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Quantum probability distributions in the early Universe. I. Equilibrium properties of the Wigner equation

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This is the first in a series of papers dealing with quantum or Wigner probability distributions and the dynamics of early-Universe phase transitions. In this paper we argue that the inflaton field (a real gauge-singlet scalar field) when spatially averaged over a causal horizon behaves as if it were a dissipative quantum-mechanical system in one dimension. Realizing that the mathematical phase-space or stochastic description of Wigner is the only consistent way to describe these systems and still preserve the canonical commutation relations we devote this paper to investigating various properties of the Wigner formalism. This framework rests on the quasiprobability distribution and its evolution equation: a generalized Fokker-Planck equation. We first show that this generalized Fokker-Planck equation is equivalent to Schrödinger's equation for nondissipative pure states. We then show how one computes ground-state energies within this formalism (for nondissipative systems) by calculating ground-state energy levels for a variety of anharmonic potentials. Finally we compute the quantum-mechanical effective potential in one dimension to order $g\hbar$.

I. INTRODUCTION

This is the first in a series of papers dealing with quantum probability distributions and the dynamics of early-Universe ($<10^{-33}$ sec) phase transitions. We are interested primarily in the quantum-mechanical evolution of the large-scale ($\gtrsim$ causal horizon) or coarse-grained real gauge-singlet scalar field (inflaton) from an out-of-equilibrium configuration in the presence of a de Sitter background.

Investigations into this problem have been carried out by Linde,[1] Starobinsky,[2] Guth and Pi,[3] Graziani and Olynyk,[4] Bardeen and Bublik,[5] and Rey.[6] All of the aforementioned find that the dynamics of the large-scale early-Universe phase transition behaves as a stochastic process. However, Graziani and Olynyk differ from the others in some important aspects. The central issue is the use of probability distributions and their time evolution to describe quantum processes. Linde, Starobinsky, Guth and Pi, Bardeen and Bublik, and Rey arrive at an evolution equation for the probability distribution of the inflation which has the form of a classical Fokker-Planck (FP) equation. Because this equation is second order, there exists an associated Langevin equation with additive Gaussian Markovian noise.[7] This noise arises from the flow of initially small-scale harmonic-oscillator quantum fluctuations across the effective particle horizon. In a nutshell, they are describing quantum mechanics with dissipation stochastically by adding a white-noise term with specific properties (i.e., fluctuation-dissipation relation) to the classical equations of motion and determining quantum expectation values by taking ensemble averages. This cannot be entirely correct except under special circumstances. The problem is that because of the uncertainty principle, a phase-space description of quantum mechanics based on real positive-definite (i.e., classical) probability distributions is impossible.[8] We will show in a subsequent paper that if one is content with stochastically describing a quantum system to order $\hbar$ which is dominated by friction then such a description can be based on classical probability distributions. Thus, the work presented here in no way invalidates the work of Linde, Starobinsky, etc. However, going beyond the order-$\hbar$ approximation inevitably leads to a breakdown of the classical phase-space description.

Graziani and Olynyk prefer to base their work on the so-called quasiprobability functions (real but not positive definite) of Wigner.[9] This formulation of quantum mechanics is well suited to dissipative systems and involves a quantum probability distribution function $W(\alpha,\beta;t)$ that can be used to compute quantum averages just as classical distributions are used to compute ensemble averages. However, the dynamical evolution of $W(\alpha,\beta;t)$ obeys a generalized FP or Wigner equation that attains the classical form in the $\hbar\to 0$ limit. Because the Wigner equation exactly describes quantum mechanics and in the presence of dissipation preserves the canonical commutation relations to all orders in $\hbar$, we feel it presents the most useful tool for analyzing the nonequilibrium nature of the dynamical inflationary phase transition.

It is obvious that quantum probability distributions and their evolution based on the Wigner equation are going to play a pivotal role in our understanding of the early Universe. However, this independent representation of quantum mechanics is not well known in the particle astrophysics community. The motivation of this paper is to bring about an understanding of the Wigner formalism by studying the stationary or equilibrium properties of the Wigner equation for nondissipative quantum systems in one dimension (i.e., ordinary quantum mechanics). Because the Wigner equation can be written as an expansion in $\hbar$, it can be truncated and solved analytically. We show in this paper how $W(\alpha,\beta;t\to\infty)$ contains all of the quantum-mechanical information of the ground state. We do this by showing how one computes to order $\hbar$ the ground-state energy and the effective potential for a variety of nonlinear tree-level potentials. Although these results are not directly relevant to the early Universe, they do illustrate the usefulness of the Wigner formalism. The purpose of this paper is to set the stage whereby the applications of this formalism to the inflationary phase transition will be set.

This paper is organized as follows. Section II contains a discussion of the relationship between quantum probability distributions and the evolution of the large-scale behavior of a scalar field. Section III contains a review of the Wigner formulation of quantum mechanics. Included is an explicit demonstration of the equivalence between the Schrödinger equation and the Wigner equation for nondissipative quantum systems. Section IV presents the calculation of the approximate ground-state energy levels for various nonlinear potentials. Detailed comparisons are made with known results. In Sec. V we present the computation of the quantum-mechanical effective potential. In Sec. VI we present our conclusions.

II. QUANTUM PROBABILITY DISTRIBUTIONS AND THE COARSE-GRAINED INFLATON FIELD

We start with a real gauge-singlet quantum scalar field in the presence of a Friedmann-Robertson-Walker (FRW) background geometry.[10] The theory is identified by its canonical position and momentum field operators in the Heisenberg representation: $\hat{\Phi}(\mathbf{x},t)$ and $\hat{\Pi}_\mu(\mathbf{x},t)$. In addition, there exists a Lagrange density $\mathcal{L}[\Phi(x), \partial_\mu\Phi(x)]$ where integration over a space-time volume yields the curved-space-time matter action.[10] We now choose $(\mathbf{x},t)$ to refer to the comoving coordinate system. If $a(t)$ is the FRW scale factor, then the causal or particle horizon is given by

$$l(t)=\int_0^t \frac{dt}{a(t)}=\begin{cases} t & \text{Minkowski,} \\ \frac{1}{2}t^{1/2} & \text{radiation dominated,} \\ \frac{1}{H}(1-e^{-Ht}) & \text{de Sitter.} \end{cases} \tag{1}$$

