Hamiltonian lattice gravity. II. Discrete moving-frame formulation

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(Received 4 April 1988)

A discrete version of a moving-frame formalism is developed and is used to obtain lattice gravity in a Hamiltonian formulation. It is more straightforward to transcribe the constraints of these theories from the continuum to the lattice using these techniques rather than the ordinary Regge calculus. The closure of the algebra of constraints has not been studied.

I. INTRODUCTION

Hamiltonian gravity is a theory of constraints. The algebra of these constraints must close, be it under Poisson brackets or under quantum-mechanical commutators, and in the continuum the classical algebra does do so. No completely satisfactory transcription of such a Hamiltonian theory exists for discrete gravity on a spatial lattice. Two recent[1] formulations have been presented, where unfortunately this algebra explicitly fails to close. In a previous publication,[2] referred to as I, we showed that should such an algebra close, it will do so in a nonlocal manner. Namely, even though the commutator of any two operators of this algebra fails to vanish only when the arguments of these operators are restricted to nearby lattice sites, the commutator itself has contribution from the entire lattice. Such nonlocality appears in a canonical Hamiltonian obtained from a Lagrangian lattice theory. In this paper we will present a different formulation of lattice Hamiltonian gravity. Although we have not checked it explicitly, there is a possibility that the algebra of constraints will close and we hope to return to this problem in a future work.

There are two significant differences in the present approach from those used in Ref. 1: One, we use a lattice version of an "n-bein" or moving-frame formalism and, two, the lattice geometry is used as a guide, especially in the structure of the momentum constraints. In a previous work[3] the introduction of a moving-frame formalism was found to be useful in the transcription of a continuum functional measure to the lattice; of course such a formalism will be crucial if ever we wish to incorporate fermion matter fields. In the continuum the momentum constraints are the generators of diffeomorphisms, and their structure, as well as commutation relations, are determined by geometry.[4] In the discrete situation we do not have an invariance under such a diffeomorphism group; in I we showed that there are lattice transformations that play analogous roles. Insisting that such transformations yield acceptable geometries, to be made specific in subsequent sections, fixes the form of these transformations. We do not have such a geometric picture of the Hamiltonian constraint, but in the present formulation the transcription from the continuum to the lattice is almost automatic. The price we pay for this seeming simplification is that we have to enlarge phase space by introducing redundant variables and canonical momenta; the elimination of these will lead to the nonlocalities alluded to earlier.

As a by-product, we get a discrete moving-frame formalism. The lattice n-ads, spin connections have interesting interpretations in terms of discrete geometries. Such a formalism will have applications, not only to a Hamiltonian formulation, but also to the Lagrangian one,[3] especially when fermion matter fields are included.

In Sec. II a moving-frame version of continuum gravity is reviewed. Instead of presenting results for an arbitrary number of space-time dimensions, for pedagogical reasons, we discuss separately the $2+1$ and $3+1$ cases. In Sec. III the lattice $(2+1)$-dimensional system is developed, while in Sec. IV the same is done for the $3+1$ situation.

II. CONTINUUM HAMILTONIAN GRAVITY IN A MOVING-FRAME FORMALISM

In a transcription of the functional-integration measure[3] from the continuum to the lattice, a moving-frame formalism was found to be useful; likewise in the transcription of continuum Hamiltonian gravity theory we shall use moving-frame coordinates. We first review continuum Hamiltonian gravity in this framework.

The geometric properties of a $(d+1)$-dimensional space, foliated into d-dimensional spacelike manifolds is described by a metric tensor ${}^{(d+1)}g_{\mu\nu}$, which we parametrize with the aid of the usual[5] lapse and shift functions N, N_A and a d-dimensional metric ${}^{(d)}g_{AB}$:

$$\begin{aligned} {}^{(d+1)}g_{TT} &= N^2 - N_B N^B , \\ {}^{(d+1)}g_{TA} &= N_A, \quad {}^{(d+1)}g_{AB} = -{}^{(d)}g_{AB} . \end{aligned} \tag{2.1}$$

The d spatial coordinates are denoted by the upper case subscripts $A, B, \ldots$ while T is reserved for the $(d+1)$th coordinate. We introduce the $(d+1)$-ad, ${}^{(d+1)}e^s_\mu$ related to the metric tensor via

$${}^{(d+1)}g_{\mu\nu} = {}^{(d+1)}e^s_\mu \, {}^{(d+1)}e^t_\nu \eta_{st} , \tag{2.2}$$

where η_{st} is a flat $(d+1)$-dimensional Minkowski metric; likewise we write the d-dimensional metric tensor ${}^{(d)}g_{AB}$ in terms of d-ads, ${}^{(d)}e^a_A$:

