Closed smooth strings on a torus

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(Received 28 July 1987)

The free energy of a gas of closed strings with extrinsic curvature (smooth strings) is evaluated on a torus. This is compared with the free energy of a collection of free particles, and hence the mass spectrum of excitations of the smooth strings is deduced. It is found that above a critical value of the coupling constant of the curvature term the spectrum is free of tachyons. Furthermore, there are no massless spin-2 excitations. The absence of massless spin-2 fields is a consequence of the fact that the smooth-string theory is not modular invariant.

I. INTRODUCTION

Recently, Polyakov[1] made the suggestion that for a realistic string model of hadrons one must add to the Nambu-Goto action a term that depends on the extrinsic curvature of the world sheet. The curvature term suppresses those string configurations which are sharply kinked, favoring those that are smooth. The term "smooth strings" has been used by many authors to describe this model and we shall follow this terminology. The extrinsic curvature term has been known in the investigations on membranes in fluids by Helfrich[2] and others.[3] The action for Polyakov's model of smooth strings may be written in the form[1] (in the first-order formalism)

$$S=T\int \sqrt{g}\,d^2\sigma+\frac{1}{2\alpha_0}\int d^2\sigma[\sqrt{g}\,(\Delta x^\mu)^2+\lambda^{ab}(\partial_a x^\mu\partial_b x_\mu-\rho\hat{g}_{ab}(\tau))]\ , \tag{1}$$

where the conformal gauge $g_{ab}=\rho\hat{g}_{ab}(\tau)$ has been used. $\hat{g}_{ab}(\tau)$ depends on the Teichmüller parameter τ. The string coordinate $x^\mu(\sigma)$ spans a d-dimensional space-time. Δx^μ is given by

$$\Delta x^\mu=\frac{1}{\sqrt{g}}\partial_a(\sqrt{g}\,g^{ab}\partial_b x^\mu)\ . \tag{2}$$

The first term in (1) is the usual Nambu-Goto action and the second term is the extrinsic curvature term. In the third term, λ^{ab} is a constraint field which fixes the metric to be the induced metric: namely, $g_{ab}=\partial_a x^\mu\partial_b x_\mu$ on the surface. The coupling constant α_0 is dimensionless and is asymptotically free.[1-3] The action in (1) has reparametrization invariance. The extrinsic curvature term is, however, not invariant under conformal transformations

$$g_{ab}\to\rho(\sigma)g_{ab}\ ,$$

where $\rho(\sigma)$ is an arbitrary function of σ.

Since Polyakov's work, there have been two main directions in understanding the physics of smooth strings. (a) To calculate the effect of the curvature term on the static quark-quark potential.[4] This involves studying open strings with quarks at end points. These calculations extend the results obtained for the Nambu-Goto action.[5] (b) To study the renormalization-group behavior of the coupling constant α_0 (Ref. 6). In addition to these studies a number of authors[7] have studied classical solutions to this action. In studies quoted above involving loop corrections, their authors have considered only flat two-dimensional topology. In this paper we undertake the study of the partition function for smooth strings on genus-one Riemann surfaces. For simplicity, we consider only closed strings so the genus-one surface is a torus. The open smooth string is considered in a separate paper.[8]

Different tori are characterized by a complex Teichmüller parameter $\tau\equiv(\tau_1,\tau_2)$. Our calculations follow that of Polchinski[9] with some important modifications which will be specified in the following. The Euclidean path integral over a string world sheet is expressed as an integral over (Teichmüller) parameter τ. From the path integral we calculate the free energy of a gas of smooth strings. This is compared with the free energy of a collection of noninteracting particles. We find that there exists a critical coupling $\alpha_0 TL^2$, where L is the length of the string, above which the mass spectrum does not contain a tachyon. The fact that smooth strings may be tachyon-free has been conjectured by Braaten, Pisarski, and Tze[4] (see also, Pisarski[10]). Another interesting property that we obtain is that there are no zero-mass excitations of spin 2. This agrees with the fact that the smooth-string theory does not have modular invariance[11] (see below). Curtright, Ghandour, Thorn, and Zachos[7] have previously conjectured on the

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basis of the shape of the classical Regge trajectory that smooth strings may have no massless excitations.