As stated in the Introduction, properties of the scalar field on scales less than $l(t)$ hold no interest for the problem at hand. We may therefore ask the following question: what effective theory governs the large-scale dynamics of the inflation? This question is of the type posed quite often in statistical mechanics and critical phenomena. Starobinsky has found that if one is interested only in scales larger than the microphysical or effective particle horizon, then the long-range dynamics is determined by a stochastically varying force and the *tree-level potential*. We wish to prove here, albeit heuristically, that the field theory (in the comoving frame), on scales larger than $l(t)$, actually behaves as a (dissipative) quantum-mechanical system. One can already see something like this going on in Starobinsky's Langevin equation:

$$\frac{\partial}{\partial t}\underset{\sim}{\phi}(\mathbf{x},t)=\frac{-1}{3H}\frac{\delta U}{\delta \underset{\sim}{\phi}}[\underset{\sim}{\phi}(\mathbf{x},t)]+\eta(\mathbf{x},t)\ ,$$
$$\langle \eta(\mathbf{x},t)\eta(\mathbf{y},t')\rangle=\frac{\hbar H^3}{4\pi^2}\frac{\sin[\epsilon\,|\,x-y\,|\,/\tau(t)]}{\epsilon\,|\,x-y\,|\,/\tau(t)}\delta(t-t')\ , \tag{2}$$

where $\epsilon \ll 1$ and $\tau(t)=(1/H)e^{-Ht}$. $\underset{\sim}{\phi}(\mathbf{x},t)$ is the quasihomogeneous inflaton field which operates only on scales greater than the microphysics horizon. $\underset{\sim}{\phi}(\mathbf{x},t)$ is called quasihomogeneous because the field is homogeneous over a microphysics horizon. In addition, there are no spatial derivatives in (2). On scales greater than the microphysics horizon, the field varies spatially. This spatial variation of $\underset{\sim}{\phi}(\mathbf{x},t)$ on large scales arises from the flow of initially small-scale harmonic-oscillator quantum fluctuations across the effective particle horizon. $U[\underset{\sim}{\phi}(\mathbf{x},t)]$ is the tree-level potential for the long-range inflaton. $\underset{\sim}{\phi}(\mathbf{x},t)$ is a classical stochastic variable. Equation (2) is what one would naively write down if one were asked to quantize a classical system stochastically (in real time) by adding a noise term to the classical equations of motion

$$\ddot{\underset{\sim}{\phi}}+3H\dot{\underset{\sim}{\phi}}+U'[\underset{\sim}{\phi}]=0 \tag{3}$$

and invoking the friction-dominated condition $\ddot{\underset{\sim}{\phi}} \ll 3H\dot{\underset{\sim}{\phi}}$. The effect of noise is to reproduce the effect of quantum fluctuations. In addition, the diffusion constant $H^3/4\pi^2$ implies that the equilibrium fluctuations attain the Bunche-Davies value.[4] This is essentially the phenomenological approach of Graziani and Olynyk. As mentioned in the Introduction however, this approach is not valid when going beyond order $\hbar$. Quantizing a classical system by adding *white noise* to the deterministic equations of motion inevitably leads to a breakdown of the canonical commutation relations.[11] We will show in a subsequent paper that a classical system can be quantized by introducing multiplicative non-Gaussian Markovian noise to the deterministic equations of motion.

To get back to the problem at hand, it should be stated that deriving a large-scale action from a microphysics action is in general a complex problem.[12] However, insight can be gained into the long-range physics by considering a few simple principles. The first step is to define a coarse-grained or smeared field operator $\hat{\phi}_{\mathbf{X}}(t)$ which is defined as the spatial average of $\hat{\Phi}(\mathbf{x},t)$ over a causal horizon volume. Although we explicitly mention only $\hat{\phi}_{\mathbf{X}}(t)$ it should be assumed that the discussion also applies to $\hat{\Pi}_{\mathbf{X}}(t)$; the coarse-grained momentum. The index $\mathbf{X}$ is a label referring to the "cell" over which $\hat{\Phi}(\mathbf{x},t)$ is averaged. $\hat{\phi}_{\mathbf{X}}(t)$ is an operator defined by

$$\hat{\phi}_{\mathbf{X}}(t)=\frac{1}{V_{\mathbf{X}}}\int_{V_{\mathbf{X}}} d^3x\ \hat{\Phi}(\mathbf{x},t)\ , \tag{4}$$

where $|\,V_{\mathbf{X}}\,|=\frac{4}{3}\pi l^3(t)$.

It is important to realize that the coarse-graining volume is constant in the comoving frame only for a system in de Sitter space where $Ht \gg 1$. Otherwise, the above procedure is complicated by the fact that the averaging volume changes in time. This is what differentiates the smearing procedure discussed here from classical statistical mechanics and even the coarse-graining procedure of Starobinsky. Although computationally difficult, spatial averaging over a causal horizon volume leads to a conceptual simplification of the quantum scalar field theory. The details are complicated and will be reported elsewhere. For our purposes, a heuristic argument will suffice.

The large-scale region (or cell) over which the smearing takes place is equal to the causal horizon. We assign a "block" averaged operator $\hat{\phi}_{\mathbf{X}}(t)$ to each cell. Because each cell lies outside of its neighbors' light cone, we expect spatial correlations $\langle \hat{\phi}_{\mathbf{X}}(t)\hat{\phi}_{\mathbf{Y}}(t)\rangle |_{\mathbf{X}\neq\mathbf{Y}}$ to be negligible. The implication is that the cell of interest becomes independent of all the other cells. We may think of each large-scale region (labeled by **X**) as being an independent quantum-mechanical system. Spatial degrees of freedom no longer are relevant, other than as a label for a particular member of the ensemble. The super-Universe [the Universe on scales $\gtrsim l(t)$], in which the scalar field theory operates, is broken up into an infinite number of identical quantum systems by coarse graining over a horizon volume (our presently observed Universe is but one of these quantum systems). With no interactions between them, each system or cell is a member of a quantum-mechanical ensemble with every element of the ensemble in a state denoted by $|\psi\rangle$. The relevant degree of freedom is now $\hat{\phi}(t)$ [the horizon averaged $\hat{\Phi}(\mathbf{x},t)$] which is measured on each element of the ensemble [i.e., cell] and averaged. (The **X** label is dropped because it is irrelevant.) This is the meaning of $\langle\hat{\phi}(t)\rangle$. Note that this procedure is identical to taking $\hat{\phi}_{\mathbf{X}}(t)$, measuring it for each **X**, and then averaging. We therefore reduce the problem under consideration from a quantum field theory which operates on all scales to a quantum-mechanical system in one dimension (t) whose canonical position and momentum are the Heisenberg operators $\hat{\phi}(t)$, $\hat{\Pi}(t)$ and which operates only on extra horizon scales.