$${}^{(d)}g_{AB} = {}^{(d)}e^a_A \, {}^{(d)}e^a_B ; \tag{2.3}$$

summation over the flat-space index a is implied. We may now relate the $(d+1)$-ads to the lapse and shift functions and to the d-ads. The Minkowski indices are labeled by 0, for the timelike component, and by $a,b,\ldots$ for the spacelike directions:

$$\begin{aligned} {}^{(d+1)}e_T^0 &= N, \quad {}^{(d+1)}e_A^a = {}^{(d)}e_A^a \ , \\ {}^{(d+1)}e_T^a &= N^a, \quad {}^{(d+1)}e_A^0 = 0 \ , \end{aligned} \tag{2.4}$$

with $N^a = {}^{(d)}e_A^a N^A$. The $(d+1)$-dimensional moving frame vectors satisfy (in the absence of fermions) the covariant curl-free condition

$$\partial_\mu {}^{(d+1)}e_\nu^s - {}^{(d+1)}\omega_\mu^{st}\, {}^{(d+1)}e_\nu^t - (\mu \leftrightarrow \nu) = 0 \ . \tag{2.5}$$

The above equation may be used to define the spin connection ${}^{(d+1)}\omega_\mu^{st}$ in terms of the $(d+1)$-ads and their inverses. The d-ads satisfy a similar relation with ${}^{(d+1)}\omega_\mu^{st}$ replaced by ${}^{(d)}\omega_A^{ab}$. The d-dimensional spin connection is related to the $(d+1)$-dimensional one by

$${}^{(d)}\omega_A^{ab} = {}^{(d+1)}\omega_A^{ab} \ . \tag{2.6}$$

As for the most part no ambiguity will arise, we shall drop the prefix (d) in front of the d-ads and d-dimensional spin connections. Although we could continue this presentation for arbitrary dimensions, it is more convenient to discuss the (2 + 1)- and the (3 + 1)-dimensional cases separately.

A. Hamiltonian gravity in 2+1 dimensions

For pedagogical reasons we will first study a theory of gravity in 2 + 1 dimensions; although in this case there are no propagating modes, it is still an interesting theory due to its relative simplicity and due to the connection it has with string theories.

With a convenient choice of units, the Lagrangian for gravity in this dimension is

$$L = \tfrac{1}{4}\epsilon^{\mu\nu\lambda}\epsilon_{abc}\left[R_{\mu\nu}^{ab} + \frac{\Lambda}{3}e_\mu^a e_\nu^b\right]e_\lambda^c \ , \tag{2.7}$$

where the three-curvature $R_{\mu\nu}^{ab}$ is related to the three-dimensional spin connection by

$$R_{\mu\nu}^{ab} = \partial_\mu \omega_\nu^{ab} - \omega_\mu^{an}\omega_\nu^{nb} - (\mu \leftrightarrow \nu) \tag{2.8}$$

and Λ is the cosmological constant. Expressed in terms of the dyads and of the spin connections, this Lagrangian takes the form

$$\begin{aligned} L = \epsilon^{AB}\epsilon_{ab}[\,&(\partial_T \omega_A^{0a} - \partial_A \omega_T^{0a} + \omega_T^{0c}\omega_A^{ca} - \omega_A^{0c}\omega_T^{ca})e_B^b \\ &+ (\partial_A \omega_B^{0a} - \omega_A^{0c}\omega_B^{ca})N^b \\ &+ \tfrac{1}{2}(\partial_A \omega_B^{ab} - \omega_A^{ac}\omega_B^{cb} - 2\Lambda e_A^a e_B^b)N] \ . \end{aligned} \tag{2.9}$$

An integration by parts yields the momentum π_a^A, conjugate to e_A^a:

$$\pi_a^A = -\epsilon^{AB}\epsilon_{ab}\omega_B^{0b} \ . \tag{2.10}$$

We shall find it convenient to use π_a^A and ω_A^{0a} interchangeably. The coefficients of N and N^b in Eq. (2.9) above give us the Hamiltonian and momentum constraints of general relativity:

$$\begin{aligned} \mathcal{H} = \tfrac{1}{2}\epsilon^{AB}\epsilon_{ab}(\,&\partial_A \omega_B^{ab} - \omega_A^{ac}\omega_B^{cb} - \omega_A^{0a}\omega_B^{0b} \\ &- 2\Lambda e_A^a e_B^b) = 0 \ , \end{aligned} \tag{2.11a}$$

$$\mathcal{H}^a = \partial_A \pi_a^A + \omega_A^{ab}\pi_b^A = 0 \ . \tag{2.11b}$$

In a moving-frame formalism we have the freedom of making independent, local Euclidean rotations on the dyads. We wish that the generators of these rotations vanish; ω_T^{ca} acts as a Lagrange multiplier whose coefficient is just this operator:

$$\mathcal{T} = \epsilon_{AB}\omega_A^{0c}e_B^c = 0 \ . \tag{2.12}$$

The coefficient of ω_T^{0a} vanishes when we impose the relations between the spin connections and the dyads:

$$\partial_A e_B^b - \omega_A^{bc}e_B^c - (A \leftrightarrow B) = 0 \ . \tag{2.13}$$

