Anomaly-free theories in $D = 4k$ dimensions

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General constraints are found for local anomaly-free theories in $D=4k$ dimensions which yield chiral four-dimensional theories after compactification: (1) the Yang-Mills group must have a complex representation; (2) most odd-order traces vanish, i.e., $\mathrm{Tr}F^1=\mathrm{Tr}F^3=\cdots=\mathrm{Tr}F^{2k-3}=0$ and $\mathrm{Tr}F^{2k+1}=c\,\mathrm{Tr}F^{2k-1}$ for $D=4k$. Any complex representation of the SO$(4l+2)$ group ($l\geq k+1$ or $l=k-1$) in $D=4k$ satisfies the constraints. Other simple solutions are also given.

I. INTRODUCTION

Recently, the investigation of particle physics has become more transparent by using the topological method. In particular, it is now well known that the [gravitational, Yang-Mills (YM), and mixed] anomalies in gauge theories at D dimensions are related to $(D+2)$-differential forms made of traces of a curvature two-form R and a gauge field-strength two-form F (Ref. 1). We hereafter call these differential forms I-forms. It turns out that anomalies defined in this way satisfy the so-called Wess-Zumino consistency condition.[2] Furthermore, generating functions of I-forms for various fermionic fields are now known.[3] Consequently, it is possible to derive anomaly-free constraints for any dimensions. So far only 10-dimensional theories were paid close attention to in the literature, because of superstrings. However, for theories of more than two-dimensional objects, the critical dimension may be different from 10, although it will take a long time to find out about it. In addition, now string theories can be formulated in dimensions different from critical dimensions.[4] Thus, we believe that it is important to classify all anomaly-free field theories for future physical theories, especially since it is possible to do so with present knowledge. For theories in $D=4k-2$, we have extended the analysis initiated by Thierry-Mieg and Schellekens[5] and have done a systematic investigation of anomaly-free theories in our previous papers.[6] However, it seems that no systematic investigations were done for theories at $D=4k$ dimensions from a general point of view. (There are some works[7] on pure gauge anomaly-free theories. Also, some works have been done from the string point of view.[4]) Therefore, in this paper, we investigate *local* anomaly free theories in $D=4k$ dimensions which yield *chiral* four-dimensional theories. We do *not* assume stringlike structures, e.g., modular invariance, and thus our solutions contain more than those which string theories predict. It is sufficient to have an anomaly-free theory at higher dimension, since anomaly-free theories remain anomaly-free at lower dimensions as long as no isometries and no U(1) symmetries are generated after compactification.[8]

We summarize simple consequences obtainable from the fact that I forms are made out of traces of both gravitational and Yang-Mills two-forms, R and F. First, note that $\mathrm{Tr}\,R^{\mathrm{odd}}=0$, since R is antisymmetric. Thus, traces of R are nonvanishing only for $4k$-forms. Meanwhile, traces of F can be nonvanishing for $2m$-forms. Second, note also the fact that if $I(R,F)$ vanishes, one can always find local counterterms to do away with anomalies. The reason for this is that we have the following relations[2] among $I(R,F)$ and the anomaly ω_D^1:

$$I(R,F)_{D+2}=d\omega_{D+1}^0\,,$$

$$\delta\omega_{D+1}^0=-d\omega_D^1,\quad \delta\omega_D^1=-d\omega_{D-1}^2\,.$$

The third equation is the Wess-Zumino consistency condition.[2] If $I(R,F)=0$, then we can find an α locally such that $\omega_{D+1}^0=d\alpha$. Then, using $\{\delta,d\}=0$, we obtain $\omega_D^1=\delta\alpha$ (modulo an exact form). Thus, by introducing α into the action, we can do away with the anomaly. Using these two facts, we obtain (i) no local anomalies exist in $D=$odd dimension, (ii) no local gravitational anomalies exist in $D=4k$ dimension, and (iii) all possible anomalies exist in $D=4k-2$. However, these anomalies can be canceled by (1) having various matter fields and/or (2) having a factorized I-form, as shown by Green and Schwarz.[9] The factorization condition yields the trace constraints for the YM contents of matter fields. Note that the anomaly-free constraints are most restrictive at $D=4k-2$ dimensions.

How about constraints of *chiral* four-dimensional theories after compactification? We must satisfy two conditions for spin-$\frac{1}{2}$ fermions with a rep Λ' for the gauge group H' in four dimensions: (i) $n_{1/2}(\Lambda')=n_{1/2}^L(\Lambda')-n_{1/2}^R(\Lambda')\neq 0$; (ii) $n_{1/2}(\Lambda'^*)\neq n_{1/2}(\Lambda')$ if fermions with the complex-conjugate representation Λ'^* exists. The second condition comes about because in four dimensions a left-handed fermion with Λ'^* can be regarded as a right-handed fermion with Λ'. The second condition immediately tells us that the rep Λ' *for the four-dimensional gauge group H' must be complex* and thus *H' must contain* U(1), SU(n) $(n\geq 3)$, SO$(4n+2)$

($n\geq 2$), or E_6, assuming H' to be compact. Witten[10] realized that the first condition applies to only zero modes and these zero modes are correlated to the zero modes in compact space. Using the index theorem by Atiyah and Singer, the zero modes in compact space can be given by the integral of the I-form over the compact space. Thus, we have the following simple consequences: (i) no four-dimensional chiral fermions exist for D=odd theories; (ii) no chiral fermions exist if F does not acquire a vacuum expectation value for the compact space for $D=4k-2$ theories; (iii) for $D=4k$ theories, $n_{1/2}(\Lambda')\neq 0$. Consequently, $D=4k$ theories also have the possibility of becoming a physically liable theory.

