Anomalous particle creation, spectral asymmetry, and superconducting strings

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We discuss the use of the spectral asymmetry of the Dirac Hamiltonian as a means to determine the rate of anomalous creation of chiral fermions in the presence of background gauge and scalar fields. The (suitably modified) spectral asymmetry is computed explicitly for a wide class of Hamiltonians in terms of functionals of the restriction of the Hamiltonian to lower-dimensional spaces. The result is used to determine the anomalous charge which gives rise to superconductivity of cosmic or axion strings (vortices). It is shown that superconductivity of cosmic strings is present even when the usual finite-energy requirement is dropped and open-space boundary conditions are adopted instead. This is shown to apply to the case of axion strings as well.

I. INTRODUCTION

The most obvious consequence of the chiral anomaly is the nonconservation of the chiral charge, i.e., the creation of chiral fermions in an external electromagnetic field. The direct computation of the rate of charge nonconservation in the second-quantized theory from the spectrum of the Dirac Hamiltonian provides indeed an intuitive way of understanding the physical origin of the anomaly.[1]

Although heuristically attractive, this approach has been thus far of little value for the sake of practical computation. Here, we shall develop a technique to compute the charge created anomalously in terms of the spectral asymmetry of the Dirac Hamiltonian, inspired to methods widely used in the context of fermion-number fractionization.[2]

These computations find a natural application to superconducting cosmic strings or vortices.[3] Indeed, it is the charge carried by massless fermions created anomalously which gives rise to superconductivity of vortexlike structures. The computation of the rate of anomalous charge creation (if any) allows then one to establish the most general conditions under which a vortex may be superconducting.

In particular, it has been recently shown[4] that several features of the Dirac Hamiltonian which describes the interaction of fermions with a vortex (and, in particular, the index theorems that describe the zero-mode sector of its projection on the plane orthogonal to the vortex) are common to various kinds of vortices (cosmic strings[3] and axion strings[5]) and, furthermore, that they hold true even when the usual finite-energy boundary conditions are dropped, in favor of weaker open-space (i.e., scattering-like) boundary conditions.

Now, zero modes of the transverse Hamiltonian lead to a dimensional reduction which implies superconductivity through a two-dimensional (time and the direction along the vortex) anomaly mechanism.[3,5] However, a naive evaluation based on spectral-asymmetry methods seems to suggest[4] that the rate of creation of the supercurrent carriers may depend critically on the boundary conditions, possibly, at the expense of superconductivity. This would imply a discrepancy between a full four-dimensional evaluation of the anomalous charge and two-dimensional arguments based on the reduction to the transverse zero-mode sector.

The computational method discussed here will allow us to settle these issues. The determination of the spectral asymmetry of an odd-dimensional Dirac operator (as the Hamiltonian of even-dimensional fermions is) in open space poses nontrivial problems. In Ref. 6 we have accomplished this for massless fermions coupled to an external gauge field; here, we shall generalize the result to the more complicated Hamiltonian appropriate to the description of the vortex-fermion interaction, which contains scalar (Higgs) fields as well.

Furthermore, in the computations of Ref. 6 it was assumed that one of the dimensions (here, the dimension along the string) be compact, i.e., in the present context, that the string be closed. Here, we shall discuss the generalization to the case of open strings. This will allow us to show that, if ultraviolet and infrared divergences are treated in a careful and consistent manner, the discrepancy between two- and four-dimensional arguments may be understood, and that superconductivity is preserved in open space, although, in the case of closed strings, in modified form. The methods developed here, furthermore, are more generally useful in the study and computation of anomalous quantum numbers.

In Sec. II we shall introduce, in a simple two-dimensional setting, the modified spectral asymmetry which describes the anomalous charge creation, and discuss in detail its regularization in open space. We shall thereby reobtain results familiar from the two-dimensional anomaly computations.

In Sec. III we shall compute the modified asymmetry, i.e., the anomalous charge, for a wide class of $2n$-dimensional Hamiltonians in compact and open space, in terms of lower-dimensional topological objects.

Finally, in Sec. IV we shall apply our results to the case of superconducting strings; we shall discuss the mechanism through which a supercurrent may arise due to anomalous particle creation, and examine the general conditions that make superconductivity possible.

II. ANOMALOUS CHARGE AND SPECTRAL ASYMMETRY: A TWO-DIMENSIONAL MODEL

If a constant electric field $\mathcal{E}$ is applied to a system of two-dimensional chiral fermions minimally coupled to the electromagnetic field, the anomaly equation $\partial_\mu J^{(\pm)\mu} = \mp(1/2\pi)\epsilon_{\mu\nu}\partial^\mu A^\nu$ ($\pm$ is the chirality of the fermions)[7] implies that particles are created at a rate per unit length

$$\partial_0 J^0 = \mp \frac{1}{2\pi}\mathcal{E} \ . \tag{1}$$

This can be seen[1] as a consequence of the fact that the second-quantized theory is defined by filling the negative-energy Dirac sea. When the electric field is switched on, the Landau level of the Dirac sea shifts at a constant velocity: the dispersion relation $p = \mp E$ (according to the chirality) is replaced by $p - A_1 = \mp E$, where the spatial component of the potential is $A_1 = \mathcal{E}t$ (in the gauge $A_0 = 0$, appropriate for a Hamiltonian description). Thus, negative-energy states close to the Landau level acquire positive energy (or vice versa, according to the chirality), thereby creating charge.

Now, in order to compute the rate of particle creation in a more general setting, we observe that the total charge created in the time Δt is equal to the difference in vacuum charge:

$$\langle \Delta Q \rangle = \langle Q(t_0+\Delta t)\rangle - \langle Q(t_0)\rangle \ . \tag{2}$$

The ground-state charge at any given time is notoriously proportional to the difference in number of positive- and negative-energy eigenstates of the Hamiltonian,[2] i.e., introducing ζ-function regularization, to its spectral asymmetry. The result is found using the expansion of ψ in eigenstates of the Hamiltonian $H, \psi_k^{(\pm)}, H\psi_k^{(\pm)} = E_k^{(\pm)}\psi^{(\pm)}$ with $\mathrm{sgn}(E_k^{(\pm)}) = \pm 1$:

$$\psi = \sum\!\!\!\!\!\!\int_k \, (b_k \psi_k^{(+)} + d_k^\dagger \psi_k^{(-)}) \tag{3}$$

in the charge-conjugation odd, normal-ordered definition $Q = \frac{1}{2}\int dx\,[\psi^\dagger, \psi]$, with the result

$$Q = \sum\!\!\!\!\!\!\int_k \, [(b_k^\dagger b_k - d_k^\dagger d_k) - \tfrac{1}{2}\mathrm{sgn}(E_k)] \ , \tag{4}$$

whence, with ζ-function regularization and the usual definition of the Fock vacuum,

$$\langle Q \rangle = -\tfrac{1}{2}\lim_{s\to 0} \sum\!\!\!\!\!\!\int_k \, \frac{\mathrm{sgn}(E_k)}{|E_k|^s} = -\tfrac{1}{2}\eta[H] \ , \tag{5}$$

$\eta[H]$ is the spectral asymmetry of the Hamiltonian H (Ref. 8).