So far, the only mention of space-time curvature effects have been made when choosing a causal horizon volume. As is well known, the classical scalar field $\Phi(\mathbf{x},t)$ in a FRW background obeys

$$\ddot{\Phi}(\mathbf{x},t)+3\left[\frac{\dot{a}(t)}{a(t)}\right]\dot{\Phi}(\mathbf{x},t)-\left[\frac{a(0)}{a(t)}\right]^2\nabla^2\Phi(\mathbf{x},t)+V'[\Phi(\mathbf{x},t)]=0\ , \qquad (5)$$

where $V[\Phi(\mathbf{x},t)]$ is the tree-level potential on all scales and $V'[\Phi(\mathbf{x},t)]$ is $\delta V[\Phi(\mathbf{x},t)]/\delta\Phi(\mathbf{x},t)$. Performing the coarse graining (4) yields

$$\frac{d^n\phi_{\mathbf{X}}}{dt^n}=\frac{1}{V_{\mathbf{X}}}\int_{V_{\mathbf{X}}}d^3x\frac{d^n\Phi}{dt^n}(\mathbf{x},t)\ ,$$
$$\frac{\delta U}{\delta\phi_{\mathbf{X}}}[\phi_{\mathbf{X}}]=\frac{1}{V_{\mathbf{X}}}\int_{V_{\mathbf{X}}}d^3x\frac{\delta V}{\delta\Phi(\mathbf{x},t)}[\Phi(\mathbf{x},t)]\ , \qquad (6)$$
$$\frac{1}{l(t)}|[\phi_{\mathbf{X}+\mathbf{Y}}(t)-\phi_{\mathbf{X}}(t)]| \simeq\frac{1}{V_{\mathbf{X}}}\int_{V_{\mathbf{X}}}d^3x\,|\nabla\phi(\mathbf{x},t)|\quad(\text{see Ref. 12})\ .$$

$U[\phi_{\mathbf{X}}]$ is the coarse-grained tree-level potential, $\phi_{\mathbf{X}}$ is the coarse-grained classical scalar field, and **Y** is the label for cells neighboring **X**. It operates on scales greater than a causal horizon. Therefore, after coarse graining, the spatial varying term in (5) is suppressed by a factor $a^{-2}(t)l^{-2}(t)$. We therefore ignore it. Equation (5) becomes, approximately

$$\ddot{\phi}_{\mathbf{X}}(t)+3\left[\frac{\dot{a}(t)}{a(t)}\right]\dot{\phi}_{\mathbf{X}}(t)+U'(\phi_{\mathbf{X}}(t))=0\ . \qquad (7)$$

We can now drop the **X**, it is irrelevant. Equation (7) describes the classical dynamics of the large-scale inflaton. It is just the one-dimensional equation of motion of a nonlinear system subjected to a time-dependent dissipation $3[\dot{a}(t)/a(t)]$. Note that $U(\phi_{\mathbf{X}}(t))$ does not necessarily have the same functional form as $V[\Phi(\mathbf{x},t)]$.

Quantizing $\Phi(\mathbf{x},t)$ leads of course to a discussion of quantum field theory in curved space-time. However, quantizing $\phi(t)$ leads to a discussion of the quantum mechanics of one-dimensional dissipative systems.[13] Therefore, the dynamics of the long-range inflaton is governed by time-dependent dissipative quantum mechanics. In de Sitter space, which will be the focus of later papers, we are dealing with a simpler case; the quantum mechanics of linearly dissipative systems.

How does one go about quantizing (7) for $\dot{a}(t)/a(t)\sim$const? One can either follow Dekker[14] and do it canonically or follow Caldeira and Leggett[15] and do it via path integration. In either case, the procedure is nontrivial. The reason is that the Hamiltonian is no longer a constant of the motion. Unless one is careful this leads to the commutator $[\hat{\phi}(t),\hat{\Pi}(t)]$ decaying in time. The end result of both Dekker's work and Caldeira and Leggett's is that it is possible to write down a time-evolution equation for the density operator $\hat{\rho}$. Fourier transforming the matrix elements of $\hat{\rho}$ leads to a function $W(\alpha,\beta;t)$, the so-called quasiprobability function of Wigner, that behaves as a classical distribution in some respects but obeys a generalized Fokker-Planck equation or Wigner equation. If the dissipation is time dependent, the Wigner equation must be modified to accommodate nonstationary random processes.[16] For the purposes of discussion we will always assume $\dot{a}(t)/a(t)\sim$const.

To sum up, we have given arguments supporting the fact that the causal horizon averaged inflaton in the comoving frame becomes a dissipative quantum system in one dimension. The quantum mechanics of dissipative systems on the other hand are governed by the Wigner equation. This is the motivation for studying Wigner distributions in their own right and applying them to the early Universe. Before we discuss the Wigner formalism, two things should be mentioned. First, why the Wigner formalism? How come one cannot simply modify the Schrödinger equation for dissipative systems? So far as we are aware, all efforts in this direction have been fruitless.[17] For example, Kostin ends up with a nonlinear Schrödinger equation that obscures the usual Hilbert-space formulation of quantum mechanics. Second, the Wigner formalism presented here is frame dependent and nonrelativistic. Are we justified in applying it to cosmology? The answer is yes.