Again, the above relation defines the spatial spin connections in term of the moving-frame coordinates. We end this section with a prescription for quantizing this theory. The vacuum-to-vacuum amplitude is given by

$$Z = \int \prod_x [de_A^a(x)d\pi_a^A(x)\delta(\mathcal{H}(x))\delta(\mathcal{H}^a(x))\delta(\mathcal{T}(x))(\text{gauge-fixing terms})]\exp\left[i\int d^3x\, \pi_a^A(x)\dot{e}\,_A^a(x)\right] \ . \tag{2.14}$$

The quantum-mechanical ordering problem is still present as Eq. (2.14) is only formal.

B. Hamiltonian gravity in 3+1 dimensions

For this dimensionality the Einstein-Hilbert Lagrangian is

$$L = \tfrac{1}{4}e^{\mu\nu\lambda\rho}\epsilon_{abcd}\left[R_{\mu\nu}^{ab} + \frac{\Lambda}{3}e_\mu^a e_\nu^b\right]e_\lambda^c e_\rho^d \ . \tag{2.15}$$

In terms of the triads and associated spin connections it may be rewritten as [cf. Eq. (2.9)]

$$\begin{aligned} L = \epsilon^{ABC}\epsilon_{abc}[\,&(\partial_T \omega_A^{0a} - \partial_A \omega_T^{0a} - \omega_T^{0d}\omega_A^{da} + \omega_A^{0d}\omega_T^{da})e_B^b e_C^c - 2(\partial_A \omega_B^{0a} - \omega_A^{0d}\omega_B^{da})e_C^b N^c \\ &+ \tfrac{1}{2}(\partial_A \omega_B^{ab} - \omega_A^{ad}\omega_B^{db} - \omega_A^{0a}\omega_B^{0b} - 2\Lambda e_\Lambda^a e_B^b)e_C^c N] \ . \end{aligned} \tag{2.16}$$

The momentum conjugate to the triad e_A^a is

$$\pi_a^A = -2\epsilon^{ABC}\epsilon_{abc}\omega_B^{0b}e_C^c \ . \tag{2.17}$$

We will use interchangeably π_a^A and ω_A^{0a}; the latter are obtainable from the former:

$$\omega_A^{0a} = \frac{e_A^a \pi_b^B e_B^b - 2 e_A^b \pi_b^B e_B^a}{4\det(e)} \ . \tag{2.18}$$

As in the previous case the coefficients of ω_T^{0a}, in Eq. (2.16), vanish as a consequence of the relations of the spin connections to the triads; these relations are of the same form as those expressed in Eq. (2.13). ω_T^{ab} acts as a Lagrange multiplier ensuring the vanishing of the local angular momenta:

$$\mathcal{T}^a = 2\epsilon^{ABC}\omega_A^{0b}e_B^b e_C^a = 0 \ . \tag{2.19}$$

The Hamiltonian constraint is

$$\mathcal{H} = \tfrac{1}{2}\epsilon^{ABC}\epsilon_{abc}(\partial_A\omega_B^{ab} - \omega_A^{ad}\omega_B^{db} - \omega_A^{0a}\omega_B^{0b} - 2\Lambda e_A^a e_B^b)e_C^c = 0 \ . \tag{2.20}$$

Because of the covariant constancy of the triads, the momentum constraints maybe expressed in two equivalent ways:

$$\mathcal{H}^c = -2\epsilon^{ABC}\epsilon_{abc}(\partial_A\omega_B^{0a} - \omega_A^{0d}\omega_B^{da})e_C^c = \partial_A\pi_c^A + \omega_A^{cd}\pi_d^A = 0 \ . \tag{2.21}$$

$$= \partial_A\pi_c^A + \omega_A^{cd}\pi_d^A = 0 \ .$$

The quantum-mechanical vacuum-to-vacuum amplitude is

$$Z = \int \prod_x [de_A^a(x)d\pi_a^A(x)\delta(\mathcal{H}(x))\delta(\mathcal{H}^a(x))\delta(\mathcal{T}^a(x))(\text{gauge-fixing terms})]\exp\left[\int d^4x\, \pi_a^A(x)\dot{e}^c{}_A(x)\right] . \tag{2.22}$$