II. YANG-MILLS TRACE CONSTRAINTS

As shown in the previous section, only Yang-Mills and mixed anomalies exist for theories in $D=4k$ dimension. Therefore, fields which have only gravitational interactions do not contribute to the anomaly at higher dimension. Furthermore, these fields do not contribute to the anomaly at lower dimension either, as long as no isometries are generated by compactification. The reason is as follows: if $D=4k$ is separated into $D'=4m-2$ space-time and $D^0=4(k-m)+2$ compact space, then the zero modes for these fields vanish. Meanwhile, if $D=4k$ is separated into $D'=4m$ space-time and $D^0=4(k-m)$ compact space, then they do not contribute to the anomaly, since the I-form for them is a $(4m+2)$-form. That is, we cannot tell whether or not a theory in $D=4k$ has a supergravity structure by just looking at anomaly-free constraints, in contrast with a theory in $D=4k-2$, where the anomaly-free constraints at both higher and lower dimensions yield constraints for the gravity sector. Thus, we obtain constraints for only those fields which have Yang-Mills interactions. For $D=4k$ space-time, there is no upper limit for space-time dimensions to be investigated, in constrast with the existence of an upper limit of $D=26$ for $N=1$ supergravitylike theories in $D=4k-2$ (Ref. 6).

For spin-$\frac{1}{2}$ chiral fermions with some representation for the gauge group G, the I-form is given by

$$I=\hat{A}(R)\mathrm{Ch}(F)\ ,$$

where $\hat{A}(R)$ denotes the Dirac genus and $\mathrm{Ch}(F)=\mathrm{Tr}\exp(iF/2\pi)$ denotes the Chern character. For theories in $D=4k$, we have the explicit form

$$I_{4k+2}=\hat{A}_k\frac{i\,\mathrm{Tr}F}{2\pi}+\hat{A}_{k-1}\frac{i^3\mathrm{Tr}F^3}{(2\pi)^3}+\cdots+\frac{i^{2k+1}\mathrm{Tr}F^{2k+1}}{(2\pi)^{2k+1}}\ , \tag{2.1}$$

where $\hat{A}_n$ denotes the Dirac class, containing polynomials of traces of $2n$ curvature two-forms, i.e., $\mathrm{Tr}R^{2n}$, $\mathrm{Tr}R^2\mathrm{Tr}R^{2(n-1)}$, etc. The explicit form for $\hat{A}_n$ is not necessary in this paper, but can be found in Refs. 3 and 6. Therefore, we have the following proposition.

Proposition 1. Any theory with $\mathrm{Tr}F^{\mathrm{odd}}=0$ (odd $\leq 2k+1$) is anomaly-free in $D=4k$.

Thus, any group which does not contain U(1), SU(n) ($n\geq 3$), SO($4n+2$) ($n\geq 2$), or E_6, can be used to make a theory anomaly-free, since in this case we have $F=-SF^tS^{-1}$ for some nonsingular matrix S, and thus $\mathrm{Tr}F^{\mathrm{odd}}=0$. However, as we will see, these groups cannot yield chiral fermions at four dimensions, if no isometries are generated after compactification.

For U(1) and a complex rep of SU(n) ($n\geq 3$), all odd-order traces do *not* vanish in general. For SO($4n+2$), only spinor-tensor reps have nonvanishing odd-order traces of order $2n+1$ or higher. For complex rep of E_6, the only nonvanishing odd-order traces are of order 5 or higher. If a rep is not complex (i.e., self-contragredient), all odd-order traces vanish for these groups also.

Now, we obtain constraints on traces, using the Green-Schwarz method. The I-form is factorized into

$$I_{4k+2}=(\hat{A}_1+\alpha)(r_{k-1}+r_{k-2}+\cdots+r_1+r_0)\ , \tag{2.2}$$

where r_n denotes a term containing $2n$ curvature tensors R and $[2(k-n)-1]$ Yang-Mills F. Comparing this with Eq. (2.1), we obtain a second proposition.

Proposition 2. In order to have an anomaly-free theory at $D=4k$ in the manner of Green and Schwarz, we must satisfy

$$\mathrm{Tr}F=\mathrm{Tr}F^3=\cdots=\mathrm{Tr}F^{2k-3}=0\ ,\qquad \mathrm{Tr}F^{2k+1}=c\,\mathrm{Tr}F^{2k-1}\ , \tag{2.3}$$

where c is an arbitrary constant in proportion to $\mathrm{Tr}F^2$ of some representation.

Note that we have for a rep $\Lambda=\sum m_j\Lambda_j$ (Λ_j: irreducible) of G:

$$\mathrm{Tr}F(\Lambda)^n=\left[\sum m_jQ_n(\Lambda_j)\right]\mathrm{Tr}F(\square)^n+[\text{products of lower-order traces of }F(\square)]\ ,$$

where indices (Casimir invariants) $Q_n(\Lambda_j)$ are normalized to a rep $\square$. Thus, it is necessary to have

$$\sum m_jQ_n(\Lambda_j)=0\quad\text{for } n=1,3,\ldots,2k+1 \tag{2.4}$$

except $n=2k-1$. Amazingly, this equation always has a solution, as long as the (complex) gauge group G is SO($4l+2$) with either $l\geq k+1$ or $l=k-1$ for $D=4k$ space-time, since the only nonvanishing odd-order index for SO($2n$) is Q_n (see Appendix B).

