPENDIX A

If $a_\mu = 1$, fractional coefficients appear in Eq. (1). In an exact calculation, it is more convenient to treat an integer coefficient than a fractional one. Therefore, to avoid the complication of fractional coefficients, we rescale the fields as follows:

$$\psi' = (1/\sqrt{2})\psi, \quad \bar\psi' = (1/\sqrt{2})\bar\psi \ . \tag{A1}$$

Then

$$Z = (\tfrac{1}{2})^{N_1 N_2} Z' \tag{A2}$$

and

$$Z' = \int \prod_x^{N_1 N_2} d\psi'(x) d\bar\psi'(x) e^{-S'} \tag{A3}$$

with

$$S' = -\sum_{x,\mu}\{\eta_\mu(x)[\bar\psi'(x+\hat\mu)\psi'(x) - \bar\psi'(x)\psi'(x+\hat\mu)] + g^2[\bar\psi'(x)\psi'(x)\bar\psi'(x+\hat\mu)\psi'(x+\hat\mu)]\} \ . \tag{A4}$$

We can calculate Eqs. (A3) and (A4), and the final expressions of Z' are multiplied by the overall factors $(\frac{1}{2})^{N_1 N_2}$ for mathematical completeness, though these are physically irrelevant. From now on we omit the prime for simplicity.

APPENDIX B

In Eq. (18), set $\bar\theta^{j} = (\mathcal{S}K\theta)^j$ and omit the tildes on the θ's for simplicity. Also the subscripts in Eq. (B1) (see below) mean the coordinates in Ω_μ. In order to yield Eq. (10) the following type of rearrangement is carried out:

$$(\theta_1^{i_1}\bar\theta_2^{j_2})(\theta_2^{i_2}\bar\theta_3^{j_3})\cdots(\theta_N^{i_N}\bar\theta_{N+1}^{j_{N+1}}) = (-1)^{F_{i_N}}(\bar\theta_1^{j_1}\theta_1^{i_1})(\bar\theta_2^{j_2}\theta_2^{i_2})\cdots(\bar\theta_N^{j_N}\theta_N^{i_N}) \ , \tag{B1}$$

where we assume the periodic boundary condition $\overset{(-)}{\theta}{}^{j_{N+1}}_{N+1} = \overset{(-)}{\theta}{}^{j_1}_1$, and $F_{i_n} + F_{j_{n+1}} = 0$ which is easily derived from Eqs. (1), (17), and (18). Therefore, if we impose the antiperiodic boundary condition

$$(-1)^{F_{j_{N+1}}}\overset{(-)}{\theta}{}^{j_{N+1}}_{N+1} = \overset{(-)}{\theta}{}^{j_1}_1 \ ,$$

the phase factor $(-1)^{F_{i_N}}$ is canceled out in Eq. (B1).

APPENDIX C

In Eq. (16), we note

$$\Omega_1\Omega_2 = \Omega_2\Omega_1 \ . \tag{C1}$$

In each term of Ω_μ, the order of θ^{p_μ} is assumed as follows:

$$\Omega_1^p=[\theta^{p_1}(1,1)\theta^{p_1}(2,1)\cdots\theta^{p_1}(N_1,1)][\theta^{p_1}(1,2)\theta^{p_1}(2,2)\cdots\theta^{p_1}(N_1,2)]\cdots[\theta^{p_1}(1,N_2)\theta^{p_1}(2,N_2)\cdots\theta^{p_1}(N_1,N_2)]\ , \tag{C2}$$

$$\Omega_2^p=[\theta^{p_2}(1,1)\theta^{p_2}(1,2)\cdots\theta^{p_2}(1,N_2)][\theta^{p_2}(2,1)\theta^{p_2}(2,2)\cdots\theta^{p_2}(2,N_2)]\cdots[\theta^{p_2}(N_1,1)\theta^{p_2}(N_1,2)\cdots\theta^{p_2}(N_1,N_2)]\ , \tag{C3}$$

where Ω_μ^p are identified with each term of Ω_μ in Eq. (19) except the coefficients which are omitted for simplicity. Starting from the right-hand side of Eq. (C1) and changing the order of all the $\theta^{p_1}(x)$ to take the order as $\theta^{p_1}(x)\theta^{p_2}(x)$ over all lattice points, we get

$$\tilde{F}_{p_1}(n_1,n_2)=F_{p_1}(n_1,n_2)\left[\sum_{m_1=n_1}^{N_1}\sum_{m_2=n_2}^{N_2}F_{p_2}(m_1,m_2)\right]_{\text{mod}2}\ . \tag{C4}$$