The focus of attention in this paper is the comoving frame or so-called synchronous gauge. In this coordinate gauge the equation of motion for the large-scale scalar field $\phi(t)$ (7) is reminiscent of a damped nonrelativistic system with a single degree of freedom which is acted upon by a force $-U'(\phi(t))$. Of course, it must be remembered that the coordinate time t that appears in (7) and the Wigner equation measures proper time along lines of constant $\mathbf{x}$. The relativistic quantum field theory in curved space-time is identified by the Heisenberg operators $\hat{\Phi}(\mathbf{x},t)$, $\hat{\Pi}_\mu(\mathbf{x},t)$, the Lagrange density $\mathcal{L}$, and the equal-time canonical commutation relations (ETCCR's) $[\hat{\Phi}(\mathbf{x},t), \hat{\Pi}(\mathbf{y},t)]=i\hbar\delta^{(3)}(\mathbf{x}-\mathbf{y})$. In this paper we are interested in the large-scale physics of this system. Performing the coarse-graining procedure, the relevant dynamical variables are now $\hat{\phi}(t)$ and $\hat{\Pi}(t)$. It is easy to verify that the ETCCR become $[\hat{\phi}_{\mathrm{X}}(t), \hat{\Pi}_{\mathrm{Y}}(t)]=i\hbar/V_{\mathrm{X}}$ which is, up to a factor of V_{X}^{-1}, nothing more than the canonical commutation relation of nonrelativistic quantum mechanics. Finally, by *choosing a gauge* (the comoving frame), the coarse-grained equation of motion for $\phi(t)$ takes on a nonrelativistic form. By restricting ourselves to de Sitter space, this equation of motion possesses linear damping. Therefore the large-scale physics of the inflaton in the comoving frame is determined by the quantum mechanics of linearly dissipative systems. The appropriate description of the dynamics of such a system is the Wigner equation. Admittedly, the equation is frame dependent; in another coordinate system, the equation for $\phi(t)$ will change and so will the Wigner equation. But, by specifying a coordinate gauge, the large-scale physics of the inflaton in the de Sitter stage can be thought of under the broader category of quantized linearly dissipative systems with a few degrees of freedom. Therefore, the tools used to study this type of statistical mechanics problem (e.g., stochastic methods) can be applied to cosmology. In addition, the results obtained by coarse graining a scalar field theory and specifying the comoving frame become easy to understand physically and amenable to comparison with previous works[2–6] (who also focus attention on the comoving frame). Although a covariant formulation of the Wigner distribution is ultimately important, at this stage, we feel the theory presented here offers clearer insight into the stochastic nature of the quantum fluctuations of the inflaton.

III. THE WIGNER FORMALISM

A. Review

In 1932, Wigner[9] presented a formulation of quantum mechanics based on a configuration-space description. The starting point for the Wigner formalism is the quasiprobability or Wigner distribution

$$W(\alpha,\beta;t)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dy \left\langle \alpha-\frac{y}{2}\right|\hat{\rho}\left|\alpha+\frac{y}{2}\right\rangle e^{i\beta y/\hbar} .$$

The ordered pair (α,β) describes a classical point in the mathematical configuration space that is analogous to the ordered (q,p) in classical phase space. $W(\alpha,\beta;t)$ contains all the information of a system in a mixed state with density operator $\hat{\rho}$. The appeal of the Wigner representation is that $W(\alpha,\beta;t)$ acts as a classical distribution in the (α,β) phase space. Because of this, it has proven to be a valuable computation tool as well as providing insights into the connections between classical and quantum mechanics.[18] To calculate the quantum expectation value of an arbitrary operator $\hat{A}(\hat{q},\hat{p})$ one associates a Weyl classical equivalent object via the correspondence

$$\alpha^n\beta^m \leftrightarrow \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\hat{q}^{\,n-k}\hat{p}^{\,m}\hat{q}^{\,k} ; \tag{8}$$

the expectation value $\langle \hat{A} \rangle$ becomes an ensemble average

$$\begin{aligned}\langle \hat{A}(\hat{q},\hat{p})\rangle &= \mathrm{Tr}[\hat{\rho}\hat{A}(\hat{q},\hat{p})] \\ &= \int d\alpha\, d\beta\, A(\alpha,\beta)W(\alpha,\beta;t) \\ &= \langle A(\alpha,\beta)\rangle ,\end{aligned} \tag{9}$$

where $A(\alpha,\beta)$ is determined from (8) and $\hat{A}(\hat{q},\hat{p})$. The dynamics are determined by the motion of $W(\alpha,\beta;t)$ which is described by a generalized Fokker-Planck equation (sometimes referred to as a Kramers-Moyal equation). For ordinary quantum systems with no dissipation, $W(\alpha,\beta;t)$ obeys

$$\frac{\partial W(\alpha,\beta;t)}{\partial t}=-\frac{\partial}{\partial\alpha}[\beta W(\alpha,\beta;t)]-\frac{i}{2\pi\hbar^2}\int_{\infty}^{\infty}dy\, e^{i\beta y/\hbar}\left[U\left[\alpha-\frac{y}{2}\right]-U\left[\alpha+\frac{y}{2}\right]\right]\left\langle\alpha-\frac{y}{2}\,|\hat{\rho}|\,\alpha+\frac{y}{2}\right\rangle , \tag{10}$$

where $U(\cdots)$ is the external classical potential. This is known as the Wigner equation. Equation (10) can be expanded in powers of $\hbar$:

$$\frac{\partial W(\alpha,\beta;t)}{\partial t}=-\frac{\partial}{\partial\alpha}[\beta W(\alpha,\beta;t)]+\sum_{k=1,3,5,\ldots}^{\infty}\frac{1}{k!}\left[\frac{\hbar}{2i}\right]^{k-1}\frac{\partial^k U(\alpha)}{\partial\alpha^k}\frac{\partial^k W(\alpha,\beta;t)}{\partial\beta^k} . \tag{11}$$

By the Pawula theorem,[19] (11) implies that $W(\alpha,\beta;t)$ is not positive definite. For the case of *linear* dissipation, Eq. (11) can be generalized to

$$\frac{\partial W(\alpha,\beta;t)}{\partial t}=-\frac{\partial}{\partial\alpha}[\beta W(\alpha,\beta;t)]+\lambda\frac{\partial}{\partial\beta}[\beta W(\alpha,\beta;t)]+D\frac{\partial^2 W(\alpha,\beta;t)}{\partial\beta^2}+\sum_{k=1,3,5,\ldots}^{\infty}\frac{1}{k!}\left[\frac{\hbar}{2i}\right]^{k-1}\frac{\partial^k U(\alpha)}{\partial\alpha^k}\frac{\partial^k W(\alpha,\beta;t)}{\partial\beta^k}\ . \tag{12}$$

λ is the constant dissipation term while D is the diffusion constant. D is an arbitrary parameter which is fixed by requiring $W(\alpha,\beta;t)$ to have some specific form. For example, classically $D=\lambda kT$ implies that $W(\alpha,\beta;t)\to\infty=e^{[\beta^2/2+V(\alpha)]/kT}$. For the long-range inflaton in de Sitter space, we have $\lambda=3H$, $D=9H^5\hbar/8\pi^2$, and $\hbar\to\hbar H^3$ in Eq. (12). The reason for the factor of H^3 in front of $\hbar$ is the fact that the commutator between the coarse-grained $\hat{\phi}(t)$ and $\hat{\Pi}(t)$ is proportional to the inverse of the coarse-grained volume $V\sim H^{-3}$. Converting (12) into a Smoluchowski equation by assuming the evolution is friction dominated[19] we obtain

$$\frac{\partial P(\alpha,t)}{\partial t}=\frac{1}{3H}\frac{\partial}{\partial\alpha}\left[U'(\alpha)+\frac{3H^4\hbar}{8\pi^2}\frac{\partial}{\partial\alpha}\right]P(\alpha,t)+O(\hbar^2)\ ,$$

where $P(\alpha,t)=\int d\beta\, W(\alpha,\beta;t)$

This is just the result from Starobinsky[2] and Rey.[6] The choice of D is dictated by demanding

$$\lim_{t\to\infty}\langle\hat{\Phi}^2(t)\rangle=\lim_{t\to\infty}\langle\alpha^2\rangle=\frac{3H^4\hbar}{8\pi^2m^2}$$

for $U(\alpha)\simeq(m^2/2)\alpha^2$ near the stationary point (i.e., the equilibrium or stationary state is the Bunche-Davies vacuum).