Equations (2) and (5) have been extensively used to compute the (fractional) vacuum fermion number of various systems;[2] however, the definition of the charge (2) and (5) is not immediately adequate to our purpose. Indeed, if the energy of one (or more) Fock states switches sign during the evolution from t_0 to $t_0+\Delta t$ it is no longer true that $\langle \Delta Q \rangle$ in (2) is the charge created in Δt. Thus, because after the transition of n_+ (n_-) levels from $E<0$ ($E>0$) to $E>0$ ($E<0$), the system is left in a state which is no longer the vacuum. The expectation value of Q in this state is

$$\langle Q \rangle = -\tfrac{1}{2}\eta + n_+ - n_- \ . \tag{6}$$

When a crossing occurs, the spectral asymmetry varies discontinuously of two units, while it is smooth elsewhere.[8] Thus, separating the continuous and the discontinuous variation of η, we can write the charge created in Δt as

$$\begin{aligned}\langle \Delta Q \rangle &= -\tfrac{1}{2}(\Delta\eta - \Delta_{\mathrm{disc}}\eta) \\ &= -\tfrac{1}{2}\int_{t_0}^{t_0+\Delta t} dt \frac{d}{dt}\eta_{\mathrm{cont}}[H(t)] \ . \end{aligned} \tag{7}$$

Namely, it is only the continuous variation of the spectral asymmetry which yields the created charge. This can also be written in terms of a modified asymmetry $\tilde{\eta}$:

$$\langle \Delta Q \rangle = -\tfrac{1}{2}(\tilde{\eta}[H(t_0+\Delta t)] - \eta[H(t_0)]) \ , \tag{8}$$

where

$$\tilde{\eta}[H(t)] \equiv \lim_{s\to 0}\mathrm{Tr}\frac{\mathrm{sgn}H(t_0)}{|H(t)|^s} = \lim_{s\to 0}\tilde{\eta}(s) \ . \tag{9}$$

We can now verify explicitly that the above prescription yields the correct result. We shall consider first the case of compact space, then the open-space problem, which requires some additional care.

Take the Dirac Hamiltonian (in the gauge $A_0 = 0$)

$$H(t) = \frac{\partial}{i\partial x} - A(x,t) \ , \tag{10}$$

which describes right-handed Weyl fermions coupled to a gauge background (chiral Schwinger model), with $0 \le x \le L$ and (say) antiperiodic boundary conditions (this is appropriate to fermions, but does not affect the result for ΔQ since a different choice of boundary conditions changes η by a constant). Time is treated as a parameter; we shall assume the time evolution to be smooth enough that the time dependence of the energy eigenvalues be a smooth function of time.

The eigenvalues of (10) are (see, e.g., Ref. 6)

$$E_k = \frac{(2k+1)\pi}{L} - \frac{2\pi}{L}\mathcal{A} \tag{11}$$

and the modified asymmetry is

$$\tilde{\eta}[H(t)] = \lim_{s\to 0} - \sum_{k=-\infty}^{\infty}\left[\frac{2\pi}{L}\right]^{-s} |k + \mathcal{A}(t) + \tfrac{1}{2}|^{-s} \times \mathrm{sgn}[k + \mathcal{A}(t_0) + \tfrac{1}{2}] \ , \tag{12}$$

where

$$\mathcal{A}(t) = \frac{1}{2\pi}\int_0^L dx\, A(x,t) \ .$$

If we set $\mathcal{A}(t_0) = 0$, then

$$\langle \Delta Q \rangle = \lim_{s\to 0}\frac{1}{2}\left[\frac{2\pi}{L}\right]^{-s}[\zeta(s;\mathcal{A}(t)+\tfrac{1}{2})-\zeta(s;\tfrac{1}{2}-\mathcal{A}(t))]$$
$$= -\mathcal{A}(t)\ , \tag{13}$$

where we introduced the generalized Riemann ζ function $\zeta(s,v)=\sum_{k=0}^{\infty}1/(k+v)^s$. This obtains a rate of creation of charge per unit length,

$$\frac{dJ_0}{dt}=-\frac{d}{dt}\frac{1}{L}\mathcal{A}(t)\ , \tag{14}$$

in agreement with the anomaly equation. Had we considered left-handed fermions, $A(x,t)$ would have had the opposite sign in the Hamiltonian (10), leading to a sign reversal in the final result (14).

In open space (i.e., when $L\to\infty$), a naive application of Eqs. (8) and (9) leads to a nonsensical result: the spectrum of H is just $E_k=k$, with $-\infty<k<\infty$, a continuous variable, regardless of the background $A(x,t)$, and the functional trace in $\tilde{\eta}=\lim_{s\to 0}\int_{-\infty}^{\infty}dk\,\mathrm{sgn}k/|k|^s$ yields an ill-defined integral for every value of s. Thus, because when $s>0$, thereby regulating the ultraviolet behavior of the functional trace, there is still an infrared divergence due to the fact that the spectrum of H stretches continuously to zero. Furthermore, the dependence of $\tilde{\eta}$ on $\mathcal{A}$ seems to have disappeared: this would imply a lack of particle creation in open space, in disagreement both with the anomaly equation and the Dirac-sea argument, which do not depend on the space topology.

To cure this disease, we define the open-space problem as the $L\to\infty$ limit of the compact-space one (i.e., we use box quantization):

$$\tilde{\eta}_{\mathrm{open}}H(t)\equiv\lim_{s\to 0}\lim_{L\to\infty}\tilde{\eta}_L(s)\ . \tag{15}$$

Let us now verify explicitly that this prescription yields the correct result, and, furthermore, that the order of the limits in (15) does not matter.

With the above Hamiltonian (10), using the asymptotic expansion of $\zeta(s,v)$ for large v:

$$\zeta(s,v)=\frac{1}{\Gamma(s)}v^{1-s}\Gamma(s-1)+O\left[\frac{1}{v}\right] \tag{16}$$

we get

$$\tilde{\eta}_{\mathrm{open}}=\lim_{s\to 0}\lim_{L\to\infty}-\left[\frac{2\pi}{L}\right]^{-s}[\zeta(s;\mathcal{A}(t)+\tfrac{1}{2})-\zeta(s;\tfrac{1}{2}-\mathcal{A}(t)]=\lim_{s\to 0}-\frac{(2\pi/L)^{-s}}{s-1}2(\mathcal{A})^{1-s}=2\mathcal{A}(t)\ , \tag{17}$$

which agrees with the compact-space result (13), and with the two-dimensional anomaly equation. Clearly, the same follows if the $s\to 0$ limit is taken first, leading to Eq. (13).