III. HAMILTONIAN LATTICE GRAVITY IN 2+1 DIMENSIONS

A. Discrete two-dimensional moving-frame formalism

As in Regge calculus,[6] we shall discretize space by approximating a curved two-dimensional manifold by a collection of flat triangles. In Fig. 1 we show a piece of such a triangulation. In the Regge calculus the link lengths are the dynamical variables. These variables do not, however, have any simple interpretation in the sense of lattice geometries; they are neither lattice scalars nor lattice vectors and have an indirect relation to a lattice metric defined on the triangles. It is simpler to obtain the lattice analogue of the moving-frame coordinates. A lattice vector lives on links, and for any vector field $V_A(x)$ defined on our piecewise flat manifold, we may define a lattice vector[7] associated with the link (i,j) by

$$V_{ij} = \int_i^j V_A(x)dx^A \ . \tag{3.1}$$

In this manner, we would like to associate with the dyads e_A^a, link vectors l_{ij}^a by using Eq. (3.1):

$$l_{ij}^a = \int_i^j e_A^a dx^A \ . \tag{3.2}$$

However, there will be different moving-frame coordinate systems associated with the triangles S and S' on either side of (ij); thus, for each triangle having (ij) as a boundary, we associate a vector $l_{ij}^a(S)$. We want the magnitude of each of these vectors to equal the corresponding link length, and thus one may be obtained from the other by a rotation:

$$l_{ij}^a(S') = O^{ab}(S',S)l_{ij}^b(S) \ . \tag{3.3}$$

As we shall see, these rotations are related to the spin connections of the continuum formalism. This result is obtained using an integrated form of Eq. (2.13). We integrate this expression on the areas bounded by contours shown in Figs. 2(a) and 2(b) and then use Stokes's theorem. The first area is entirely inside one of the triangles; inside such a flat region the spin connection vanishes:

$$\sum_{(ij)\in S} \int_i^j e_A^a dx^A = \sum_{(ij)\in S} l_{ij}^a(S) = 0 \ . \tag{3.4}$$

This yields the expected result that the sum of the link vectors around the sides of a triangle vanishes. Let us, also, look at the situation depicted in Fig. 2(c) where we assume that the spin connection has a nonvanishing component only in the B direction and that it is relatively constant over the rectangle (i,j,k,l). Again, using Stokes's theorem, we obtain

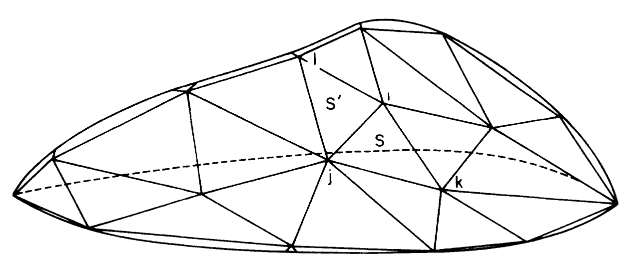

FIG. 1. Triangulation of a curved surface. Triangles S and S' share a common link (ij).

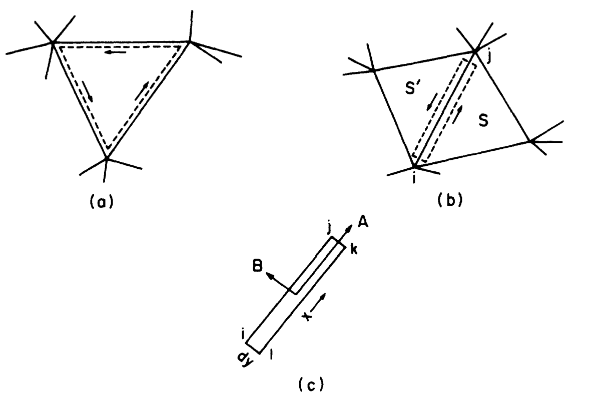

FIG. 2. Two-dimensional regions over which the expression relating the spin connections to the dyads is integrated. (a) Area totally within one flat triangle. (b) Area straddling two triangles. (c) Generic thin sliver illustrating the notation used in Eq. (3.5).

$$\int_i^j e_A^a dx^A = (\delta^{ab} + \omega_B^{ab} dy) \int_l^k e_A^b dx^A \ . \tag{3.5}$$

Note that $\omega_B^{ab} dy$ may be viewed as the infinitesimal part of a two-dimensional rotation. As shown in Fig. 2(b), we have such an area straddling a link joining two triangles. The spin connection is concentrated on this link and this results in Eq. (3.3). The independent variables associated with each link are the magnitude of the link vector l_{ij} and the two unit vectors $\hat{l}_{ij}(S)$ and $\hat{l}_{ij}(S')$. The rotation matrix $O^{ab}(S',S)$ is defined by the relation

$$\hat{l}_{ij}^a(S') = O^{ab}(S',S) \hat{l}_{ij}^b(S) \ . \tag{3.6}$$

Similarly to the way Eq. (2.8) relates the spin connections to the curvature tensor, there is a relation of these rotations and the lattice curvature. We remember that in Regge calculus, the curvature scalar is concentrated at the vertices and is twice the deficit angle ε; in the discrete case, it is easy to show that, for a particular vertex i,

$$2 \sin \varepsilon_i = R_i^{ab} \epsilon_{ab} \ , \tag{3.7}$$

where the curvature matrix is defined as

$$R_i^{ab} = \left[\prod_n O(S_{n+1}, S_n) \right]^{ab} ; \tag{3.8}$$

in the above, the product extends over all pairs of triangles emanating from vertex i.