As a general reference we list the various properties of $W(\alpha,\beta;t)$ for a pure state (and no dissipation) as listed by Hillery *et al.*[9]

(1) $W(\alpha,\beta;t)$ is a Hermitian form of the state vector $|\psi\rangle$. This implies that (a) $W(\alpha,\beta;t)$ is real, (b) $\int d\alpha\, W(\alpha,\beta;t)=\langle\beta|\hat{\rho}|\beta\rangle$, and (c) $\int d\alpha\, d\beta\, W(\alpha,\beta;t)=\mathrm{Tr}(\hat{\rho})=1$.

(3) $W(\alpha,\beta;t)$ is Galilei invariant and invariant with respect to space and time reflections.

(4) $\int d\alpha\, d\beta[W(\alpha,\beta;t)]^2$
$<1/2\pi\hbar$ (mixed state)
$=1/2\pi\hbar$ (pure state) .

(5) $\int d\alpha\, d\beta\, A(\alpha,\beta)B(\alpha,\beta)=(2\pi\hbar)\mathrm{Tr}(\hat{A}\hat{B})$.

Item (4) is especially important because it shows that not all $W(\alpha,\beta;t)$ are allowed. For example, it rules out highly peaked distributions such as the classical distribution $\delta(\alpha-\alpha_0)\delta(\beta-\beta_0)$. Item (4) is a constraint equation that is used in conjunction with (11) in determining admissible Wigner distributions. For Gaussian states it is the integral equivalent of the uncertainty principle. In the next section we will prove its usefulness by showing that it can be used to determine ground-state energies.

B. The equivalence between the Wigner and Schrödinger equations

For pedagogic purposes we demonstrate that (11) is identical to

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2}\frac{\partial^2\psi}{\partial\alpha^2}+U(\alpha)\psi\ . \tag{13}$$

For a pure state our starting point is Eq. (10) with

$$W(\alpha,\beta;t)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dy\, e^{i\beta y/\hbar}\psi^*\left[\alpha+\frac{y}{2}\right]\psi\left[\alpha-\frac{y}{2}\right]\ . \tag{14}$$

Defining $F(\alpha,y)\equiv\psi^*(\alpha+y/2)\psi(\alpha-y/2)$ and observing that (14) is a Fourier transform we have

$$\frac{\partial F(\alpha,y)}{\partial t}=-i\hbar\frac{\partial^2 F(\alpha,y)}{\partial\alpha\partial y}-\frac{i}{\hbar}\left[U\left[\alpha-\frac{y}{2}\right]-U\left[\alpha+\frac{y}{2}\right]\right]F(\alpha,y)\ . \tag{15}$$

We now expand (15) in powers of y:

$$\begin{aligned}
&F(\alpha,y)=C_0P_0(\alpha)+C_1yP_1(\alpha)+C_2y^2P_2(\alpha)+\cdots\ ,\\
&C_0=1,\quad P_0=\psi^*(\alpha)\psi(\alpha)\ ,\\
&C_1=\tfrac{1}{2},\quad P_1=\psi\frac{\partial\psi^*}{\partial\alpha}-\psi^*\frac{\partial\psi}{\partial\alpha}\ ,\\
&C_2=\tfrac{1}{8},\quad P_2=\psi\frac{\partial^2\psi^*}{\partial\alpha^2}-2\frac{\partial\psi}{\partial\alpha}\frac{\partial\psi^*}{\partial\alpha}+\psi^*\frac{\partial^2\psi}{\partial\alpha^2}\ ,\\
&C_3=\frac{1}{48},\quad P_3=\psi\frac{\partial^3\psi^*}{\partial\alpha^3}-3\frac{\partial\psi}{\partial\alpha}\frac{\partial^2\psi^*}{\partial\alpha^2}+3\frac{\partial\psi^2}{\partial\alpha}\frac{\partial^2\psi}{\partial\alpha^2}-\psi^*\frac{\partial^3\psi}{\partial\alpha^3}\ .
\end{aligned} \tag{16}$$

Let $\theta\equiv iy/\hbar$ and denote (16) by

$$F(\alpha,\theta)=\sum_{n=0}^{\infty}\hbar^n\theta^n d_nP_n(\alpha)\ . \tag{17}$$

Substituting (17) into (15) we obtain a hierarchy of equations. We write down the first four.

(1) $n=0$

$$\frac{\partial P_0(\alpha)}{\partial t}=\hbar(-ic_1)\frac{\partial P_1(\alpha)}{\partial\alpha}\ . \tag{18}$$

This is just the continuity equation.

(2) $n=1$

$$(-ic_1)\frac{\partial P_1(\alpha)}{\partial t}=2\hbar(-c_2)\frac{\partial P_2(\alpha)}{\partial\alpha}+U'(\alpha)\left[\frac{c_0}{\hbar}\right]P_0(\alpha)\ , \tag{19}$$

(3) $n=2$

$$(-c_2)\frac{\partial P_2(\alpha)}{\partial t}=3\hbar(ic_3)\frac{\partial P_3(\alpha)}{\partial\alpha}+U'(\alpha)\left[\frac{-ic_1}{\hbar}\right]P_1(\alpha) \tag{20}$$

(4) $n=3$

$$(ic_2)\frac{\partial P_3(\alpha)}{\partial\alpha}=4\hbar(c_4)\frac{\partial P_4(\alpha)}{\partial\alpha}+U'(\alpha)\left[\frac{-c_2}{\hbar}\right]P_2(\alpha)-\frac{1}{24}U(\alpha)\left[\frac{c_0}{\hbar}\right]P_0(\alpha)\ , \tag{21}$$

etc.