The path integral we calculate is not modular invariant. This implies that the region of integration over the Teichmüller parameter τ is not restricted to the fundamental domain[12] $-\frac{1}{2}<\tau_1<\frac{1}{2}, \mathrm{Im}\tau_2>0;\ |\tau|>1$. Instead the region of integration is $\mathrm{Im}\tau_2>0, -\frac{1}{2}<\tau_1<\frac{1}{2}$. The restriction $-\frac{1}{2}<\tau_1<\frac{1}{2}$ arises from the fact that the path integral is still invariant under translations $\tau \to \tau+1$.

We now turn to the evaluation of the path integral and free energy of smooth strings on a torus.

II. PARTITION FUNCTION ON A TORUS

Following Polyakov,[1] we split all fields into slow and fast parts and integrate out the fast components:

$$x=x_0+x_1\ ,$$
$$\rho=\rho_0+\rho_1\ , \tag{3}$$
$$\lambda^{ab}=\lambda_0^{ab}+\lambda_1^{ab}\ ,$$

where the wave vectors of fast quantities lie between $\tilde{\Lambda}$ and Λ. In the one-loop approximation, one obtains the renormalized action in the form (for details, see Ref. 1)

$$S=T\int d^2\sigma\sqrt{g_0}+\frac{1}{2\alpha_0}\int d^2\sigma\left\{\left[1-\frac{\alpha_0}{2\pi}\ln\frac{\Lambda}{\tilde{\Lambda}}\right]\sqrt{g_0}(\Delta x_0)^2+\lambda_0^{ab}\left[\partial_a x_0\partial_b x_0-\rho_0\hat{g}_{ab}(\tau)\left[1-\frac{D-2}{4\pi}\alpha_0\ln\frac{\Lambda}{\tilde{\Lambda}}\right]\right]\right\}\ . \tag{4}$$

After renormalization of the x field,

$$x_0\to z^{1/2}x_0,\quad \lambda_0^{ab}\to z^{-1}\lambda_0^{ab}\ ,$$
$$z=1-\frac{D-2}{4\pi}\alpha_0\ln\frac{\Lambda}{\tilde{\Lambda}}\ , \tag{5}$$

one obtains

$$S=T\int d^2\sigma\sqrt{g_0}+\frac{1}{2\tilde{\alpha}_0}\int d^2\sigma[\sqrt{g_0}(\Delta x_0)^2+\lambda_0^{ab}(\partial_a x_0\partial_b x_0-\rho_0\hat{g}_{ab}(\tau))]\ , \tag{6}$$

where

$$\frac{1}{\tilde{\alpha}_0}=\frac{1}{\alpha_0}-\frac{D}{2}\frac{1}{2\pi}\ln\frac{\Lambda}{\tilde{\Lambda}}\ ,$$
$$g_0^{ab}=\rho_0\hat{g}^{ab}(\tau)\ . \tag{7}$$

The quantum theory of smooth strings involves evaluating the path integral over the space of all metrics of a given topology and over all configurations x_0^μ as in Polyakov's proposal[13] for the Nambu-Goto theory.

The action in (6) is manifestly invariant under reparametrizations of the world sheet but not conformally invariant. We can nevertheless evaluate the conformal anomaly in one loop by the method of Fujikawa.[14] This has been done in Ref. 15. It is found that the Liouville term has the form

$$\frac{26-d}{48\pi}(\partial_\mu\phi)^2+\frac{M}{8\sqrt{2\pi}a}e^{\phi}\ ,$$

where $a^2=\alpha_0 T$ and $e^\phi=\rho_0$. In $d=26$, the Liouville mode becomes nonpropagating. However, it has been pointed out by Pisarski[10] that smooth strings may not be stable for $d=26$ and, in fact, they will be stable for $d<26$. In our evaluation of the path integral we will work in $d<26$ dimensions and ignore the effect of conformal fields that fluctuate on the world sheet. We shall choose the space of metrics on a torus to be conformal to the metric with $\rho_0=1$. The metric can then be set in the form

$$\hat{g}_{ab}=\begin{bmatrix}1 & \tau_1\\ \tau_1 & \tau_1^2+\tau_2^2\end{bmatrix}\ , \tag{8}$$

where $\tau\equiv(\tau_1,\tau_2)$ is a complex Teichmüller parameter that characterizes the torus.