It is interesting to compare these two constraints with those of nonsupergravity theories in $D=4k-2$ dimensions:[6,11]

$$\mathrm{Tr}F^2=\mathrm{Tr}F^2=\cdots=\mathrm{Tr}F^{2k-4}=0\ ,\qquad \mathrm{Tr}F^{2k}=c'\mathrm{Tr}F^{2k-2}\ , \tag{2.5}$$

where c' is an arbitrary constant in proportion to $\mathrm{Tr}F^2$ of some representation. One solution in $D=10$ ($k=3$) is SO(16)×SO(16) with the rep (16,16)⊕(128,1)⊕(1,128) (Ref. 11). We have found solutions for SO(N) (N: arbi-

III. CHIRAL FOUR-DIMENSIONAL FERMION CONSTRAINTS

We assume the following: (1) a $D=4k$ theory has a gauge group G with the rep Λ; (2) G is broken into

$H'\times H^0$ with the rep $\sum_i(\Lambda_i',\Lambda_i^0)$ after compactification, where F_i^0 is the vacuum expectation value for F; (3) no isometries are generated after compactification. The last assumption is needed to prove the anomaly-free property after compactification.[8] Then, the index for fermions with the rep Λ_i' is given by

$$\begin{aligned} n_{1/2}(\Lambda_i') &= \text{index}(\Lambda_i^0)_K \\ &= \int_K \hat{A}(R_0)\text{Ch}(F_i^0) \\ &= \int_K \left[\hat{A}_m + \hat{A}_{m-1}\frac{i^2\text{Tr}(F_i^0)^2}{(2\pi)^4} + \cdots \right. \\ &\qquad \left. + i\frac{i^{2m}\text{Tr}(F_i^0)^{2m}}{(2\pi)^{2m}} \right], \end{aligned} \tag{3.1}$$

where the compact space has the dimension $4m=4(k-1)$. Because traces of Yang-Mills F appear only as even-order ones in the index for the compact space, we have

$$\text{index}\,(\Lambda_i^0)_K = \text{index}(\Lambda_i^{0*})_K \ .$$

which implies

$$n_{1/2}(\Lambda_i') = n_{1/2}(\Lambda_i'^*) \quad \text{for } D=4k \text{ theories}, \tag{3.2}$$

provided that Λ'^* *exists in the decomposition of* Λ. Consequently, in order to have a chiral four-dimensional theory, the original rep Λ must be complex, since otherwise the decomposition always contains both (Λ_i',Λ_i^0) and $(\Lambda_i'^*,\Lambda_i^{0*})$. Therefore, we must use the group G which contains U(1), SU(n) ($n\geq 3$), SO($4n+2$)($m\geq 2$), or E_6. Note that we do not have to use a complex rep for Λ_i^0 of H^0. Summarizing, we have proposition 3.

Proposition 3. In order to have a chiral four-dimensional theory from a ($D=4k$)-dimensional theory, both the original gauge group G and the four-dimensional gauge group H' must contain U(1), SU(n) ($n\geq 3$), SO($4n+2$) ($n\geq 2$), or E_6. The rep must be complex in both G and H'.

However, in the case of $D=4k-2$ theories, we always have

$$n_{1/2}(\Lambda_i') = -n_{1/2}(\Lambda_i'^*) \ ,$$

provided that Λ_i^0 is a complex rep, since only odd order traces appear in the index for the compact space. Thus, we do *not* have to use a complex rep for the original group G, but we must use a complex rep for both H' and H^0. Note that the famous superstring theory in $D=10$ uses the rep **(248,1)**$\oplus$**(1,248)** of $E_8\times E_8$, which is not complex, and uses SU(3) and $E_8\times E_6$ for H^0 and H', which have complex reps. This fact is one of the major differences between theories in $D=4k$ and theories in $D=4k-2$. Note that in either case, four-dimensional gauge group H' must be those which have complex reps.

IV. SOLUTIONS

In this section, we fill find solutions to two constraints: (1) an anomaly-free property and (2) chiral four-dimensional theories. For simplicity, we assume that all the groups, G, H', and H^0, be *simple*, unless otherwise stated.

Because G uses a complex rep and still satisfies vanishing of most of odd-order traces, it is usually hard to find a solution with just one irrep, except for the case of SO($4l+2$) as mentioned in Sec. II. Even with two irreps, it becomes harder to find a solution as one looks at a higher dimension. Here, we are satisfied with solutions up to $D=12$ dimension. Our strategy is (1) try to find solutions with a single irreducible representation (irrep) and (2) if we cannot find them, then try to find solutions with two irreps. Hereafter, λ_j denote the fundamental weights in Dynkin notation. We also use the Dynkin notation for an irrep:

$$\Lambda=(m_1,m_2,\ldots,m_n)=m_1\lambda_1+m_2\lambda_2+\cdots m_n\lambda_n \ .$$

A. $D=4$

The Yang-Mills constraint is given by

$$\text{Tr}F^3 = c\,\text{Tr}F \ . \tag{4.1}$$

This means that the third-order index Q_3 must vanish. In the case of G being semisimple, we have $\text{Tr}F^3=0$. Solutions of this constraint have been discussed in the context of grand unified theories in the past. Only SU(n) ($n\geq 3$) has to be discussed. For E_6 and SO ($4n+2$) ($n\geq 2$), any complex rep is a solution, since these groups do not have a genuine third-order Casimir invariant.