The box-quantization prescription (15) is actually equivalent to treating the infrared singularity in the functional trace (9) with a symmetric prescription, as it can be seen by calculating alternatively the open-space asymmetry as

$$\begin{aligned}\tilde{\eta}_{\mathrm{open}}&=\lim_{s\to 0}\lim_{L\to\infty}\sum_{k=0}^{\infty}\left[\left|\frac{2\pi}{L}(k-\mathcal{A}')\right|^{-s}-\left|\frac{2\pi}{L}(k+\mathcal{A}'+1)\right|^{-s}\right]\\&=\lim_{s\to 0}\lim_{L\to\infty}\frac{L}{2\pi}\left[\int_{-\infty}^{\mathcal{A}'-\epsilon(L)}dk+\int_{\mathcal{A}'+\epsilon(L)}^{\infty}dk+\int_{\mathcal{A}'+\epsilon'(L)}^{\mathcal{A}'+\epsilon(L)}dk\frac{\mathrm{sgn}k}{\left|k-\frac{2\pi}{L}\mathcal{A}'\right|^{s}}\right]\ ,\end{aligned} \tag{18}$$

where $\mathcal{A}'=\mathcal{A}-\frac{1}{2}$, $\epsilon(L)=(2\pi/L)(\mathcal{A}'-[\mathcal{A}'])$, $\epsilon'(L)=(2\pi/L)\{1-(\mathcal{A}'-[\mathcal{A}'])\}$, and $[\mathcal{A}']$ denotes the largest integer smaller than or equal to $\mathcal{A}'$. Clearly as $L\to\infty$, $\epsilon\to 0$, $\epsilon'\to 0$, and the third integral vanishes [it is $O(1/L)$ with respect to the first two]. Computing the integrals explicitly, of course, yields again the result (17).

Finally, it is worth noticing that, as long as no crossings occur, the conventional form of the asymmetry (5) may be used as well. In particular, if one is interested in computing the rate of particle creation (which depends only on the continuous variation of η, up to an infinite constant) the standard spectral asymmetry (5) may be used even in open space, despite the fact that because of the presence of a continuous spectrum, crossings occur with continuity for every value of the background variable $\mathcal{A}$.

Indeed, a straightforward calculation yields

$$\begin{aligned}\lim_{s\to 0}\lim_{L\to\infty}\eta_L(s)&=\lim_{s\to 0}\lim_{L\to\infty}-\sum_{k=0}^{\infty}\left[\left[\frac{2\pi}{L}(k+\tfrac{1}{2}+\mathcal{A}-[\mathcal{A}+\tfrac{1}{2}])\right]^{-s}-\left[\frac{2\pi}{L}\{k+\tfrac{1}{2}-(\mathcal{A}-[\mathcal{A}+\tfrac{1}{2}])\}\right]^{-s}\right]\\&=\lim_{s\to 0}\lim_{L\to\infty}-\frac{L}{2\pi}\left[\int_{2\pi/L}^{\infty}dk\left[\frac{1}{k^s}-\frac{1}{k^s}\right]\right]+s\frac{L}{2\pi}\left[\int_{2\pi/L}^{\infty}dk\frac{1}{k^{s+1}}\right]\frac{2\pi}{L}(2\mathcal{A}-2[\mathcal{A}+\tfrac{1}{2}])\\&=\lim_{s\to 0}\lim_{L\to\infty}\left[\frac{2\pi}{L}\right]^{-s}2\{\mathcal{A}(t)-[\mathcal{A}(t)+\tfrac{1}{2}]\}\ .\end{aligned} \tag{19}$$

Although clearly this does not have a well-defined limit as $L \to \infty$ ($\mathcal{A} - [\mathcal{A}]$ has a highly discontinuous behavior as L grows) taking a derivative with respect to t leads again to the correct result (14) for the particle creation rate.

This concludes the general discussion of our approach to the computation of the anomalous charge in terms of a suitably modified and (in open space) regulated spectral asymmetry. In the following we shall apply this to realistic four-dimensional (or generally even-dimensional) models.

III. COMPUTATION OF THE SPECTRAL ASYMMETRY IN HIGHER DIMENSIONS

We shall consider $2n$-dimensional fermionic models described by Dirac Hamiltonians of the form

$$H = \begin{bmatrix} D(x_3;t) & L(x^\mu;t) \\ L^\dagger(x^\mu;t) & -D(x_3;t) \end{bmatrix}, \qquad (20)$$

where D and L are $2^{n-1}\times 2^{n-1}$ matrix differential operators, the index μ runs from 1 to $(2n-2)$, x_3 denotes the $(2n+1)$st space coordinate, and t is the time coordinate, which is treated as a parameter. Furthermore, we require D to have the form

$$D(x_3;t) = \mathbb{1}\left[\frac{\partial}{i\partial x_3} - A_3(x_3;t)\right] = \mathbb{1}H_1(x_3;t) , \qquad (21)$$

where $\mathbb{1}$ is the $2^{n-1}\times 2^{n-1}$ unit matrix and H_1 is the one-dimensional Hamiltonian (10).

Clearly, any Dirac Hamiltonian that describes the coupling of fermions to an arbitrary number of scalar, vector, and higher-spin (Abelian) fields can be cast in the form (10) if it may be written as

$$H(x^\mu,x_3;t) = \Gamma H_1(x_3;t) + H_T(x^\mu;t) , \qquad (22)$$

where x^μ, x_3, t, and H_1 are defined as above and Γ is a $2^n\times 2^n$ element of the Clifford algebra which anticommutes with H_T: $\{\Gamma, H_T\} = 0$. H_T will be henceforth referred to as the transverse Hamiltonian. The form (20) is achieved by choosing a representation of the Clifford algebra where

$$\Gamma = \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix},$$

and defining $D = H_1$; $L = H_T[(1-\Gamma)/2]$.

The simplest nontrivial example of a model with Hamiltonian of the form (20) is that of Dirac fermions coupled to an Abelian gauge field in four-dimensional space, provided the space dependence of the background fields is factorized, i.e., $A = (A^\mu(x^\mu), A_3(x_3))$ (a simple realization is the case when the magnetic and electric fields are constant and parallel). More complicated examples are provided by fermions coupled to both gauged and ungauged vector fields, and scalar fields in the presence of vortexlike structures, with x_3 the dimension along the vortex. These we shall discuss in somewhat greater detail in Sec. IV.