APPENDIX D

Proposition. $\{(\hat{p}_{n_1n_2})\}$ means the set of all nonvanishing (nontraceless) terms of

$$z_1=\sum_{\{p_{n_1n_2}\}=1}^{4}\prod_{n_2=1}^{N_2}\text{Sp}\left[\prod_{n_1=1}^{N_1}R^0_{p_{n_1n_2}}\right]\prod_{n_1=1}^{N_1}\text{Tr}\left[\prod_{n_2=1}^{N_2}R^0_{q_{n_1n_2}}\right]\ , \tag{D1}$$

where $R^0_{p_{n_1n_2}}$ are defined by Eq. (10), and $p_{n_1n_2}+q_{n_1n_2}=5$. Also, $\{(\bar{p}_{n_1n_2})\}$ means the set of all nonvanishing terms of

$$z_2=\sum_{\{p'_{n_1n_2}\}=1}^{4}\prod_{n_2=1}^{N_2}\text{Sp}\left[\prod_{n_1=1}^{N_1}KR^0_{p'_{n_1n_2}}\right]\prod_{n_1=1}^{N_1}\text{Tr}\left[\prod_{n_2=1}^{N_2}KR^0_{q'_{n_1n_2}}\right]\ , \tag{D2}$$

where $p'_{n_1n_2}+q'_{n_1n_2}=5$. Then it can be shown that the set $\{(\hat{p}_{n_1n_2})\}$ is uniquely transformed into $\{(\bar{p}_{n_1n_2})\}$ by a permutation of the values of $\hat{p}$'s satisfying $n_1+n_2=$odd (or equivalently even), such as $(2\leftrightarrow3)$ and vice versa.

Proof. In Eq. (D1), we note the following equalities for a trace of one term in the spatial direction (the same equalities are also satisfied in the temporal direction):

$$\begin{aligned}\text{tr}\left[\prod_{n_1=1}^{N_1}R^0_{p_{n_1n_2}}\right]&=\text{tr}\left[\prod_{n_1=\text{odd}}^{N_1-1}R^0_{p_{n_1n_2}}KKR^0_{p_{n_1+1n_2}}\right]\\&=\text{tr}\left[\prod_{n_1=\text{even}}^{N_1}R^0_{p_{n_1n_2}}KKR^0_{p_{n_1+1n_2}}\right]\\&=\text{tr}\left[\prod_{n_1=\text{odd}}^{N_1-1}KR^0_{\dot{p}_{n_1n_2}}KR^0_{p_{n_1+1n_2}}\right]=\text{tr}\left[\prod_{n_1=\text{even}}^{N_1}KR^0_{\dot{p}_{n_1n_2}}KR^0_{p_{n_1+1n_2}}\right]\quad(N_1+1\equiv1)\ ,\end{aligned} \tag{D3}$$

where we use

$$R^0_pK=K(K_{pp'}R^0_{p'})=KR^0_{\dot{p}} \tag{D4}$$

and $K^2=1$. From the explicit form for the K, $\dot{p}$ which is defined by Eq. (D4) is identified with the permutation of the values of p, such as $(2\leftrightarrow3)$.

When applying Eq. (D3) to all of the columns and rows, there are at least 2 degrees of freedom of choice: i.e., we can take the choice of either of the last two equalities. The above enables us to transform all p belonging to the site of $n_1+n_2=$odd (or equivalently even) to $\dot{p}$, and at the same time q of $n_1+n_2=$odd (or even) to $\dot{q}$. Then Eq. (D1) can be uniquely transformed into Eq. (D2), because only this transformation enables it to satisfy $p'_{n_1n_2}+q'_{n_1n_2}=5$ for all sites.