First, we look for a *single irrep solution* for SU(n) ($n\geq 3$). For a totally antisymmetric Young tableau of f boxes, Λ_f, and a totally symmetric Young tableau of f boxes, $f\Lambda_1$, we have[12]

$$\begin{aligned} Q_3(\Lambda_f) &= (n-2f)\frac{(n-3)!}{(f-1)!(n-f-1)!} \ , \\ Q_3(f\Lambda_1) &= (n+2f)\frac{(n+f)!}{(f-1)!(n+2)!} \ . \end{aligned} \tag{4.2}$$

Therefore, for these reps, there exist no complex rep solutions satisfying $Q_3=0$. Note that $n=2f$ corresponds to a self-contragredient rep, which is not complex. In general, all $Q_p(\Lambda_f)$ (p: odd and ≥ 3) contain the factor $(n-2f)$. Thus, we look for other reps. As can be seen from Appendix A, a systematic search for solutions of Young tableaux made of up to six boxes yields only self-contragredient irreps. We found two complicated and extremely high-dimensional *single irrep* solutions:[13] $\Lambda=\lambda_3+\lambda_{21}$ of SU(32) and $\Lambda=5\lambda_1+\lambda_2+8\lambda_3+4\lambda_4$ of SU(5). The first one is the only solution up to SU(300) of the type $\Lambda=\lambda_j+\lambda_k$ ($j+k\neq n$).

For a rep made of two irreps, it is easy to find solutions: Since Q_3 for an irrep is always a rational number (actually integer) we can always find two integers (m_1,m_2) such that $m_1Q_3(\Lambda_1)+m_2Q_3(\Lambda_2)=0$ where both Λ_1 and Λ_2 are both complex and irreducible. Then, $\Lambda=m_1\Lambda_1\oplus m_2\Lambda_2$ is a solution. A special type of two irrep solutions occurs when $m_1=m_2=1$. We found that there exist two classes of such solutions:[13] For $\Lambda=\lambda_j\oplus\lambda_k$,

$$(1)\ j=\tfrac{1}{2}(n\pm1-\sqrt{n-1}),\quad k=\tfrac{1}{2}(n\pm1+\sqrt{n-1}),$$
$$(2)\ j=\tfrac{1}{2}(n\pm2-\sqrt{n}),\quad k=\tfrac{1}{2}(n\pm2+\sqrt{n}), \tag{4.3}$$

where n,j, and k must be integers. One of the solutions is the famous SU(5): $\mathbf{5}^*\oplus\mathbf{10}$.

B. $D=8$

For $D=8$ theories, we have two constraints: (1) both G and H' must be one of complex groups and (2) $\mathrm{Tr}F=0$ and $\mathrm{Tr}F^5=c\,\mathrm{Tr}F^3$ for G. The second constraint means that the fifth-order index Q_5 must vanish (see Appendix B).

Among irreps of SU(n) ($n\geq3$) with up to six boxes of Young tableaux, we have found the following single-irrep solutions (see Appendix A).

(i) Any complex irrep of SU(3) and SU(4).

(ii) Two boxes: $(010\ldots)=\mathbf{120}$ of SU(16).

(iii) Three boxes: $(110\ldots)=\mathbf{240}$ of SU(9); $(0010\ldots)=\mathbf{2925}$ of SU(27).

(iv) Four boxes: $(020\ldots)=\mathbf{3185}$ of SU(14); $(1010\ldots)=\mathbf{7140}$ of SU(16).

(v) Five boxes: $(10010\ldots)=\mathbf{263\,120}$ of SU(25).

(vi) Six boxes: $(030\ldots)=\mathbf{41405}$ of SU(13); $(0020\ldots)=\mathbf{1\,163\,800}$ of SU(24).

In Ref. 7, solutions made of more than two irreps are found with a stronger gauge anomaly cancellation constraint, i.e., $\mathrm{Tr}F^5=0$. The solutions given above do not satisfy this strong constraint, but a weaker constraint, $\mathrm{Tr}F^5\propto\mathrm{Tr}F^2\mathrm{Tr}F^3$, which is required for the Green-Schwarz mechanism.

For E_6, the only nontrivial odd-order index is Q_5. For a complex irrep Λ, we have $Q_5(\Lambda)=-Q_5(\Lambda^*)\neq0$, since otherwise independent Casimir invariants cannot distinguish Λ from Λ^*. Thus, there exist no single irrep solutions for E_6. It is easy to find two irrep solutions of the type $\Lambda=m_1\Lambda_1\oplus m_2\Lambda_2$, since Q_5 is a rational for an irrep. For example, the following two are solutions:

$$\Lambda=(010\,000)-11(100\,000)=\mathbf{351}-11\times\mathbf{27},$$
$$\Lambda=(200\,000)-4(010\,000)=\mathbf{351}'-4\times\mathbf{351}.$$

However, we have failed to find two irrep solutions with $m_1=m_2=1$.