Now, we are interested in determining the spectral asymmetry of the Hamiltonian (20), and in studying its application to particle creation, in light of the discussion of Sec. II. If the space is compact, the problem is trivial, since the eigenspectrum of H is discrete. The eigenvalues $\mu_{n,k}$ can be expressed in term of the (discrete) eigenvalues λ_n of the transverse Hamiltonian H_T and of the eigenvalues m_k of H_1.

$$\mu_{\pm n,k} = \pm(\lambda_n^2 + m_k^2)^{1/2} , \quad \mu_{i,k}^0 = \epsilon^i m_k , \qquad (23)$$

where $1 \le i \le n_z$ counts the zero modes of the transverse Hamiltonian, which are eigenstates of Γ with eigenvalue $\epsilon^i = \pm 1$. The problem then reduces to an effectively one-dimensional one, since[6]

$$\eta[H] = n_0 \eta_1[H_1], \quad n_0 = \sum_{i=1}^{n_z} \epsilon^i , \qquad (24)$$

where η_1 is the one-dimensional spectral asymmetry computed in Sec. II, and n_0 is called the Γ-index of H_T.

Things get more complicated in open space, where the spectrum of H contains both discrete bound states and a continuum. The computation of η in open space has been accomplished in Ref. 6 in the particular case of three-dimensional massless fermions coupled to a gauge background [earlier open-space determinations of η (Ref. 9) assumed the operator H_1 to be just a constant]. The approach of Ref. 6, however, is of more general applicability, thus, we briefly review it, and generalize it.

The open-space η invariant is defined as

$$\eta(2s) = \int_{-\infty}^{\infty} d\mu\, \rho(\mu) \frac{\operatorname{sgn}\mu}{|\mu|^{2s}} = \operatorname{Tr}\left[\frac{H}{(H^2)^{s+1/2}}\right], \qquad (25)$$

where $\rho(\mu)$ is the spectral density[10] of H and the integral over μ should be understood as an integration over the continuum and a sum over the bound states. We may recast the functional trace in (25) by separating the trace of the transverse Hamiltonian (eigenvalues λ_n) and the trace of H_1 (eigenvalues m_k). Introducing spectral densities for the operators $LL^\dagger$ and $L^\dagger L$ in (20) we get[6]

$$\begin{aligned} \eta(2s) &= \frac{s}{\sqrt{\pi}\,\Gamma(s+1/2)} \int_0^\infty d\beta \int_{-\infty}^{\infty} d\omega\, e^{-\beta\omega^2} \beta^{s-1} \operatorname{Tr}\left[\frac{H}{H^2+\omega^2} e^{-\beta H^2}\right] \\ &= \frac{s}{\sqrt{\pi}\,\Gamma(s+1/2)} \int_0^\infty d\beta \int_{-\infty}^{\infty} d\lambda\, \beta^{s-1} [\rho_{LL^\dagger}(\lambda) - \rho_{L^\dagger L}(\lambda)] \sum\!\!\!\!\!\!\int_k \int_{-\infty}^{\infty} d\omega\, e^{-\beta(\omega^2+\lambda^2+m_k^2)} \frac{m_k}{m_k^2+\omega^2+\lambda^2} \\ &= \frac{1}{\Gamma(s+\frac{1}{2})} \int_0^\infty d\beta\, \beta^{s-1} \bar{\eta}(\beta) \bar{\xi}(\beta) . \end{aligned} \qquad (26)$$

Here, we have defined the inverse Mellin transforms of the spectral asymmetry of H_1 and of the ζ function of the operator $[\Gamma H_T^2]$:

$$\begin{aligned}\bar{\eta}(\beta)&=\int_{c-i\infty}^{c+i\infty}\frac{dt}{2\pi i}\beta^{-t}\Gamma(t+\tfrac{1}{2})\eta_1[H_1]\\&=\int_{c-i\infty}^{c+i\infty}\frac{dt}{2\pi i}\beta^{-t}\Gamma(t+\tfrac{1}{2})\mathrm{Tr}\frac{H_1}{(H_1^2)^{s+1/2}}\ ,\end{aligned}\tag{27}$$

$$\begin{aligned}\bar{\zeta}(\beta)&=\int_{c-i\infty}^{c+i\infty}\frac{dt}{2\pi i}\beta^{-t}\Gamma(t)\zeta[\Gamma H_T^2]\\&=\int_{c-i\infty}^{c+i\infty}\frac{dt}{2\pi i}\beta^{-t}\Gamma(t)\mathrm{Tr}\left[\frac{1}{(LL^\dagger)^t}-\frac{1}{(L^\dagger L)^t}\right]\ ,\end{aligned}\tag{28}$$

where, in both cases, $\mathrm{Re}(c)>M$ with M the real part of the pole with largest real part of η_1 and ζ, respectively (see Ref. 8 for a proof that both ζ and η are meromorphic with a finite number of poles in the $\mathrm{Re}(z)>0$ half-plane).

The latter quantity is just the heat-kernel regularization of the Γ-index of the transverse Hamiltonian

$$\bar{\zeta}(\beta)=\mathrm{Tr}(e^{-\beta(LL^\dagger)}-e^{-\beta(L^\dagger L)})=\mathrm{Tr}\Gamma e^{-\beta H_T^2}\ ,\tag{29}$$

i.e., in the $\beta\to\infty$ limit it yields n_0 in (24); the former is explicitly

$$\bar{\eta}(\beta)=\mathrm{Tr}[(\beta H_1^2)^{1/2}e^{-\beta H_1^2}]\ .\tag{30}$$

Computing the integral in (26) leads to

$$\eta(2s)=\sum\!\!\!\!\!\!\int_k\int d\lambda\frac{m_k}{(m_k^2+\lambda^2)^{s+1/2}}[\rho_{LL^\dagger}(\lambda)-\rho_{L^\dagger L}(\lambda)]\ .\tag{31}$$

It is now convenient to separate the bound-state sum and the continuum integration in the trace over the eigenvalues λ of the transverse Hamiltonian. The nonzero bound-state eigenvalues of $LL^\dagger$ and $L^\dagger L$ are paired, i.e., to each eigenmode ψ of $LL^\dagger$ there corresponds an eigenmode $\psi'=L^\dagger\psi$ of $L^\dagger L$ with the same eigenvalue (see, e.g., Ref. 2 for a proof that ψ' is a normalizable bound state if ψ is). Thus, the λ traces in (26) and (31) receive only contributions from the sum over zero-mode bound states, and from the continuum integration:

$$\begin{aligned}&\eta(2s)=n_0\eta_1[H_1]+\eta_c(2s)\ ,\\&\eta_c(2s)=\int_{E_{\mathrm{th}}}^{\infty}d\lambda\sum\!\!\!\!\!\!\int_k\frac{m_k}{(m^2+\lambda^2)^{s+1/2}}[\rho_{LL^\dagger}(\lambda)-\rho_{L^\dagger L}(\lambda)]\ ,\end{aligned}\tag{32}$$

where n_0 is the Γ-index (24), the λ integration is now extended over the continuum, and E_{th} is the threshold energy of the transverse Hamiltonian.