The last geometric topic we wish to discuss concerns the definition on the lattice of functions that are Euclidean vectors but curvilinear scalars, such as, for example, the lapse function N^a. In the continuum, to any such function we may associate a Euclidean scalar but a curvilinear vector, namely, $N_A = N^a e_A^a$, with an inverse relation, $N^a = N_A e_a^A$. Now consider the two triangles S and S' on either side of the link (ij) in Fig. 1. Curvilinear vectors are associated with links; on the link (ij) we have a vector N_{ij}. The Euclidean vector functions $N_i^a(S)$ and $N_i^a(S')$ are obtained by contracting N_{ij} with vectors belonging to the dual basis: namely,

$$\begin{aligned} N_i^a(S) &= N_{ij} \frac{\epsilon^{ab} l_{ik}^b(S)}{\epsilon^{ab} l_{ij}^a(S) l_{ik}^b(S)} \ , \\ N_i^a(S') &= N_{ij} \frac{\epsilon^{ab} l_{il}^b(S')}{\epsilon^{ab} l_{ij}^a(S') l_{il}^b(S')} \ . \end{aligned} \tag{3.9}$$

In addition to being labeled by i, the vector $N_{ij}^a(S)$ also depends on the link (ij); as in most cases, no ambiguity arises; we shall, for notational compactness, drop this dependence. Note specifically that this $N_i^a(S)$ is orthogonal to $l_{ik}^a(S)$, with a similar expression for S' and that $N_i^a(S) l_{ij}^a(S) = N_i^a(S') l_{ij}^a(S')$.

B. Hamiltonian gravity on a discrete two-dimensional manifold

As there are three variables associated with each of the links so there will be three canonical momenta π_{ij}, the momentum canonical to l_{ij}, and two angular momenta, $L_{ij}(S)$ and $L_{ij}(S')$, canonically related to the two unit vectors. We may also associate a momentum vector $\pi_{ij}^a(S)$ with each link vector:

$$\pi_{ij}^a(S) = \hat{l}_{ij}^a(S) \pi_{ij} - \frac{\epsilon^{ab} \hat{l}_{ij}^b(S) L_{ij}(S)}{l_{ij}} \ , \tag{3.10}$$

with a similar expression for $\pi_{ij}^a(S')$. The following relation for phase-space volumes, for each link (ij),

$$d\pi \, dL(S) dL(S') dl \, d\hat{l}(S) d\hat{l}(S') = d^2\pi(S) d^2\pi(S') d^2 l(S) d^2 l(S') \delta(l(S) - l(S')) \delta(\hat{l}^a(S)\pi^a(S) - \hat{l}^a(S')\pi^a(S')) \ , \tag{3.11}$$

is easy to show. As the sum of the link vectors around any triangle vanishes, Eq. (3.4), we may choose

$$\sum_{(ij) \in S} \pi_{ij}^a(S) = 0 \ . \tag{3.12}$$

Ideally, the above relation should be used to eliminate redundant momentum variables and retain only an independent set; we cannot accomplish this explicitly and even in principle the solutions would introduce nonlocal expressions in that the eliminated momenta, when expressed in terms of the retained ones, would involve variables from the whole lattice and not just from nearby links. *This is the nonlocality mentioned in Sec. I.* We shall keep Eqs. (3.4) and (3.12) as constraints on phase space. We now turn our attention to rewriting the constraints of Eqs. (2.11) and (2.12) on the lattice.

(a) Angular momentum constraint. The angular momentum constraint is easy; for each triangle we have

$$\mathcal{T}(S)=\sum_{(ij)\in S} L_{ij}(S)=0 \ . \tag{3.13}$$

(b) Momentum constraints. In the continuum, the momentum constraints are the generators of diffeomorphism transformations, which do not change, but just reparametrize the spatial manifold. On the lattice, such transformations will change the intrinsic geometry. What we shall require is that neither the Hamilton-Jacobi functional nor the quantum-mechanical wave functional change under such transformations. What is crucial, however, is that these transformations generate an acceptable lattice geometry. Namely, whatever these transformations do to the link vectors, the sum of the transformed vectors around any triangle must vanish and the lengths of vectors belonging to two triangles sharing a common link must be the same. In the continuum the generator of diffeomorphism is just the covariant divergence of the momentum operator. The lattice divergence at a vertex[7] is related to the sum of vectors emanating from that vertex. A typical situation is illustrated in Fig. 3. We have several triangles, labeled by n, with $n=1,2,\ldots$, and with 0 as a common vertex; the other vertices of this system are denoted by $m=1,2,\ldots$, with vertices m and $m+1$ belonging to triangle S_m. Using the procedure outlined in the discussion surrounding Eq. (3.9) we can define on each triangle a Euclidean vector $N^a(S_n)$. With an appropriate choice of the $N^a(S_n)$'s, setting