For SO($4n+2$)($n\geq2$), the only nontrivial odd-order index is Q_{2n+1} and $Q_{2n+1}(\Lambda)=-Q_{2n+1}(\Lambda^*)\neq0$ for a complex irrep Λ. Thus, any complex irrep of SO($4n+2$)($n\geq3$) is a solution (see Appendix B). For SO(10), it is easy to find two irrep solutions of the type $m_1\Lambda_1\oplus m_2\Lambda_2$. However, we have failed to find two irrep solutions of the type $\Lambda=\Lambda_1\oplus\Lambda_2$ where Λ_j is an irrep.

C. $D=12$

The trace constraints are

$$\mathrm{Tr}F=\mathrm{Tr}F^3=0,$$
$$\mathrm{Tr}F^7=c\,\mathrm{Tr}F^5,$$

which requires that $Q_3=Q_7=0$.

For SU(n)($n\geq3$), we could not find single irrep solutions among those irreps with up to six boxes of Young tableaux. For two irrep solutions, two classes of solutions, Eq. (4.3), do not satisfy $Q_7=0$ either, by using

$$Q_7(\Lambda_f)=\frac{n(n^3+42n^2+119n-42)-60f(n-f)n(n+7)+360f^2(n-f)^2}{(n-3)(n-4)(n-5)(n-6)}Q_3(\Lambda_f).$$

For E_6, we have $Q_3=Q_7=0$ for any irreps. Thus, any complex irrep of E_6 is a solution.

For SO($4n+2$)($n\geq2$), any complex irrep is a solution, except those of SO(14) which have nonvanishing Q_7. For SO(14), we have not found a two irrep solution of the type $\Lambda=\Lambda_1\oplus\Lambda_2$ where Λ_j is an irrep.

D. Global anomalies

So far we have looked at only local anomalies. The global gauge anomalies may exist, if the homotopy group $\Pi_D(G)$ does not vanish for the D-dimensional spacetime. In this case, we have to investigate the global gauge anomaly-free condition carefully.[7,14,15] That is, it so happens that a theory can be globally anomaly-free, even though the homotopy group is nonzero. Therefore, we define *safe* gauge groups as those which do not have nontrivial homotopy groups. For the table of homotopy groups of various Lie groups, see Ref. 16. In the following, we discuss only those possibilities. Global gravitational anomalies are more subtle[14] and are left as a future project.

For $D=4$, we have

$$\Pi_4(\mathrm{SU}(n))=0\quad\text{for } n\geq3,$$
$$\Pi_4(\mathrm{E}_6)=0,$$
$$\Pi_4(\mathrm{SO}(n))=0\quad\text{for } n\geq6.$$

Thus, we are safe.

For $D=8$, we have

$$\Pi_8(\mathrm{SU}(3))=Z_{12},\quad \Pi_8(\mathrm{SU}(4))=Z_{24},$$
$$\Pi_8(\mathrm{SU}(n))=0\quad\text{for } n\geq5,$$
$$\Pi_8(\mathrm{E}_6)=0,$$
$$\Pi_8(\mathrm{SO}(n))=Z_2\quad\text{for } n\geq10.$$

Thus, SU(3), SU(4), and SO($4n+2$) ($n\geq2$) are not safe groups.

For $D=12$, we have

$$\Pi_{12}(\mathrm{SU}(3))=Z_{60},\quad \Pi_{12}(\mathrm{SU}(4))=Z_{60}\ ,$$

$$\Pi_{12}(\mathrm{SU}(5))=Z_{360},\quad \Pi_{12}(\mathrm{SU}(6))=Z_{720}\ ,$$

$$\Pi_{12}(\mathrm{SU}(n))=0 \quad \text{for } n\geq 7\ ,$$

$$\Pi_{12}(\mathrm{E}_6)=Z_{12}\ ,$$

$$\Pi_{12}(\mathrm{SO}(10))=Z_{12},\quad \Pi_{12}(\mathrm{SO}(n))=0 \quad \text{for } n\geq 14\ .$$

Thus, $\mathrm{SU}(n)(n\leq 6)$, SO(10), and E_6, are not safe groups.

Note that for higher dimensions, $\mathrm{SU}(n)$ with suitably high rank and $\mathrm{SO}(n)$ with *both* suitably high rank *and* $D\equiv 4 \bmod 8$ are always safe, while, even with suitably high rank, $\mathrm{SO}(n)$ is not safe for $D\equiv 0 \bmod 8$. This conclusion follows from the Bott periodicity theorem for classical groups:[17]

$$\Pi_D(\mathrm{SU}(n))=0 \quad \text{for } n\geq \frac{D+1}{2} (D\text{: even})\ ,$$

$$\Pi_D(\mathrm{SO}(n))\ (n\geq D+2)=\begin{cases}0 & (D\equiv 4 \bmod 8)\ ,\\ Z_2 & (D\equiv 0 \bmod 8)\ .\end{cases}$$