In the case of Weyl fermions coupled to a background U(1) gauge field, $E_{\mathrm{th}}=0$ and the continuum is peaked at threshold:[11]

$$\begin{aligned}&[\rho_{LL^\dagger}(\lambda)-\rho_{L^\dagger L}(\lambda)]=\phi_c\delta(\lambda)\ ,\\&\eta(2s)=(n_0+\phi_c)\eta_1[H_1]\ .\end{aligned}\tag{33}$$

If scalar fields are present as well (as in the models which we shall examine in Sec. IV) a mass gap may separate the continuum from the bound states, preventing the factorization of the asymmetry (33).

It is now apparent how the spectral asymmetry ought to be modified in order to be used for the computation of particle creation. In the factorized cases (24) and (33) (compact space and coupling to gauge fields only) the problem reduces to an effective one-dimensional one, since only the zero-mode sector [the μ_i^0 eigenmodes (23)] or the zero-energy threshold of the continuum of the transverse Hamiltonian contribute to the asymmetry. The resulting one-dimensional asymmetry is to be replaced by the modified form $\bar{\eta}$, and computed with the symmetric integration prescription discussed in the previous section; the Γ-index and the continuum contribution ϕ_c may be calculated by means of open-space index theorems[12] (see also Ref. 6). The result, substituted in Eq. (8), yields the anomalous charge.

If, instead, there is a mass gap, the zero-mode contribution in Eq. (32) should be still treated in this guise, but the continuum contribution does not require either an infrared prescription [since the mass gap prevents the integrand in (32) from having infrared singularities], or a prescription to account for zero crossing [since the mass gap endows continuum eigenvalues of the form (23) with a definite sign along the evolution of the system]. Equivalently, the created charge is again found by using the Hamiltonian (20) in Eqs. (8) and (9), since the asymmetry $\bar{\eta}$ in (9) satisfies all of the above requirements.

Finally, we evaluate the continuum contribution to the asymmetry η_c in (32) explicitly. We consider two distinct cases: while we define the transverse Hamiltonian to act on an open $(2n-2)$-dimensional space, we take H_1 to act on a compact space first (corresponding to an overall space topology $S^1\times\mathbb{R}^{2n-2}$), then on open space.

In the former case, the trace over H_1 is a discrete sum, which we can compute formally by introducing an intermediate regulator

$$\begin{aligned}\eta_c(2s)&=\lim_{\beta\to0}\int_{E_{\mathrm{th}}}^{\infty}d\lambda\sum_k\frac{m_k}{(m_k^2+\lambda^2e^{-\beta\lambda^2})^{s+1/2}}[\rho_{LL^\dagger}(\lambda)-\rho_{L^\dagger L}(\lambda)]\\&=\lim_{\beta\to0}\int_{E_{\mathrm{th}}}^{\infty}d\lambda\sum_k\frac{\mathrm{sgn}(m_k)}{(m_k)^{2s}}[\rho_{LL^\dagger}(\lambda)-\rho_{L^\dagger L}(\lambda)]\\&\qquad\times\left[1-(s+\tfrac{1}{2})\frac{\lambda^2}{m_k^2}e^{-\beta\lambda^2}+\cdots+\frac{(-1)^n}{n!}\frac{(2s+1)(2s+3)\cdots(2s+2n-1)}{2^n}\left[\frac{\lambda^2}{m_k^2}e^{-\beta\lambda^2}\right]^n\right]\\&=\eta_1(2s)\zeta_c(0)-\tfrac{1}{2}\eta_1(2s+2)\zeta_c(-1)+\cdots+\frac{(-1)^n}{n!}\frac{(2s+1)(2s+3)\cdots(2s+2n-1)}{2^n}\eta_1(2n+1)\zeta_c(-n)\ ,\end{aligned}\tag{34}$$

where ζ_c is obtained from ζ, Eq. (28), by subtracting the zero-mode contribution [equal to the constant in n_0 in (24)]. Note that $\zeta_c(t)$ is regular if t is a nonpositive integer, and $\eta_1(2s)$ is regular for Re(s) large enough, provided the pertinent differential operators are elliptic, positive, and self-adjoint and have a positive-definite symbol.[8] The expression (34) of $\eta_c(0)$ is reminiscent of the low-temperature limit of the anomalous part of the effective action of four-dimensional massive fermions coupled to an Abelian-gauge background.[13]

Although we cannot calculate, in general, the infinite sum in Eq. (34), we may set bounds on $\eta_c(0)$ in (34). In particular, we prove that

$$\operatorname{sgn}[\eta_c(0)]=\operatorname{sgn}[\eta_1(0)\zeta(0)], \quad |\eta_c(0)| < |\eta_1(0)\zeta(0)| \ . \tag{35}$$

This means that the continuum contribution in (32) is subleading as compared to the bound-state one. We prove this by showing that

$$\operatorname{sgn}\left[\lim_{s\to 0}\left[\sum_k \frac{m_k}{(m_k^2+\lambda^2)^{s+1/2}}\right]\right]=\operatorname{sgn}\left[\lim_{s\to 0}\left[\sum_k \frac{\operatorname{sgn}(m_k)}{|m_k|^{2s}}\right]\right], \tag{36}$$

$$\left|\lim_{s\to 0}\sum_k \frac{m_k}{(m_k^2+\lambda^2)^{s+1/2}}\right| < \left|\lim_{s\to 0}\sum_k \frac{\operatorname{sgn}(m_k)}{|m_k|^{2s}}\right|, \tag{37}$$

where $\lambda^2>0$.

First, we rewrite the sum on the left-hand sides of (36) and (37) using the explicit form (11) of the eigenvalues m_k:

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\frac{m_k}{(m_k^2+\lambda^2)^{s+1/2}}&=-\left[\frac{2\pi}{L}\right]^{-2s}\sum_{k=0}^{\infty}\left[\frac{k+f}{[(k+f)^2+\lambda^2]^{s+1/2}}-\frac{k+(1-f)}{\{[k+(1-f)]^2+\lambda^2\}^{s+1/2}}\right],\\ f&=\mathcal{A}-[\mathcal{A}+\tfrac{1}{2}]+\tfrac{1}{2}\ .\end{aligned} \tag{38}$$

It is trivial to verify that the equality (36) and the inequality (37) are satisfied term by term in the series (38); it follows that they are satisfied by the sum of the series when Re(s) is large enough to ensure convergence. To show that they hold true when $s\to 0$, it is sufficient to show that the series converges when $s=0$.