Although it is sufficient to prove it within this argument for the present model, it will lead to a more general proof which is applicable to other cases having more fermionic components or to other lattice fermion formulations. Since Eq. (D2) corresponds to Eq. (26), the proof is sufficient if the following equation can be proved for each site of $n_1+n_2=$odd (or even):

$$\sum_{p,q=1}^{4}(KR_{\dot p}^{0})\otimes(KR_{\dot q}^{0})(R_4^0)^{pq}=\sum_{\dot p,\dot q=1}^{4}(KR_{\dot p}^{0})\otimes(KR_{\dot q}^{0})(R_4^0)^{\dot p\dot q}\ , \tag{D5}$$

where the notations p and q on the left-hand side correspond to p_1 and p_2 in Eq. (24), respectively (also see Appendix E for the technique of removing $\mathcal{S}$). The proof is as follows. The left-hand side in Eq. (D5) can be rewritten as

$$\begin{aligned}\sum_{p,q=1}^{4}\sum_{\dot p,\dot q=1}^{4}[K(K_{p\dot p}R_{\dot p}^{0})]\otimes[K(K_{q\dot q}R_{\dot q}^{0})](R_4^0)^{pq}&=\sum_{\dot p,\dot q=1}^{4}(KR_{\dot p}^{0})\otimes(KR_{\dot q}^{0})\sum_{p,q,=1}^{4}[K_{\dot p p}(R_4^0)^{pq}K_{q\dot q}]\\&=\sum_{\dot p,\dot q=1}^{4}(KR_{\dot p}^{0})\otimes(KR_{\dot q}^{0})(R_4^0)^{\dot p\dot q}\ ,\end{aligned} \tag{D6}$$

where we use Eq. (D4) and $K^T=K$. Q.E.D.

From the above fact, it is obvious that the set $\{(\hat{p}_{n_1n_2})\}$ of nonvanishing terms $(\hat{p}_{n_1n_2})$ is uniquely transformed into the set $\{(\bar{p}_{n_1n_2})\}$, and vice versa. Q.E.D.

We can apply the above proposition to Eq. (26), paying relevant attention to the difference between Eqs. (D2) and (26). The difference is the existence of the diagonal matrices $\mathcal{S}$, $\mathcal{S}'$, and G [which are defined by Eqs. (20), (11), and (15), respectively] in Eq. (26) for each corresponding term in Eq. (D2) except for the overall factors. The present argument, essentially, gives the conditions for the traceless property since the final expression of the matrix has no diagonal elements. (The argument is independent of the existence of such diagonal matrices, because those diagonal elements are all nonzero elements.)

On the other hand, the argument ignores the case in which the traceless property occurs due to cancellation among diagonal elements. For the above case, detailed information and discussions about the diagonal matrices $\mathcal{S}$, $\mathcal{S}'$, G are required. The cancellation possibly occurs since the trace includes an odd number of $\mathcal{S}$ due to the periodic boundary condition (although the rate is small). On the other hand, it can be proved that the cancellation cannot occur when the trace includes an even number of $\mathcal{S}$ in the case of the antiperiodic boundary condition (see Appendix E for a concrete calculation). Therefore the cancellation can be interpreted as a kind of manifestation of the boundary effect.

However, we omit a discussion of the case of the cancellation, since it has no practical merit compared with requiring detailed discussions, nor will we focus our attention now on the boundary effect. Finally we emphasize that the terms excluded by this rule never contain nonvanishing terms.

APPENDIX E

It may be instructive to illustrate an example for explicit evaluation for a nonvanishing term of Eq. (26) (the partition function). Here we list some useful relations:

$$[K,\mathcal{S}]=0\ , \tag{E1}$$

$$\mathcal{S}\tilde{R}_p\mathcal{S}=(-1)^{F_p}\tilde{R}_p\ , \tag{E2}$$

$$\sum_{n_\mu=1}^{N_\mu}F_{p_\mu}(n_\mu)=0\ . \tag{E3}$$

Also we note that $\mathcal{S}^{n_1}$ in Eq. (26), which arises from the factor $\eta_\mu(n)$ in Eq. (1) (it originates from Dirac matrices), further gives the sign factors

$$(-1)^{f_\eta(p)}\ . \tag{E4}$$

Here

$$f_\eta(p)=\sum_{n_1=\text{odd(even)}}^{N_{1\max}}\ \sum_{n_2=\text{odd(even)}}^{N_{2\max}}F_p(n_1,n_2)\ , \tag{E5}$$

for each nonvanishing term of $(\tilde{p}_{n_1n_2})$, is given by the summations of the fermion number of $(\tilde{p}_{n_1n_2})$. Here, $N_{\mu\max}$ means the maximal odd or even numbers less than or equal to N_μ, and we use Eqs. (E1), (E2), and (E3).