For E_6, we have[16]

$$\Pi_{16}(\mathrm{E}_6)=0,\quad \Pi_{20}(\mathrm{E}_6)=Z_{1512}$$

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APPENDIX A: INDICES FOR VARIOUS YOUNG TABLEAUX

We tabulate odd-order indices (up to six boxes of Young tableaux and up to seventh order) where n is the dimension of the single box, using the formulas given in Ref. 12. Indices of various orders are normalized to the one-box Young tableaux: $Q_p(\square)=1$. We denote the pth-order index for the Young tableau $\langle f_1,f_2,f_3,\ldots\rangle$ by $Q_p[f_1,f_2,f_3,\ldots]$, where f_j denotes the number of boxes at the jth row of the Young tableau. Note that the relation between the Young tableau Γ and its conjugate tableau Γ^*:[12]

$$Q_p(\Gamma^*,n)=(-)^{(\sum f_j-1)}Q_p(\Gamma,-n)\ .$$

Here, a conjugate tableau means that the tableau which is obtained from the original by the mirror reflection along its diagonal. That is, $\langle 3,1,1,1\rangle$ is conjugate to $\langle 4,1,1\rangle$. Note that for $\mathrm{SU}(n)$, the Young tableau $\langle f_1,f_2,\ldots,f_n\rangle$ corresponds to the irrep, $(f_1-f_2,f_2-f_3,\ldots,f_{n-1}-f_n)$ of Dynkin notation. A self-contragredient rep (noncomplex rep) of $\mathrm{SU}(n)$ satisfies the relation: $f_1-f_2=f_{n-1}-f_n$, $f_2-f_3=f_{n-2}-f_{n-1},\ldots$

Two boxes:

$$Q_3[1,1]=n-4,\quad Q_5[1,1]=n-16,\quad Q_7[1,1]=n-64\ ,$$

$$Q_3[2]=n+4,\quad Q_5[2]=n+16,\quad Q_7[2]=n+64\ .$$

Three boxes:

$$Q_3[1,1,1]=\frac{(n-6)(n-3)}{2}\ ,$$

$$Q_5[1,1,1]=\frac{(n-27)(n-6)}{2}\ ,$$

$$Q_7[1,1,1]=\frac{n^2-129n+1458}{2}\ ,$$

$$Q_3[2,1]=(n-3)(n+3)\ ,$$

$$Q_5[2,1]=(n-9)(n+9)\ ,$$

$$Q_7[2,1]=(n-27)(n+27)\ ,$$

$$Q_3[3]=\frac{(n+3)(n+6)}{2}\ ,$$

$$Q_5[3]=\frac{(n+6)(n+27)}{2}\ ,$$

$$Q_7[3]=\frac{n^2+129n+1458}{2}\ .$$

Four boxes:

$$Q_3[1,1,1,1]=\frac{(n-3)(n-4)(n-8)}{6}\ ,$$

$$Q_5[1,1,1,1]=\frac{(n-8)(n^2-43n+192)}{6}\ ,$$

$$Q_7[1,1,1,1]=\frac{(n-8)(n^2-187n+3072)}{6}\ ,$$

$$Q_3[2,1,1]=\frac{(n-4)(n^2-n-8)}{2}\ ,$$

$$Q_5[2,1,1]=\frac{(n-16)(n^2-n-32)}{2}\ ,$$

$$Q_7[2,1,1]=\frac{(n-64)(n^2-n-128)}{2}\ ,$$

$$Q_3[2,2]=\frac{(n-4)n(n+4)}{3}\ ,$$

$$Q_5[2,2]=\frac{(n-14)n(n+14)}{3}\ ,$$

$$Q_7[2,2]=\frac{n(n^2-1996)}{3}\ ,$$

$$Q_3[3,1]=\frac{(n+4)(n^2+n-8)}{2}\ ,$$

$$Q_5[3,1]=\frac{(n+16)(n^2+n-32)}{2}\ ,$$

$$Q_7[3,1]=\frac{(n+64)(n^2+n-128)}{2}\ ,$$

$$Q_3[4]=\frac{(n+3)(n+4)(n+8)}{6}\ ,$$

$$Q_5[4]=\frac{(n+8)(n^2+43n+192)}{6}\ ,$$

$$Q_7[4]=\frac{(n+8)(n^2+187n+3072)}{6}\ .$$

Five boxes:

$$Q_3[1,1,1,1,1]=\frac{(n-10)(n-5)(n-4)(n-3)}{24},$$

$$Q_5[1,1,1,1,1]=\frac{(n-10)(n-5)(n^2-55n+300)}{24},$$

$$Q_7[1,1,1,1]=\frac{(n-10)(n^3-252n^2+7007n-37\,500)}{24},$$

$$Q_3[2,1,1,1]=\frac{(n+2)(n-3)(n-5)^2}{6},$$

$$Q_5[2,1,1,1]=\frac{(n-5)(n-25)(n^2-5n-30)}{6},$$

$$Q_7[2,1,1,1]=\frac{n^4-131n^3+2189n^2+2315n-93\,750}{6},$$

$$Q_3[2,2,1]=\frac{(n-5)n(5n^2+3n-50)}{24},$$

$$Q_5[2,2,1]=\frac{5(n-5)n(n^2-9n-202)}{24},$$

$$Q_7[2,2,1]=\frac{n(5n^3-262n^2-7985n+89\,050)}{24},$$

$$Q_3[3,1,1]=\frac{n^4-17n^2+100}{4},$$

$$Q_5[3,1,1]=\frac{n^4-65n^2+2500}{4},$$

$$Q_7[3,1,1]=\frac{n^4-257n^2+62\,500}{4},$$

$$Q_3[3,2]=\frac{n(n+5)(5n^2-3n-50)}{24},$$

$$Q_5[3,2]=\frac{5n(n+5)(n^2+9n-202)}{24},$$

$$Q_7[3,2]=\frac{n(5n^3+262n^2-7985n-89\,050)}{24},$$

$$Q_3[4,1]=\frac{(n-2)(n+3)(n+5)^2}{6},$$

$$Q_5[4,1]=\frac{(n+5)(n+25)(n^2+5n-30)}{6},$$

$$Q_7[4,1]=\frac{n^4+131n^3+2189n^2-2315n-93\,750}{6},$$

$$Q_3[5]=\frac{(n+3)(n+4)(n+5)(n+10)}{24},$$

$$Q_5[5]=\frac{(n+5)(n+10)(n^2+55n+300)}{24},$$

$$Q_7[5]=\frac{(n+10)(n^3+252n^2+7007n+37\,500)}{24}.$$

Six boxes:

$$Q_3[1,1,1,1,1,1]=\frac{(n-3)(n-4)(n-5)(n-6)(n-12)}{120},$$

$$Q_5[1,1,1,1,1,1]=\frac{(n-5)(n-6)(n-12)(n^2-67n+432)}{120},$$

$$Q_7[1,1,1,1,1,1]=\frac{(n-12)(n^4-318n^3+12719n^2-140\,442n+466\,560)}{120},$$

$$Q_3[2,1,1,1,1]=\frac{(n-3)(n-4)(n-6)(n^2-5n-12)}{24},$$

$$Q_5[2,1,1,1,1]=\frac{(n-6)(n^4-48n^3+467n^2-228n-5184)}{24},$$

$$Q_7[2,1,1,1,1]=\frac{n^5-198n^4+6227n^3-48\,966n^2-55\,368n+1\,119\,744}{24},$$

$$Q_5[2,2,1,1]=\frac{3n(n-6)(n^3-24n^2-89n+1096)}{40},$$

$$Q_5[2,2,1,1]=\frac{3n(n-6)(N^3-24n^2-89n+1096)}{40},$$

$$Q_7[2,2,1,1]=\frac{3n(n^4-110n^3+215n^2+26\,990n-180\,816)}{40},$$

$$Q_3[2,2,2]=\frac{(n+4)n(n-1)(n-3)(n-6)}{24},$$

$$Q_5[2,2,2]=\frac{(n+13)n(n-1)(n-6)(n-24)}{24},$$

$$Q_7[2,2,2]=\frac{n(n-1)(n^3-65n^2-2598n+37\,872)}{24},$$

$$Q_3[3,1,1,1]=\frac{(n-3)(n^4-3n^3-16n^2+36n+144)}{12},$$

$$Q_5[3,1,1,1]=\frac{n^5-18n^4+65n^3+600n^2+324n-15\,552}{12},$$

$$Q_7[3,1,1,1]=\frac{n^5-66n^4+1073n^3+4824n^2+2916n-559\,872}{12},$$

$$Q_3[3,2,1]=\frac{2(n-4)(n-3)n(n+3)(n+4)}{15},$$

$$Q_5[3,2,1]=\frac{2n(n^4-205n^2+4284)}{15},$$

$$Q_7[3,2,1]=\frac{2n(n^4-1825n^2+113\,544)}{15},$$

$$Q_3[3,3]=\frac{(n-4)n(n+1)(n+3)(n+6)}{24},$$

$$Q_5[3,3]=\frac{(n-13)n(n+1)(n+6)(n+24)}{24},$$

$$Q_7[3,3]=\frac{n(n+1)(n^3+65n^2-2598n-37\,872)}{24},$$

$$Q_3[4,1,1]=\frac{(n+3)(n^4+3n^3-16n^2-36n+144)}{12},$$

$$Q_5[4,1,1]=\frac{n^5+18n^4+65n^3-600n^2+324n+15\,552}{12},$$

$$Q_7[4,1,1]=\frac{n^5+66n^4+1073n^3-4824n^2+2916n+559\,872}{12},$$

$$Q_3[4,2]=\frac{3(n-3)n(n+3)(n+4)(n+6)}{40},$$

$$Q_5[4,2]=\frac{3n(n+6)(n^3+24n^2-89n-1096)}{40},$$

$$Q_7[4,2]=\frac{3n(n^4+110n^3+215n^2-26\,990n-180\,816)}{40},$$

$$Q_3[5,1]=\frac{(n+3)(n+4)(n+6)(n^2+5n-12)}{24},$$

$$Q_5[5,1]=\frac{(n+6)(n^4+48n^3+467n^2+228n-5184)}{24},$$

$$Q_7[5,1]=\frac{n^5+198n^4+6227n^3+48\,966n^2-55\,368n-1\,119\,744}{24},$$

$$Q_3[6]=\frac{(n+3)(n+4)(n+5)(n+6)(n+12)}{120},$$

$$Q_5[6]=\frac{(n+5)(n+6)(n+12)(n^2+67n+432)}{120},$$

$$Q_7[6]=\frac{(n+12)(n^4+318n^3+12\,719n^2+140\,442n+466\,560)}{120}.$$

APPENDIX B: ODD-ORDER TRACE IDENTITIES

In this appendix, we give odd-order trace identities for simple Lie algebras, which will be useful for the full construction of theories at $D=4k$. Our notation is

X_μ: generators for an irreducible rep Λ ;

$F=\xi^\mu X_\mu$: arbitrary element of a Lie algebra ;

$f_p(\xi)=g_{\mu_1\cdots\mu_p}\xi^{\mu_1}\xi^{\mu_2}\cdots\xi^{\mu_p}$;

$D_p(\Lambda)=g^{\mu_1\cdots\mu_p}\mathrm{Tr}X_{\mu_1}\cdots X_{\mu_p}$;

$$Q_p(\Lambda)=\frac{D_p(\Lambda)}{D_p(\Box)} \quad [\text{provided } D_p(\Box)\neq 0] ;$$

$Q_p^0=Q_p$ (adjoint rep) ;

C_p: some constant ;

d_0: dimension of an adjoint rep .