This can be accomplished by expanding the summand in (38) in powers of $1/k$, then truncating the expansion (which is possible, since the series has alternating signs and, after a finite number of terms, monotonically decreasing summand). It is thus easy to show that

$$\begin{aligned}&\left|\sum_{k=0}^{\infty}\frac{k+f}{[(k+f)^2+\lambda^2]^{s+1/2}}-\frac{k+(1-f)}{\{[k+(1-f)]^2+\lambda^2\}^{s+1/2}}\right|\\ &\qquad < \left|\frac{1}{k^{2s+1}}\left[2s(1-2f)-\frac{1}{k}2s(s+\tfrac{1}{2})(1-2f)+O\left[\frac{1}{k^2}\right]\right]\right|,\end{aligned} \tag{39}$$

which proves that the series converges if $s\geq 0$, thereby completing the argument.

We come now to the case when H_1 acts on open space as well. It follows immediately that the continuum contribution vanishes identically:

$$\eta_c(2s)=\int_{E_{\rm th}}^{\infty}d\lambda\frac{k}{(k^2+\lambda^2)^{s+1/2}}[\rho_{LL^\dagger}(\lambda)-\rho_{L^\dagger L}(\lambda)]=0\ . \tag{40}$$

Of course, the same result is found taking $0\leq x_3\leq L$ and computing the large-L limit:

$$\begin{aligned}\lim_{L\to\infty}\sum_{k=-\infty}^{\infty}\frac{m_k(L)}{\{[m_k(L)]^2+\lambda^2\}^{s+1/2}}&=\lim_{L\to\infty}\frac{L}{2\pi}\int_{\epsilon'(L)}^{\epsilon(L)}dk\frac{k}{(k^2+\lambda^2)^{s+1/2}}\\ &=\lim_{L\to\infty}\frac{1}{1-2s}\frac{L}{2\pi}\left[[\sqrt{\epsilon(L)^2+\lambda^2}]^{1-2s}-[\sqrt{\epsilon'(L)^2+\lambda^2}]^{1-2s}\right],\end{aligned} \tag{41}$$

where $\epsilon(L),\epsilon'(L)$ are defined as in Eq. (18) and tend to zero as $L\to\infty$.

This shows that if the Hamiltonian (20) is defined on $(2n-1)$-dimensional open space and a mass gap separates the continuum from the zero modes, only the latter contribute to the spectral asymmetry, which is given by Eq. (24) as in the compact-space case. The anomalous charge is just obtained replacing η_1 with $\tilde{\eta}_1$ in (9) in Eq. (24), and substituting the result in Eq. (8).

IV. APPLICATION TO SUPERCONDUCTING STRINGS

We shall now apply the results of the previous sections to the determination of the rate of fermion creation in the background of a stringlike or vortexlike structure; this is related to the superconducting properties of these objects.

Stable vortices arise in many models due to spontaneous breaking of gauged or ungauged symmetries. Most notably, we consider the coupling with fermions of vortices arising in the spontaneous breaking of the gauged U(1) of a theory with gauge group $U(1)\times\tilde{U}(1)$ (cosmic strings[3]), and of a global (ungauged) U(1) symmetry of a theory with gauge group $\tilde{U}(1)$ (axion strings[5]). We shall only examine the fermion couplings, and refer to the original literature[3,5] for a more detailed description of these models.

The fermion part of the Lagrangian of the former model is (in Minkowski space-time)

$$\mathcal{L}=\psi^\dagger \not{D}\psi+\chi^\dagger \not{D}'\chi-[\phi\psi^T\sigma_2\chi+\phi^*\psi^\dagger\sigma_2(\chi^\dagger)^T] , \quad (42)$$

where ψ,χ are two-component, four-dimensional left-handed Weyl fermions, the covariant derivatives, defined with the usual Dirac-Weyl matrices $\sigma^\mu=(\mathbb{1},-\boldsymbol{\sigma})$, contain the coupling with the U(1) gauge fields A_μ and R_μ: $D_\mu=i\partial_\mu-q_A A_\mu-q_R R_\mu$; $D'_\mu=i\partial_\mu-q'_A A_\mu-q'_R R_\mu$, and ϕ is the Higgs field which couples to the R field and is thus responsible for the breaking of $U(1)_R$.

The charges $q_{A,R},q'_{A,R}$ are adjusted so as to cancel the chiral gauge anomalies that would make the theory inconsistent. In order to remove the chiral anomaly due to the A coupling, it is sufficient to set $q_A=-q'_A$. Cancellation of the anomalies due to the R coupling and to the mixed triangle diagrams (AAR, ARR) requires the introduction of (at least) one more pair of fermions with complex-conjugate coupling with ϕ, that is, with action obtained replacing ϕ with ϕ^* in Eq. (42). In the cosmic string model, $q'_R=-(q_R+e)$, where e is the gauge–Higgs-field coupling constant.

The Lagrangian of the axion model is obtained from (42) by setting $R_\mu=0$, and $q_A=-q'_A$. If ψ and $\sigma_2\chi^*$ are arranged in a single four-component Dirac spinor

$$\Psi=\begin{bmatrix}\psi\\ \sigma_2\chi^*\end{bmatrix} ,$$

this is seen to be equivalent to coupling Ψ to A_μ with a single charge q through $\mathcal{L}=\bar{\Psi}\gamma^\mu(i\partial_\mu-qA_\mu)\Psi$, with the usual Dirac matrices.

The single-particle Dirac Hamiltonian implied by (42), with respect to the basis of Dirac spinors Ψ defined above, has the form (in the gauge $A_0=R_0=0$)

$$\begin{aligned}H=&\alpha^i[i\partial_i-(s_A+\gamma_5 e_A)A_i-(s_R+\gamma_5 e_R)R_i]\\&+\beta(\mathrm{Re}\phi+i\gamma_5\mathrm{Im}\phi) , \end{aligned}\quad (43)$$

where $i=1,2,3$; $\beta=\gamma^0$, $\alpha^i=\gamma^0\gamma^i$, and $s_{A,R}=\frac{1}{2}(q_{A,R}-q'_{A,R})$, $e_{A,R}=\frac{1}{2}(q_{A,R}+q'_{A,R})$. In the axion model, $s_R=e_R=e_A=0$; in the cosmic-string model, $e_A=0$, $s_R=s+e/2$, and $e_R=-e/2$.

A vortexlike background configuration has $A_3=R_3\equiv 0$; $A^\beta=A^\beta(x^\alpha)$, $R^\beta=R^\beta(x^\alpha)$ with $\alpha,\beta=1,2$, and $\phi=\phi(x^\alpha)$, i.e., purely transverse gauge and Higgs fields. Furthermore, the Higgs field vanishes in the core of the vortex [say, $\phi(0,0)=0$] and satisfies the boundary condition

$$\phi(x^\alpha)\underset{|x|\to\infty}{\longrightarrow}\phi_\infty=\mathrm{const} .$$

Topologically stable vortices exist, and are classified by the winding number of the Higgs field

$$n=\frac{1}{2\pi}\oint\frac{\phi^* i\partial\phi}{|\phi|^2}\cdot dl ,$$

where the integration is performed over a loop which encircles the string.