$$\mathcal{H}_0^a N_0^a=\sum_n N_0^a(S_n)[\pi_{0i}^a(S_n)+\pi_{0j}^a(S_n)] \tag{3.14}$$

to zero yields the momentum constraints. Again we should remember that $N_0^a(S_n)$ depends on the link $(0i)$. To show that the transformations generated by this operator preserve the geometry, consider the set $N^a(S_1)$ and $N^a(S_2)$ originating from an N_{02} [cf. Eq. (3.9)]. It is easy to show that l_{12}^a and l_{23}^a do not change; $l_{01}^a(S_1)$ and $l_{03}^a(S_2)$ are just rotated and thus do not change their lengths while the change in length of $l_{01}^a(S_1)$ is equal to that of $l_{01}^a(S_2)$.

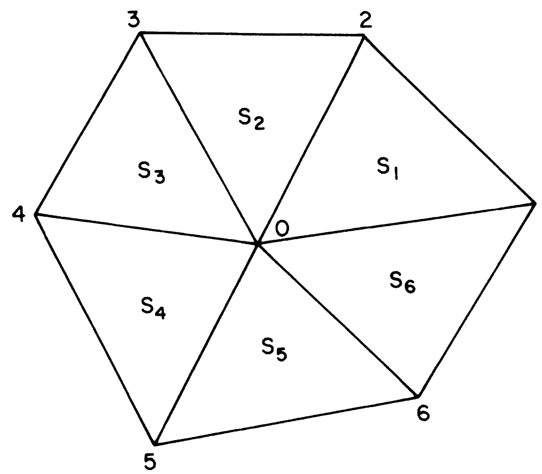

FIG. 3. Complex of triangles surrounding the vertex 0.

(c) Hamiltonian constraints. As the structure of lattice curvature is described in the discussion surrounding Eq. (3.7), the transcription of the Hamiltonian constraints, Eq. (2.11a), to the lattice is obvious once the cross product is defined. This is particularly simple for vectors whose sum around any basic triangle vanishes. In general, let $V_A^r(x)$, $r=1,2$ be a two-vector field, whose discrete versions V_{ij}^r satisfy $\sum_{(ij)\in S} V_{ij}^r=0$; similarly, let N_i be the discretization of a scalar field $N(x)$. The contribution of a triangle S with vertices i,j,k to the integral of the cross product of the V's is

$$\int_S d^2x\, N(x)V_A^r(x)V_B^s(x)\epsilon^{AB}\epsilon_{rs}\rightarrow \tfrac{1}{9}(N_i+N_j+N_k)(V_{ij}^r V_{jk}^s+V_{jk}^r V_{ki}^s+V_{ki}^r V_{ij}^s)\epsilon_{rs} \ . \tag{3.15}$$

Referring to Fig. 3, the Hamiltonian constraint at the vertex 0 is

$$\mathcal{H}_0=\varepsilon_0-\tfrac{1}{18}\sum_n [\pi_{0i}^a(S_n)\pi_{ij}^b(S_n)+\pi_{ij}^a(S_n)\pi_{j0}^b(S_n)+\pi_{j0}^a(S_n)\pi_{0i}^b(S_n)]\epsilon_{ab}$$
$$-\frac{\Lambda}{9}\sum_n [l_{0i}^a(S_n)l_{ij}^b(S_n)+l_{ij}^a(S_n)l_{j0}^b(S_n)+l_{j0}^a(S_n)l_{0i}^b(S_n)]\epsilon_{ab} \ . \tag{3.16}$$

By using the phase space constraints on the $l_{ij}^a(S)$'s and the $\pi_{ij}^a(S)$'s as well as the three dynamical constraints $\mathcal{H}$, $\mathcal{H}^a$, and $\mathcal{T}$, we can write the discrete version of Eq. (2.14); it is somewhat tedious and we refrain from doing this explicitly.