At first, in Fig. 3, we show a schematic illustration for

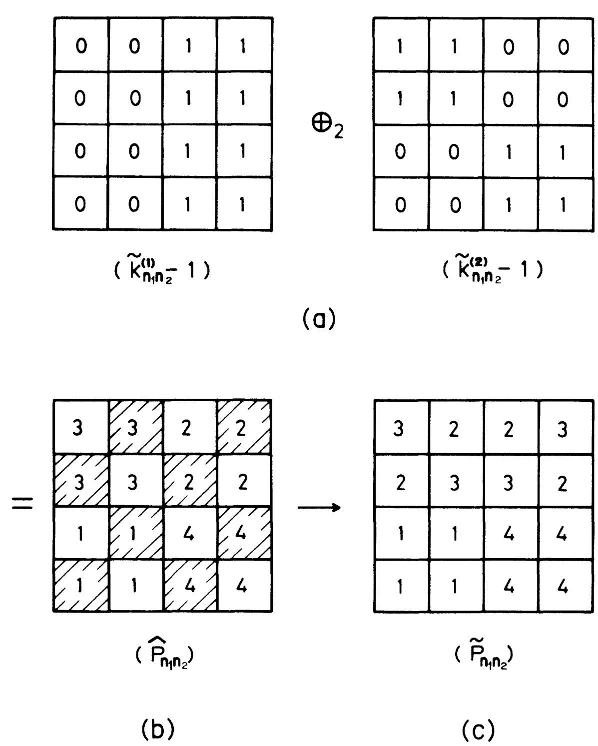

FIG. 3. A schematic illustration for generating a nonvanishing term of Eq. (26) by combining two nonvanishing terms of Eq. (27). Here unit squares mean the lattice points (n_1,n_2) and the coordinates are assumed to be in the same arrangement of matrix elements. Also $\oplus_2$ means a binary sum, and the shaded lattice points in the checkerboard mean $n_1+n_2=$odd.

generating the nonvanishing term $(\tilde{p}_{n_1 n_2})$ from the set of nonvanishing terms $(\tilde{k}_{n_1 n_2})$ of z_0 in Eq. (27). In Fig. 3(a), the $(k^{(i)}_{n_1 n_2})$ $(i=1,2)$ are two arbitrary elements of the set $\{(\tilde{k}_{n_1 n_2})\}$, and $\oplus_2$ means a binary sum. In Fig. 3(b), the numbers are $(\hat{p}_{n_1 n_2})$, and the shaded sites in the pattern of a checkerboard mean $n_1+n_2=\text{odd}$. In Fig. 3(c), the $(\tilde{p}_{n_1 n_2})$ is obtained by the permutation $(2\leftrightarrow 3)$ of the values of $\hat{p}$'s for the shaded sites in Fig. 3(b). This element of the set $\{(\tilde{p}_{n_1 n_2})\}$ is identified with one of the nonvanishing terms in Eq. (26) (see Appendix D). Next, we show the steps for the evaluation of Fig. 3(c).

Step 1. For the factors $(-1)^{\tilde{F}_p(n_1,n_2)}$: for example, $(-1)^{\tilde{F}_p(1,1)}=1$, etc., and the total contribution is

$$(-1)^{\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\tilde{F}_p(n_1,n_2)}=1$$

[if Eq. (E3) is suitably applied to $\tilde{F}_p(n_1,n_2)$, this is evaluated more easily].

Step 2. For $(-1)^{\delta_{3p}}$: there are four $p=3$, and therefore the total contribution becomes $(-1)^4=1$.

Step 3. For $\mathcal{S}^{n_1}$: from Eqs. (E4) and (E5), the total contribution is given by the factor

$$(-1)^{f_\eta(p)}=(-1)^{F_p(1,1)+F_p(1,3)+F_p(3,1)+F_p(3,3)},$$

and therefore the total contribution becomes unity.