The Hamiltonian (43) is of the form (22) discussed in Sec. III, with the following identifications:

$$\Gamma=-\alpha_3,\quad H_T=\alpha^i D_i+\beta(\mathrm{Re}\phi+i\gamma_5\mathrm{Im}\phi) . \quad (44)$$

The transverse Hamiltonian has a nonvanishing Γ-index (24). Indeed, its zero modes[14] (when A_μ and R_μ are rotationally invariant up to a gauge transformation) and the relevant index theorem[15] have been known explicitly for a long time in the axion case, and were recently determined[4] in the more general case of models of the form (42), with an arbitrary number of fermion flavors. The result is

$$\mathrm{ind}(H_T)=n_0=n , \quad (45)$$

where n is the winding number defined above. This is true provided the eigenstates of the transverse Hamiltonian obey scatteringlike boundary conditions, i.e., if the gauge potentials and the covariant derivatives of the Higgs field fall off at least as $1/|x|$ as $x\to\infty$ in the transverse plane.

Projecting the Hamiltonian on the zero-mode sector yields an effective low-energy one-dimensional operator, of the form (10), which describes motion along the vortex. The eigenmodes of the Hamiltonian in the transverse zero-mode sector thus describe massless particles moving along the string. If an external electric field directed along the string is switched on, i.e., $A_3=\mathcal{E}(x_3)t\neq 0$, the vacuum gets populated by anomalously created particles, whose number builds up at a rate given by the anomaly equation (1). These are bound to move along the string, at the speed of light, and with direction of motion determined by their two-dimensional chirality (i.e., by the eigenvalue of $\Gamma=\alpha_3$). Hence, a current builds up, also, proportional to the anomalous charge. Since the charge increases linearly with time, so does the current, thus displaying superconducting behavior (see Refs. 3 and 4 for a detailed discussion).

However, we know from the computations in the previous section that in open space the anomalous charge receives, in general, contributions from the entire spectrum, and not only from the zero modes. In particular, the anomalous charge has been estimated in Ref. 4 with somewhat surprising results. The computation, however, is based on a rough and ready way of evaluating the spectral asymmetry, and it is thus presented as a conjecture, which requires further test. The idea is that the η invariant can be written as

$$\eta[H_k]=\sum_k\!\!\!\!\!\!\int\ \eta[H_k] , \quad (46)$$

where H_k is the operator obtained projecting H_1 in (22) on its eigenmodes $|a_k\rangle$, which satisfy $H_1|a_k\rangle=m_k|a_k\rangle$:

$$H_k \equiv \langle a_k \mid H \mid a_k \rangle \ . \tag{47}$$

Strictly speaking, Eq. (46) is true only before all regulators are removed. Nevertheless, in Ref. 4, $\eta[H_k]$ was computed in the $s \to 0$ limit, with the result

$$\eta_{H_k}(0) = \text{sgn}(m_k) \left[n - (n+\phi_R) \left[\frac{m_k^2}{m_k^2 + |\phi_\infty|^2} \right]^{1/2} \right] ,$$
$$\phi_R = \frac{e}{2\pi} \int dx_1 dx_2 \epsilon^{\alpha\beta} \partial_\alpha R_\beta \ . \tag{48}$$

The spectral asymmetry (48) was substituted in Eq. (46). Despite the fact that, in open space, setting $m_k = k$ and integrating over all k leads to a linear divergence, it was observed that the derivative of the charge with respect to time is finite, even if it is computed using in Eq. (5) the spectral asymmetry determined in this fashion. This is true at least if the time dependence is entirely contained in the eigenvalues of H_1, m_k, because then

$$\frac{\partial \eta_H}{\partial t} \propto \frac{\partial \eta_H}{\partial k} = \eta_{H_k} \mid_{-\infty}^{\infty} = -2\phi_R \ . \tag{49}$$

Moreover, it follows that the rate of charge creation is $\partial \langle \Delta Q \rangle / \partial t \propto \phi_R$.

This result, if correct, has a dramatic import. For axion strings, $\phi_R = 0$ identically. This would imply a discrepancy with the two-dimensional anomaly computation in the transverse zero-mode sector, since axion strings would not be superconducting, despite the fact that the transverse Hamiltonian has a nonvanishing Γ-index. More generally, the two-dimensional argument would agree with the full calculation only when $\phi_R = -n$, as will soon become apparent. This is actually true if one requires the string to carry finite energy per unit length, since it is equivalent to requiring that the covariant derivatives of the Higgs field fall off *more rapidly* than $1/|x|$; but this is not necessarily the case if scattering boundary conditions are enforced instead.[4,14]

Let us now test these conjectures using the machinery developed in the previous sections. We shall treat both the case of a closed and an open vortex in open space. If the string is closed, its interaction with fermions is still described locally by the Hamiltonian (43), but the spectrum of H_1, the Hamiltonian along the string, is affected by the long-range behavior of the fields. Thus, use of (43) with the space topology $S^1 \times \mathbb{R}^2$, discussed in Sec. III, is appropriate.

In any case, the determination of the anomalous charge is based on the computation of the spectral asymmetry of the Hamiltonian. As we have seen in Sec. III [Eq. (32)], the η invariant, thus, the vacuum charge, consists of a zero mode and a continuum contribution

$$\langle \Delta Q \rangle = -\frac{s_A}{2} \{ n_0 \tilde{\eta}_1 [H_1] + \eta_c(0) \} \ . \tag{50}$$

In writing down Eq. (50) we assumed $\langle Q(t_0) \rangle$ [Eq. (2)] to vanish. This is true when $A_3(t_0) = 0$ [see (31)]; it is thus apparent that the charge creation takes place by switching the electric field on. Note that the above is the electric charge, and it depends explicitly on the coupling s_A between the fermions and the electromagnetic potential A_μ.

The zero-mode contribution coincides with the result from the two-dimensional anomaly computation. According to Eq. (45), $n_0 = n$; thus, the result of Ref. 4 would suggest $\eta_c = -(n+\phi_R)\tilde{\eta}_1[H_1]$. This vanishes only when $\phi_R = -n$ (as advertised); otherwise, it modifies the zero-mode contribution, and, if $\phi_R = 0$ it cancels it completely.