IV. HAMILTONIAN LATTICE GRAVITY IN 3+1 DIMENSIONS

A. Discrete three-dimensional moving-frame formalism

Many of the arguments used in the previous section may be carried over to this dimensionality; we shall not present them in as great detail as before. A curved three-dimensional manifold is approximated by piecewise flat tetrahedra. Each link (ij) belongs to several such tetrahedra, the precise number depending on the specific tessellation. Again, for each tetrahedron S we define a three-dimensional link vector $l_{ij}^a(S)$. Vectors belonging to different tetrahedra, but to the same link are related by Eq. (3.3), where this time $O^{ab}(S',S)$ is a three-dimensional rotation associated with the triangle common to the two tetrahedra. As we require that the sum of the link vectors around any triangle vanish, we have, for each tetrahedron, three independent link vectors that form the bases of the moving frames, or triads. As the rotation matrices $O^{ab}(S',S)$ are associated with triangles, we have three relations of the type of Eq. (3.6), only two of which are independent; given the l_{ij}^a's, these relations determine the rotation matrices.

In the two-dimensional situation the curvature matrices were associated with points, whereas in this case they will be defined on links. Consider all the tetrahedra S_n containing a common link (ij) and labeled sequentially in a counterclockwise manner. We define

$$R_{ij}^{ab} = \left[\prod_n O(S_n, S_{n-1})\right]^{ab} . \tag{4.1}$$

The deficit angle ε_{ij} around this link is obtained from

$$2\sin\varepsilon_{ij} = R_{ij}^{ab}\hat{l}_{ij}^{c}\epsilon_{abc} . \tag{4.2}$$

As in the two-dimensional case, we close this section with a discussion of the proper definition of a Euclidean vector field N^a. Consider the situation depicted in Fig. 4 where we have several tetrahedra surrounding a common link (01) and links $(0m)$, $m=2,3,\ldots$, forming the edges of these tetrahedra. [We should have indicated a deficit angle around the link (01); for clarity of presentation we have not done so.] To a vector field N_{01} we associate the Euclidean fields $N_0^a(S_m)$ by

$$N_0^a(S_m) = N_{01}\frac{\epsilon^{abc}l_{0m}^b(S_m)l_{0(m+1)}^c(S_m)}{\epsilon^{abc}l_{01}^a(S_m)l_{0m}^b(S_m)l_{0(m+1)}^c(S_m)} . \tag{4.3}$$

As in the previous section, the explicit dependence on the link (01) is suppressed. For each tetrahedron, $N_0^a(S_m)$ is orthogonal to l_{0m}^a and to $l_{0(m+1)}^a$ and its scalar product with l_{01}^a is independent of the particular tetrahedron.

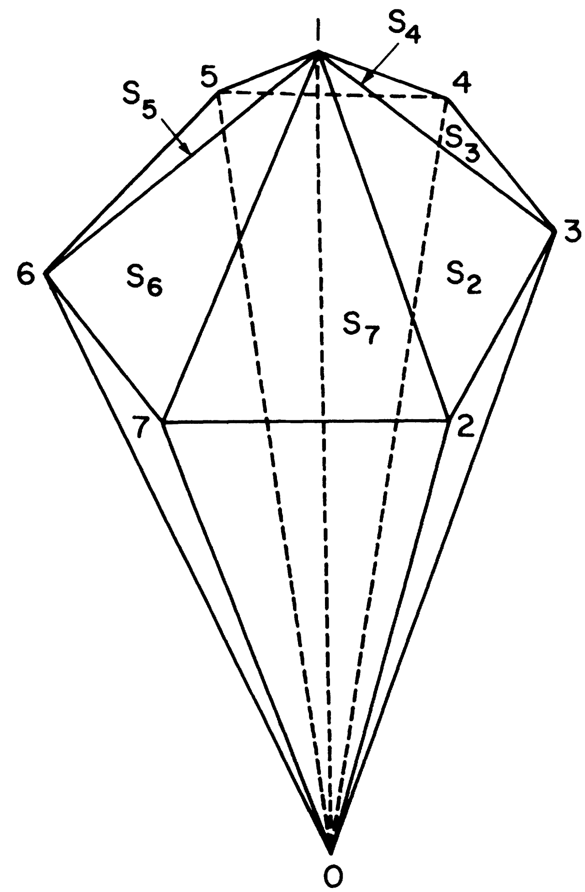

FIG. 4. Complex of tetrahedra sharing a common link (01). Tetrahedron S_2 has vertices (0,1,2,3).

B. Hamiltonian gravity on a discrete three-dimensional manifold

On each link (ij) we have l_{ij} and n unit vectors, the $\hat{l}^a(S_n)$'s, as dynamical coordinates. Their canonical momenta are π_{ij} and n angular momenta, $L_{ij}^a(S_n)$. The index n refers to the various tetrahedra containing the link in question. For each of these tetrahedra we may define a momentum vector π_{ij}^a associated with (ij):

$$\pi_{ij}^a(S_n) = \hat{l}_{ij}^a(S_n)\pi_{ij} - \frac{\epsilon^{abc}\hat{l}_{ij}^b(S_n)L_{ij}^c(S_n)}{l_{ij}} . \tag{4.4}$$

For each link the following relation among phase-space volumes,

$$d\pi\, dl \prod_n dL^a(S_n)d\hat{l}^a(S_n) = \prod_n d^3\pi(S_n)d^3l(S_n)\delta(l(S_n)-l(S_{n-1}))\delta(\hat{l}^a(S_n)\pi^a(S_n)-\hat{l}^a(S_{n-1})\pi^a(S_{n-1})) , \tag{4.5}$$

holds. We require Eq. (3.12) hold on all triangles.