(i) $p=3$[useful for SU(3)]:

$$\mathrm{Tr}F^3=C_3 f_3(\xi)D_3(\Lambda) .$$

(ii) $p=5$[useful for $SU(n)(n\geq 3)$,E_6, and SO(10)]:

$$\mathrm{Tr}F^5=C_5 f_5(\xi)D_5(\Lambda) + \frac{10}{6+d_0}\left[\frac{d_0}{d(\Lambda)}-\frac{1}{4}\frac{Q_2^0}{Q_2(\Lambda)}\right]\mathrm{Tr}F^2\mathrm{Tr}F^3 .$$

This equation is especially useful for SU(3) and SU(4), since $D_5(\Lambda)=0$.

(iii) $p=7$ for SO(10), SO(14), and E_6 where $D_3(\Lambda)=D_4(\Lambda)=0$ identically:

$$\mathrm{Tr}F^7=C_7 f_7(\xi)D_7(\Lambda) + \frac{21}{10+d_0}\left[\frac{d_0}{d(\Lambda)}-\frac{5}{12}\frac{Q_2^0}{Q_2(\Lambda)}\right]\mathrm{Tr}F^2\mathrm{Tr}F^5 .$$

Note that $D_7(\Lambda)=0$ for SO(10) and E_6.

(iv) $p=2n+3$ $(n\geq 1)$ for SO $(4n+2)$ $(n\geq 1)$:

$$\mathrm{Tr}F^{2n+3}=\frac{(2n+3)(n+1)}{(2n+1)(4n+3)}\left[\frac{d_0}{d(\Lambda)}-\frac{2n+1}{12}\frac{Q_2^0}{Q_2(\Lambda)}\right] \times\mathrm{Tr}F^2\mathrm{Tr}F^{2n+1} ,$$

with $\mathrm{Tr}F=\mathrm{Tr}F^3=\mathrm{Tr}F^5=\dots=\mathrm{Tr}F^{2n-1}=0$.

In order to use formulas given above, we must know odd-order indices. General index formulas for simple classical Lie algebras are given in Ref. 12. Among exceptional Lie algebras, only E_6 has the odd-order index Q_5, whose explicit expression for any irrep is given below. See also Ref. 12.

Define $f_j (j=0,1,\dots,6)$ for an irrep $\Lambda=\sum_{j=1}^{6} m_j\lambda_j$ as

$$f_1=\tfrac{1}{3}(m_1-m_2-3m_3-2m_4-m_5)-\tfrac{1}{2}m_6 ,$$
$$f_2=\tfrac{1}{3}(-2m_1-m_2-3m_3-2m_4-m_5)-\tfrac{1}{2}m_6 ,$$
$$f_3=\tfrac{1}{3}(-2m_1-4m_2-3m_3-2m_4-m_5)-\tfrac{1}{2}m_6 ,$$
$$f_4=\tfrac{1}{3}(m_1+2m_2+3m_3+4m_4+2m_5)+\tfrac{1}{2}m_6 ,$$
$$f_5=\tfrac{1}{3}(m_1+2m_2+3m_3+m_4+2m_5)+\tfrac{1}{2}m_6 ,$$
$$f_6=\tfrac{1}{3}(m_1+2m_2+3m_3+m_4-m_5)+\tfrac{1}{2}m_6 ,$$
$$f_0=\frac{1}{\sqrt{2}}m_6 ,$$

where $f_4\geq f_5\geq f_6\geq f_1\geq f_2\geq f_3$ and $f_1+f_2+f_3+f_4+f_5+f_6=0$. Using these f's, define $l_j (j=0,1,\dots,6)$ by

$$l_1=f_1-\tfrac{5}{2},\quad l_2=f_2-\tfrac{7}{2},\quad l_3=f_3-\tfrac{9}{2} ,$$
$$l_4=f_4+\tfrac{9}{2},\quad l_5=f_5+\tfrac{7}{2},\quad l_6=f_6+\tfrac{5}{2} ,$$
$$l_0=f_0+\frac{1}{\sqrt{2}} .$$

Then, we still have $l_1+l_2+l_3+_4+l_5+l_6=0$. Now, the fifth-order Casimir invariant for E_6 with $D_5(\Lambda)=d(\Lambda)J_5(\Lambda)$, is given by

$$J_5(\Lambda)=\sum_{j=1}^{6}(l_j)^5 -\frac{5}{6}\left[\sum_{j=1}^{6}(l_j)^3\right]\left[\sum_{j=1}^{6}(l_j)^2-(l_0)^2\right] .$$

By the way, the second order Casimir invariant for E_6 with $D_2(\Lambda)=d(\Lambda)J_2(\Lambda)$, is given by

$$J_2(\Lambda)=\sum_{j=1}^{6}(l_j)^2+(l_0)^2-78 .$$

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