We now show how one generates this hierarchy from the Schrödinger equation. The $n=0$ equation is obvious and well known. To generate the $n=1$ equation we first write

$$\frac{\partial P_1(\alpha)}{\partial t}=\frac{\partial}{\partial t}\left[\psi\frac{\partial\psi^*}{\partial\alpha}-\psi^*\frac{\partial\psi}{\partial\alpha}\right]. \tag{22}$$

We then use the Schrödinger equations

$$\frac{\partial\psi}{\partial t}=\frac{i\hbar}{2}\frac{\partial^2\psi}{\partial\alpha^2}-\frac{i}{\hbar}U(\alpha)\psi\ ,\qquad \frac{\partial\psi^*}{\partial t}=-\frac{i\hbar}{2}\frac{\partial^2\psi^*}{\partial\alpha^2}-\frac{i}{\hbar}U(\alpha)\psi^*\ , \tag{23}$$

to replace the time derivatives in (22) by α derivatives. We then arrange terms and find that Eq. 22) becomes Eq. (19). This procedure can be repeated for any $(\partial/\partial t)P_m(\alpha)$. The upshot is that generating $\partial P_m(\alpha)/\partial t$ via the Schrödinger equation is equivalent to generating it via the Wigner equation. It is important to note that the non-Gaussian terms in (11) are mandatory if one is to make the Wigner and Schrödinger equations equivalent.

IV. THE COMPUTATION OF THE GROUND-STATE ENERGY IN THE WIGNER FORMALISM

The previous calculation demonstrated the equivalence of the Wigner and Schrödinger's equation (11) for nondissipative systems. As such, the distribution $W(\alpha,\beta;t)$ contains all of the quantum-mechanical information of the system. In particular, the stationary or equilibrium distribution $W(\alpha,\beta;t\to\infty)=W(\alpha,\beta)$ should therefore determine the properties of the stationary states of the system. Of central interest here is how $W(\alpha,\beta)$ can be used to yield ground-state energies. To start, we must solve

$$\frac{\partial W(\alpha,\beta;t)}{\partial t}=-\frac{\partial}{\partial\alpha}[\beta W(\alpha,\beta;t)]+\sum_{k=1,3,5,\ldots}\frac{1}{k!}\left[\frac{\hbar}{2i}\right]^{k-1}\frac{\partial^k U(\alpha)}{\partial\alpha^k}\frac{\partial^k W(\alpha,\beta;t)}{\partial\beta^k} \tag{24}$$

subject to the classical constraints

$$\int d\alpha\, d\beta W(\alpha,\beta;t)=1\ , \tag{25}$$

$$\frac{\partial W(\alpha,\beta;t)}{\partial t}=0\ , \tag{26}$$

and the quantum constraint (we assume a pure state)

$$\int d\alpha\, d\beta[W(\alpha,\beta;t)]^2=\frac{1}{2\pi\hbar}\ . \tag{27}$$

Equation (24) is in general difficult to solve even for stationary states.[20] The complication is the presence of non-Gaussian terms $\partial^k W(\alpha,\beta;t)/\partial\beta^k$ for $k\geq 3$. These terms are always present whenever $U(\alpha)$ deviates from an oscillator. However, because (24) is an expansion in $\hbar$, we may find an approximate solution to the Wigner equation. To order $\hbar$, the Wigner equation takes on the deceptively classical form

$$\frac{\partial W(\alpha,\beta;t)}{\partial t}=-\frac{\partial}{\partial\alpha}[\beta W(\alpha,\beta;t)]+U'(\alpha)\frac{\partial W(\alpha,\beta;t)}{\partial\beta} \tag{28}$$

subject to the constraints (25)–(27). What prevents $W(\alpha,\beta;t)$ from equaling its classical ($\hbar\to 0$) value is the quantum constraint (27). Assuming (28) is normalized we wish to find

$$0=-\beta\frac{\partial W(\alpha,\beta)}{\partial\alpha}+U'(\alpha)\frac{\partial W(\alpha,\beta)}{\partial\beta}\ , \tag{29}$$

where

$$\int d\alpha\, d\beta[W(\alpha,\beta)]^2=\frac{1}{2\pi\hbar}\ .$$

We find

$$W(\alpha,\beta)=Ne^{-\xi[\beta^2/2+U(\alpha)]} \tag{30}$$

(N is the normalization constant).

Classically ξ is an arbitrary parameter which is fixed by the equipartition theorem (i.e., $\xi\sim 1/kT$). As $T\to 0$ this obviously breaks down. Instead, as $T\to 0$, ξ is determined by the quantum constraint (27). Note that the order $\hbar$ distribution $W(\alpha,\beta)$ is positive definite.

Although ξ has the dimensions of 1/energy, we as yet cannot be sure that ξ^{-1} is exactly the ground-state energy E_{gs}. We know E_{gs} is given by

$$E_{gs}=\tfrac{1}{2}\langle\hat{p}^{\,2}\rangle_{gs}+\langle U(\hat{q})\rangle_{gs} \tag{31}$$

or by the Weyl correspondence

$$E_{gs}=\tfrac{1}{2}\langle\beta^2\rangle+\langle U(\alpha)\rangle\ , \tag{32}$$

where we have assumed $U(\hat{q})$ is some polynomial in $\hat{q}$. The ensemble averages in (32) are to be evaluated with respect to $W(\alpha,\beta)$. Let $U(\alpha)=\lambda\alpha^k$, then

$$\langle U(\alpha)\rangle=\frac{1}{\xi k}$$

and

$$\left\langle\frac{\beta^2}{2}\right\rangle=\frac{1}{2\xi}\ .$$

Therefore,

$$E_{\rm gs}=\frac{1}{\xi}\left[\frac{1}{2}+\frac{1}{k}\right]\ . \tag{33}$$

To find ξ, we substitute (30) into (27) and obtain

$$\sqrt{\pi}\xi\frac{\int d\alpha\, e^{-2\xi U(\alpha)}}{\left[\int d\alpha\, e^{-\xi U(\alpha)}\right]^2}=\frac{1}{\hbar}\ . \tag{34}$$

For the harmonic-oscillator ($k=2$) Eq. (30) is exact and we expect (34) to yield the correct ground-state energy. Let $U(\alpha)=(\omega^2/2)\alpha^2$, performing the Gaussian integrations, we obtain $E_{\rm gs}=1/\xi=\hbar\omega/2$. For $k>2$, the solution (30) is only an approximation and the value for $E_{\rm gs}$ obtained from (33) and (34) is therefore only approximate. For $U(\alpha)=\lambda\alpha^k$, the integrals in (34) can be solved analytically. We find

$$\frac{1}{\xi}=\left[\frac{k\sqrt{\pi}}{2^{1+1/k}\Gamma(1/k)}\right]^{2k/(k+2)}\hbar^{2k/(k+2)}\lambda^{2/(k+2)}\ . \tag{35}$$

From (33), $E_{\rm gs}$ is therefore

$$E_{\rm gs}=\left[\frac{1}{2}+\frac{1}{k}\right]\left[\frac{k\sqrt{\pi}}{2^{1+1/k}\Gamma(1/k)}\right]^{2k/(k+2)}\times\hbar^{2k/(k+2)}\lambda^{2/(k+2)}\ . \tag{36}$$

We are now in a position to compare (36) with the results of Hioe, MacMillen, and Montroll.[21] They computed the energy levels of oscillators with α^{2n} anharmonicity. We will take $k=4$, 6, 8 ($n=2,3,4$) and compare our $E_{\rm gs}$ with the exact result.