(a) Angular momentum constraints. For each tetrahedron we have

$$\mathcal{T}^a(S) = \sum_{(ij)\in S} L_{ij}^a(S) = 0 . \tag{4.6}$$

(b) Momentum constraints. The argument for the construction of the operator giving the momentum constraints is the same as the one presented for the two-dimensional case. Let S_n be the set of tetrahedra having a common vertex, say 0; denote the other vertices of each tetrahedron by i,j,k. Then,

$$\mathcal{H}^a N_0^a = \sum_n N^a(S_n)[\pi_{0i}^a(S_n)+\pi_{0j}^a(S_n)+\pi_{0k}^a(S_n)] , \tag{4.7}$$

with an appropriate choice of the $N^a(S_n)$'s, yields, upon setting it equal to zero, the desired momentum constraints. As in the lower-dimensional case, the $N^a(S_n)$'s are determined by Eq. (4.3); under infinitesimal transformations by $\mathcal{H}^a$ lattice geometries are preserved.

(c) Hamiltonian constraints. The lattice version of the integral over a tetrahedron S_2, with vertices 0, 1, 2, and 3 (cf. Fig. 4) of a triple product of the vector fields, $V_A^r(x)$, with $r=1,2,3$, is

$$\int_{S_2} d^3x\, N(x)V_A^r(x)V_B^s(x)V_C^t(x)\epsilon^{ABC}\epsilon_{rst} \to \tfrac{1}{16}(N_0+N_1+N_2+N_3)\times(V_{01}^r V_{02}^s V_{03}^t + V_{12}^r V_{10}^s V_{13}^t + V_{20}^r V_{21}^s V_{23}^t + V_{31}^r V_{30}^s V_{32}^t)\epsilon_{rst} \,. \tag{4.8}$$

We have written the above in a form symmetric in all the V_{ij}^r's. Because of the condition that the sum of the V^r's vanishes around any triangle we could have just kept one of the four products on the right-hand side above and multiplied the answer by four; we shall do thus below. The Hamiltonian constraints will be presented for the set of tetrahedra discussed in connection with the momentum constraints, namely, all tetrahedra S_n, emanating from a common vertex 0. Within each tetrahedron we will label the sites by $0,i,j,k$; we will use l to denote any of the sites connected by a link to 0:

$$\mathcal{H}_0 = \tfrac{1}{2}\sum_l l_{0l}\varepsilon_{0l} - \tfrac{1}{8}\sum_n [\omega_{0i}^a(S_n)\omega_{0j}^b(S_n)\omega_{0k}^c(S_n)]\epsilon_{abc} - \frac{\Lambda}{4}\sum_n [l_{0i}^a(S_n)l_{0j}^b(S_n)l_{0k}^c(S_n)]\epsilon_{abc} \,, \tag{4.9}$$

where the ω_{ij}^r's are related to the momenta by [cf. Eq. (2.18)]

$$\omega_{ij}^a = \frac{\sum_{(kl)\in S}(l_{ij}^a \pi_{kl}^b l_{kl}^b - 2 l_{ij}^b \pi_{kl}^b l_{kl}^a)}{8\,\mathrm{vol}(S)} \,. \tag{4.10}$$

The factor $\frac{1}{2}$ in front of the deficit angle term in Eq. (4.9) reflects the fact that ε_{0l} appears twice: once in $\mathcal{H}_0$ and once in $\mathcal{H}_l$.

V. CONCLUSION

Through the use of a lattice version of a moving-frame formalism we obtained a discrete version of Hamiltonian gravity. As mentioned in Sec. I, we paid a price in that the phase space had to be enlarged; even though for each triangle the sum of the link vectors and the sum of their canonical momenta vanishes, these are treated in a symmetric fashion. It is not possible to implement Eqs. (3.4) and (3.12) or their three-dimensional analogues, in a closed form. The redundant variables will be nonlocal functions of the retained ones. It is this that hinders an immediate check of the closure or nonclosure of the constraint algebra. We hope to return to this problem soon.

The discrete moving frame formalism has its own geometric aesthetic appeal; the spin connections are explicitly related to Euclidean rotations between coordinate systems defined on contiguous simplices, and the curvature is related to a product of such rotations around a subsimplex. This is reminiscent of the definition of field strengths in lattice gauge theories.

ACKNOWLEDGMENT

This work was supported in part by the National Science Foundation under Grant No. PHY-86-05552.

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