However, we already know that if the string is open η_c vanishes identically, and only the zero-mode contribution survives, contrary to the conjecture. More generally, we can determine η_c from Eq. (31), observing that if the eigenstates of H_T reduce to scattering states at infinity in the transverse plane, the left-right spectral density is peaked at threshold, i.e.,

$$[\rho_{LL^\dagger}(\lambda) - \rho_{L^\dagger L}(\lambda)] = \phi_c \delta(\lambda - E_{\text{th}}) \ . \tag{51}$$

To prove this, we drop the sum (or integral) over k in Eq. (32), and take the resulting expression

$$\eta_{c,k} = \int d\lambda\, f(\lambda) [\rho_{LL^\dagger}(\lambda) - \rho_{L^\dagger L}(\lambda)] \ ,$$
$$f(\lambda) = \frac{m_k}{(m_k^2 + \lambda^2)^{1/2}} \ , \tag{52}$$

as a definition[10] of the action of the spectral density (which is a measure, not an ordinary function) on the test function $f(\lambda)$. From Eq. (32) and the index theorem (45) we get

$$\eta_{c,k} = \eta_{H_k}(0) - n\ \text{sgn}(m_k) \ , \tag{53}$$

where $\eta_{H_k}(s)$ is the η invariant (25) of H_k in (47). The result (51) follows immediately by comparing with the determination of $\eta_{H_k}(0)$, Eq. (48), with

$$E_{\text{th}} = |\phi_\infty| , \quad \phi_c = -(\phi_R + n) \ . \tag{54}$$

Note that, because of the coupling to the Higgs field, the fermions acquire a mass; the threshold energy is just the fermion mass in the vacuum (i.e., no-vortex) sector. Also, the normalization of the continuum contribution $\zeta_c(0) = \phi_c = -(\phi_R + n)$ is fixed by sum rules[16] in terms of the high- and low-β limits of the heat kernel (29), which are, respectively, the index (45) and[4] $\lim_{\beta \to 0} \bar{\zeta}(\beta) = -\phi_R$, since $\zeta_c(0) + \bar{\zeta}(0) = \bar{\zeta}(\infty)$, with ζ_c and $\bar{\zeta}$ defined in (28) and (34).

At any rate, we get

$$\eta_c(2s) = -\left[\sum_k \frac{m_k}{(m_k^2 + |\phi_\infty|^2)^{s+1/2}} \right] (\phi_R + n)$$
$$= -(\phi_R + n)[\eta_1(2s) - \tfrac{1}{2} |\phi_\infty|^2 \eta_1(2s+2) + \cdots] \ , \tag{55}$$

where we have also used Eq. (34). Substitution of this in Eq. (50) yields immediately the charge.

The continuum thus contributes to the spectral asymmetry, i.e., to the charge, because of an imbalance in the density of states, and not because of particle creation. The relevant states are massive fermions, bound to move along the string, since their transverse momentum vanishes (the only contribution to their energy besides the mass is from the Hamiltonian along the string H_1). Contrary to what happens, e.g., to fermions coupled to a gauge vortex, and, in general, when the continuum stretches to zero, there is no continuum contribution to the creation of massless fermions due to redefinition of the Dirac sea.

Thus, the continuum contribution, because of the inequality (35), cannot cancel the contribution from the transverse zero modes, even when $\phi_R = 0$. It follows that the anomalous charge creation is present even for axion strings. Finally, the continuum states, being massive, do not contribute to the superconductivity proper (since they do not move at the speed of light, neither with direction fixed by their helicity). In the particular case of open strings, there is no continuum contribution at all, and the two-dimensional computation in the zero-mode sector is fully adequate to determine both the supercurrent and the anomalous charge.

It should now be clear why the naive computation (49) fails in this case, whereas it would work in two dimensions. Taking a derivative with respect to k amounts to extracting the ultraviolet behavior of the integral over the momentum along the string. This does not lead to the correct result, in general, if it is done, as above, after removing the regulator, since the regulator affects precisely the ultraviolet behavior: for example, zero, and not ϕ_R is (correctly) obtained if a derivative with respect to k of $\eta_c(2s)$, Eq. (40), is taken when $s > 0$.

In the two-dimensional case (or, equivalently, in the transverse zero-mode sector of a higher-dimensional problem) an infrared singularity is present aside of the ultraviolet one. The charge creation is due to shifting of the infrared singularity [recall (19)]; since, however, the spectral density of H_1 shifts rigidly (it is a constant for every k) the effect is equivalently detected by measuring the shift of its ultraviolet limit.

We conclude that superconductivity is present, both for closed and open strings, whenever the Γ-index of the transverse Hamiltonian is nonzero. For models without a mass gap, and with a spectral asymmetry peaked at threshold, as described by Eq. (33) (for example, the gauge vortex, reviewed, e.g., in Ref. 16) superconductivity is present even when the Γ-index vanishes, provided the continuum contribution is nonzero. Since[6] $(n_0 + \phi_c) \propto \phi_B$, if ϕ_B is the flux of the magnetic field in the plane orthogonal to the vortex, superconductivity is present whenever $\phi_B \neq 0$.

V. CONCLUSION

In a recent paper,[6] we have shown how the spectral asymmetry of an odd-dimensional Dirac operator can be computed in terms of lower-dimensional topological invariants, and we have used this to determine the parity-anomalous part of the effective action of odd-dimensional fermions coupled to a gauge background.

Here, we have turned our attention to the use of the spectral asymmetry as a means to compute vacuum quantum numbers: the odd-dimensional Dirac operator now being the Hamiltonian of an even-dimensional theory. This has required a generalization of the computational technique of Ref. 6 in several respects.

First, we modified the usual η invariant in order to adapt it to the computation of charge creation, then we discussed the infrared problems that arise when all the dimensions are taken to be open, and finally we computed the η invariant for a more general class of operators, including the case when a mass gap is present. The peculiarity of our approach is the possibility of disentangling the contributions from bound states and continuum modes to the spectral asymmetry.

These, which are our main results, we applied to a simple problem, namely, the computation of the charge which gives rise to superconductivity of vortices. Of course, the discussion of the conditions for anomalous charge creation may be used as a determination of sufficient conditions for the presence of chiral anomalies; above, we have explicitly verified that our method agrees with the anomaly computations which have been widely discussed in the literature from this point of view.[1]

A more interesting possible application of our methods is their use to compute different anomalous quantum numbers in addition to the electric charge. In this case, the spectral asymmetry is weighted by the eigenvalues of the pertinent operator. A systematic study of the anomalous vacuum quantum numbers, and even a more direct determination of the vacuum current and supercurrent would thus be possible. The computation of the anomalous angular momentum in the presence of a superconducting string, in particular, is of peculiar interest since the conjectural results of Ref. 4 suggest unexpected anomalous effects. Finally, it is interesting to ask whether these methods could be used to shed light on the gravitational interaction of fermions with a vortex, and, in general, on anomalous quantum numbers due to gravitational anomalies.

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