(a) $k=4$,

$$E_{\rm gs}=0.578\hbar^{4/3}\lambda^{1/3},\quad E_{\rm gs}^{\rm exact}=0.668\hbar^{4/3}\lambda^{1/3}\ ,$$

% difference = 13.5%.

(b) $k=6$,

$$E_{\rm gs}=0.524\hbar^{3/2}\lambda^{1/4},\quad E_{\rm gs}^{\rm exact}=0.681\hbar^{3/2}\lambda^{1/4}\ ,$$

% difference = 23%.

(c) $k=8$,

$$E_{\rm gs}=0.495\hbar^{8/5}\lambda^{1/5},\quad E_{\rm gs}^{\rm exact}=0.704\hbar^{8/5}\lambda^{1/5}\ ,$$

% difference = 29%. It is clear that the estimates of $E_{\rm gs}$ based on the truncated Wigner equation and the quantum constraint get worse as the order of the nonlinearity increases. This is to be expected. As k increases, we are effectively dropping more and more terms from the Wigner equation. What is comforting to know is that the simple procedure of the truncated Wigner equation plus quantum constraint yields reasonable ground-state properties.

As a last example, we compute the approximate $E_{\rm gs}$ for the bistable potential $V(\hat{q})=-(\gamma/2)\hat{q}^2+(g/4)\hat{q}^4$. From (34),

$$\sqrt{\pi}\xi\int d\alpha\exp\left[-2\xi\left[\frac{-\gamma}{2}\alpha^2+\frac{g}{4}\alpha^4\right]\right]=\frac{1}{\hbar}\ .$$

The integrals can be evaluated analytically and we obtain

$$2\sqrt{\xi\pi}\frac{g}{2\gamma}\frac{K_{1/4}\left[\frac{\xi\gamma^2}{4g}\right]}{K_{1/4}^2\left[\frac{\xi\gamma^2}{8g}\right]}=\frac{1}{\hbar} \tag{37}$$

[$K_\nu(x)$ is the Bessel function].

Solving (37) for ξ in general can only be done numerically. However, for $\xi\gamma^2/4g\gg 1 K_\nu(x)\sim\sqrt{\pi}/2xe^{-x}$ and (37) can be solved for ξ explicitly. We obtain

$$\frac{1}{\xi}=\frac{\sqrt{2\gamma}}{2}\hbar\ . \tag{38}$$

The last step to be performed is to find the relationship between ξ and $E_{\rm gs}$. As before,

$$E_{\rm gs}=\left\langle\frac{\beta^2}{2}\right\rangle+\langle U(\alpha)\rangle\ ,$$

where $\langle\beta^2/2\rangle=1/2\xi$. $\langle U(\alpha)\rangle$ is more difficult to compute but it is still tractable:

$$\langle U(\alpha)\rangle=-\frac{1}{I(\xi)}\frac{d}{d\xi}I(\xi)\ , \tag{39}$$

where

$$I(\xi)=\int d\alpha\exp\left[-\xi\left[\frac{-\gamma}{2}\alpha^2+\frac{g}{4}\alpha^4\right]\right]\ . \tag{40}$$

Equation (40) can be evaluated explicitly and substituted into (39). For $\xi\gamma^2/4g\gg 1$, (39) reduces to $\langle U(\alpha)\rangle\simeq 1/2\xi$. Therefore,

$$E_{\rm gs}=\frac{1}{\xi}=\frac{\sqrt{2\gamma}}{2}\hbar\ . \tag{41}$$

This is just what we would expect. The condition $\gamma^2/4g\gg 1/\xi$ implies that the ground-state energy lies well below the top of the energy barrier that separates the two ground states. Approximating the potential around $\alpha=\pm\sqrt{\gamma/g}$ by a harmonic oscillator, the effective frequency squared is just 2γ. Writing $E_{\rm gs}=\hbar\omega/2$, we obtain (41).

V. THE EFFECTIVE POTENTIAL IN ONE DIMENSION

The effective potential $V_{\rm eff}[\langle\hat{q}\rangle]$ is by now a standard tool used in theoretical physics. It represents the minimum expectation value of the energy for stationary

$\langle \hat{q}(t) \rangle$. The functional computation based on path integrals is by now also standard lore. In this section we wish to show how the effective potential in one dimension is computed within the mathematical phase-space formalism of Wigner. Again, we are discussing nondissipative quantum systems.

For our computation we will take

$$U(\hat{q}) = \frac{\mu^2}{2}\hat{q}^2 + \frac{g}{k}\hat{q}^k$$

or the Weyl equivalent $U(\alpha) = (\mu^2/2)\alpha^2 + (g/k)\alpha^k$. In addition, things will be simplified if we take g to be small. We again use the truncated form of the Wigner equation. This means our calculation of $V_{\rm eff}[\langle \hat{q} \rangle]$ will be performed to order $\hbar g$.