Step 4. For the trace part

$$\begin{aligned}\prod_{n_2=1}^{N_2}\mathrm{Sp}\left[\prod_{n_1=1}^{N_1}\mathcal{R}_p(n_1,n_2)\right]\prod_{n_1=1}^{N_1}\mathrm{Tr}\left[\prod_{n_2=1}^{N_2}\mathcal{R}_{5-p}(n_1,n_2)\right] &=\mathrm{Sp}(\bar{R}_3\bar{R}_2\bar{R}_1\bar{R}_1\mathcal{S})\mathrm{Sp}(\bar{R}_2\bar{R}_3\bar{R}_1\bar{R}_1\mathcal{S})\mathrm{Sp}(\bar{R}_2\bar{R}_3\bar{R}_4\bar{R}_4\mathcal{S})\mathrm{Sp}(\bar{R}_3\bar{R}_2\bar{R}_4\bar{R}_4\mathcal{S})\\ &\quad\times\mathrm{Tr}(\bar{R}_2\bar{R}_3\bar{R}_3\bar{R}_2)\mathrm{Tr}(\bar{R}_3\bar{R}_2\bar{R}_2\bar{R}_3)\mathrm{Tr}(\bar{R}_4\bar{R}_4\bar{R}_1\bar{R}_1)\mathrm{Tr}(\bar{R}_4\bar{R}_4\bar{R}_1\bar{R}_1)\\ &=1\times 1\times(\tilde{g}^2-1)\times(\tilde{g}^2-1)\times 1\times 1\times\tilde{g}^2\times\tilde{g}^2 . \end{aligned}\tag{E6}$$

Here $\bar{R}_p=K\tilde{R}_p$, and we use Eqs. (E1), (E2), and (E3) in the first equality.

Step 5. For the total contribution of Fig. 3(c): from the product of steps 1–4 we have

$$\tilde{g}^4(\tilde{g}^2-1)^2=g^8\pm 2g^6+g^4 . \tag{E7}$$

Although we have calculated the 4×4 partition function by the FORTRAN program as stated in the text, we also emphasize that now the partition function on an even 4×4 lattice can be calculated by hand. It is only required to repeat the above same calculations about 500 times (approximately $90^2/4^2$), because translation invariance reduces the amount of calculations.

However, since a few $(\tilde{p}_{n_1 n_2})$, which have an exceptional length of cycle, exist, it is necessary to take care of the symmetry factor [for example, the case which has unity for all the elements of $(\tilde{p}_{n_1 n_2})$ differs from the case of Fig. 3(c) with the length of cycle]. If described in detail, the translation invariance is of course assured by Eqs. (E2) and (E3) in the case where $\mathcal{S}$ exists at the boundary as it was in Eq. (E6).

APPENDIX F

It may not be obvious that g^2 is regarded as temperature as in pure gauge theories. However, for the following reason, we regard g^2 as temperature. If we rescale as

$$\psi\to g\psi \quad\text{and}\quad \bar{\psi}\to g\bar{\psi} , \tag{F1}$$

and drop an overall multiplicative factor, then the partition function becomes

$$Z=\int\prod_x d\psi(x)d\bar{\psi}(x)e^{-S_0/g^2} , \tag{F2}$$

where S_0 is $S\,|_{g^2=1}$ in Eq. (1) and therefore does not include the parameter g^2. Thus, S_0/g^2 is equivalent to the original action S in Eq. (1) under the finite g^2, and therefore g^2 can be regarded as temperature. Also, it is shown that some versions of the two-dimensional many-flavor lattice (chiral) Gross-Neveu models in the $N_f\to\infty$ limit correspond to some spin systems with temperature[5,21] $T\propto g^2$.

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[13] Strictly speaking, this is the maximum not depending on details such as boundary conditions. This number just applies to the case of antiperiodic boundary conditions which both temporal and spatial directions satisfy. Since the spatial direction is a periodic boundary condition in the present case, the nonvanishing terms are more diminishing owing to traces including an odd number of $\mathcal{S}$ (see also Appendix D).

[14] The following is an extreme example in such a case:

$$\frac{\partial^2}{\partial G_m^2}\ln\left[\int\prod_{n=1}^{N}d\psi_n d\bar{\psi}_n \exp\left[-\sum_{n=1}^{N}G_n\bar{\psi}_n\psi_n - S'\right]\right] = \langle(\bar{\psi}_m\psi_m)^2\rangle - \langle\bar{\psi}_m\psi_m\rangle^2 \le 0\ ,$$

owing to $(\bar{\psi}_m\psi_m)^2=0$, where S' is independent of G_n. In statistical mechanics, the leveled exponential function also shows the same property in the second-order cumulant [R. Kubo, J. Phys. Soc. Jpn. **17**, 1100 (1962)].

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