The approximate stationary solution to (24) is

$$W(\alpha,\beta) = N \exp\left[-\xi\left[\frac{\beta^2}{2} + U(\alpha)\right]\right]. \tag{42}$$

We define a generating function $T[J]$ for the stationary ensemble averages $\langle \alpha^n \rangle = \langle \hat{q}^n \rangle$:

$$T[J] = \int d\alpha\, d\beta\, e^{J\alpha} W(\alpha,\beta). \tag{43}$$

Equation (42) is now expanded in powers of g:

$$W(\alpha,\beta) = N \exp\left[-\xi\left[\frac{\beta^2}{2} + \frac{\mu^2}{2}\alpha^2\right]\right]\left[1 - \frac{\xi g}{k}\alpha^k\right] \tag{44}$$

and substituted into (43). We evaluate N via the normalization constraint and find

$$T[J] = e^{J/2\xi\mu^2}\left[1 - \frac{g}{k}\sum_{\substack{l=2\\ \text{even}}}^{k} \begin{bmatrix} k \\ l \end{bmatrix} \frac{(k-l-1)!!J^l}{\xi^{(k+l)/2-1}\mu^{k+l}}\right]. \tag{45}$$

The generating function for the nth-order cumulants (or in field-theory language, the connected Green's functions) is given by

$$W[J] = \ln Z[J]. \tag{46}$$

From (45) the lowest nontrivial term is

$$W[J] = \frac{J^2}{2\xi\mu^2} - \frac{g}{k}\sum_{\substack{l=2\\ \text{even}}}^{k} \begin{bmatrix} k \\ l \end{bmatrix} \frac{(k-l-1)!!J^l}{\xi^{(k+l)/2-1}\mu^{k+l}}. \tag{47}$$

The ground-state expectation value (in the presence of J) of $\hat{q}$ is given by

$$\begin{aligned} \langle \hat{q} \rangle &= \langle \alpha \rangle = \frac{\delta W[J]}{\delta J}, \\ \langle \hat{q} \rangle &= \langle \alpha \rangle \\ &= \frac{J^2}{\xi\mu^2} - \frac{g}{k}\sum_{\substack{l=2\\ \text{even}}}^{k} \begin{bmatrix} k \\ l \end{bmatrix} \frac{(k-l-1)!!J^l}{\xi^{(k+l)/2-1}\mu^{k+l}}. \end{aligned} \tag{48}$$

We now wish to solve for J as a function of $\langle \hat{q} \rangle$. To do this we solve (48) perturbatively and obtain

$$J = \xi\mu^2\langle q \rangle + \frac{g}{k}\sum_{\substack{l=2\\ \text{even}}}^{k} \begin{bmatrix} k \\ l \end{bmatrix} \frac{(k-l-1)!!}{\mu^{k-l}} l\,\xi^{(l-k)/2+1}\langle \hat{q} \rangle^{l-1}. \tag{49}$$

We perform a Legendre transform of $W[J]$ by

$$\theta[\langle \hat{q} \rangle] = J\langle \hat{q} \rangle - W[J]. \tag{50}$$

Using Eqs. (47) and (49) we have

$$\frac{\theta[\langle \hat{q} \rangle]}{\xi} = \frac{\mu^2}{2}\langle \hat{q} \rangle^2 + \frac{g}{k}\sum_{\substack{l=2\\ \text{even}}}^{k} \begin{bmatrix} k \\ l \end{bmatrix} \frac{(k-l-1)!!}{\mu^{k-l}\xi^{(k-l)/2}}\langle \hat{q} \rangle^l. \tag{51}$$

Performing the summation by writing the term with $l = k$ first and then $l = k-2$ second, etc., we obtain

$$\frac{\theta[\langle \hat{q} \rangle]}{\xi} = \frac{\mu^2}{2}\langle \hat{q} \rangle^2 + \frac{g}{k}\langle \hat{q} \rangle^k + \frac{g}{k}\begin{bmatrix} k \\ k-2 \end{bmatrix} \frac{\langle \hat{q} \rangle^{k-2}}{\mu^2\xi} + O(g/\xi^2). \tag{52}$$

Realizing that $\binom{k}{k-2} = k(k-1)/2$ and $1/\xi = \hbar\mu/2 + O(g)$ we have

$$\frac{\theta[\langle \hat{q} \rangle]}{\xi} = \frac{\mu^2}{2}\langle \hat{q} \rangle^2 + \frac{g}{k}\langle \hat{q} \rangle^k + \frac{\hbar g}{4\mu}(k-1)\langle \hat{q} \rangle^{k-2} + \cdots. \tag{53}$$

The first two terms in (53) represent the classical potential while the third term is the lowest-order perturbative quantum correction.

Coleman[22] has computed the effective potential associated with the tree-level $U(\hat{q})$ using standard summing-over-loops techniques. He finds

$$V_{\rm eff}[\langle \hat{q} \rangle] = U[\langle \hat{q} \rangle] + \frac{\hbar}{2}\sqrt{U''}[\langle \hat{q} \rangle]. \tag{54}$$

To see how this compares with (53), substituting $U[\langle \hat{q} \rangle] = (\mu^2/2)\langle \hat{q} \rangle^2 + (g/k)\langle \hat{q} \rangle^k$ into (54) yields

$$V_{\rm eff}[\langle \hat{q} \rangle] = \frac{\mu^2}{2}\langle \hat{q} \rangle^2 + \frac{g}{k}\langle \hat{q} \rangle^k + \frac{\hbar}{2}[\mu^2 + g(k-1)\langle \hat{q} \rangle^{k-2}]^{1/2}. \tag{55}$$

Expanding (55) about $g = 0$, we have

$$V_{\rm eff}[\langle \hat{q} \rangle] = \frac{\mu^2}{2}\langle \hat{q} \rangle^2 + \frac{g}{k}\langle \hat{q} \rangle^k + \frac{\hbar\mu}{2} + \frac{\hbar g}{4\mu}(k-1)\langle \hat{q} \rangle^{k-2}. \tag{56}$$

Comparing (53) and (56), we see that they are identical up to the constant zero-point energy:

$$\frac{\theta[\langle \hat{q} \rangle]}{\xi} = V_{\rm eff}[\langle \hat{q} \rangle] - V_{\rm eff}[0]. \tag{57}$$

This completes the calculation of the effective potential in one dimension within the Wigner framework. It should be mentioned that the above procedure can be easily extended to composite operators. A subsequent paper will deal with this for a dissipative quantum system: the free large-scale inflaton in de Sitter space.

VI. CONCLUSION

We have presented arguments, based on causal horizon coarse graining, that the long-range scalar field (inflaton) dynamics in a FRW universe is governed by dissipative quantum mechanics in one dimension. As far as we are aware, the Wigner representation of quantum mechanics (which is based on quantum or quasiprobability functions) is the only consistent way of describing these dissipative quantum systems. Thus, we have initiated our study of the dynamics of early-Universe phase transitions by investigating the approximate equilibrium solutions of the Wigner equation. We have shown that these equilibrium solutions describe the ground-state properties of various nondissipative quantum systems reasonably well. We have done this by showing how one computes ground-state energies and the effective potential within the mathematical phase-space framework of Wigner. It should be stressed that if one were able to construct the full $W(\alpha,\beta)$ solution to (11) the calculations of the ground-state energy and effective potential would be exact.

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