Since the smooth string theory is not conformally invariant we introduce length scales (L_1,L_2) explicitly. The volume of the torus is $\int g\, d^2\sigma=L_1L_2\tau_2$ for the metric (5). We shall, however, see later that the path integral depends only on the physical length of the strings taken to be L_1. So, without loss of generality, we set $L_1=L_2=L$. We proceed to treat λ_0^{ab} as a background field[6,10] in the form

$$\lambda_0^{ab}=T\tilde{\alpha}_0\sqrt{g_0}\hat{g}_{ab}(\tau)\ . \tag{9}$$

The path over torus can be written as

$$W_{\mathrm{torus}}=\int\frac{[dg_{ab}][dx^\mu]}{V_{\mathrm{GC}}}\exp(-S_{\mathrm{SS}})\ . \tag{10}$$

We work in Euclidean space. V_{GC} is the volume of the general coordinate group. Integration over x_0^μ may be carried out as follows. The metric for small variations in x^μ is defined as (we omit the subscript 0 from now on)

$$\|\delta x^\mu\|^2=\frac{1}{L^2}\int d^2\sigma\sqrt{g}\,\delta x^\mu\delta x_\mu\ . \tag{11a}$$

The measure for x^μ integration is defined in terms of the Gaussian integral:

$$\prod_\mu\int[d\delta x^\mu]e^{-\|\delta x^\mu\|^2/2}=1\ . \tag{11b}$$

Then a straightforward calculation yields[8]

$$W_{\text{torus}}=\int \frac{[dg_{ab}][dx^{\mu}]}{V_{\text{GC}}}\exp\left[-\frac{T}{2}\int d^2\sigma\sqrt{g}\left[g^{ab}\partial_a x^{\mu}\partial_b x_{\mu}-\frac{1}{\tilde{\alpha}_0 T}(\Delta x^{\mu})^2\right]\right]$$
$$=\int \frac{[dg_{ab}]}{V_{\text{GC}}}\left[\prod_{\mu}L^{\mu}\right]T^{d/2}\left[\frac{2\pi L^2}{\int d^2\sigma\sqrt{g}}\det{}'(-\Delta)\det{}'\left[-\frac{1}{\tilde{\alpha}_0 T}\Delta+1\right]\right]^{-d/2}. \quad (12)$$

In Eq. (12) the prime on determinants implies that the zero modes are to be excluded and L^{μ} ($\mu=1,\ldots,d$) denote the lengths of the box in d dimensions.

In order to carry out the metric integration we have to make a change of variables. For smooth strings, variation of the metric can be resolved into changes arising from general coordinate and τ transformations. (Weyl transformations are omitted here.) We write

$$\delta g_{ab}=\delta g_{ab}^{(1)}+\delta g_{ab}^{(2)}, \quad (13)$$

$$\delta g_{ab}^{(1)}=D_a\delta\xi_b+D_b\delta\xi_a+g_{ab,i}d\tau_i, \quad (14)$$

$$\delta g_{ab}^{(2)}=B(D_a\Delta\delta\xi_b+D_b\Delta\delta\xi_a), \quad (15)$$

where we have introduced an arbitrary dimensionful parameter B. That the term $\delta g_{ab}^{(2)}$ in the variation of metric should be added to $\delta g_{ab}^{(1)}$ is due to the higher-derivative term (curvature term) in the action (1) or (6). It should be mentioned that the total change δg_{ab} given by (13), (14), and (15) belongs to infinitesimal general coordinate transformations on the world sheet. Therefore, the actions (1) and (6) are invariant under this transformation simply because they are covariant scalars. It will be seen later in the paper that the term δg_{ab}^{2} in the variation of the metric plays the crucial role in canceling the longitudinal modes of the string.

The Jacobian of these restricted transformations (13)–(15) is defined by

$$[dg_{ab}]=(d\xi)'d^2\tau J(\tau). \quad (16)$$

We now define the metric for small δg_{ab} as

$$\|\delta g_{ab}\|^2=\int d^2\sigma\sqrt{g}g^{ac}g^{bd}(\delta g_{ab}^{(1)}\delta g_{cd}^{(1)}+\delta g_{ab}^{(1)}\delta g_{cd}^{(2)}+\delta g_{ab}^{(2)}\delta g_{cd}^{(1)}). \quad (17)$$

The term $\delta g_{ab}^{(2)}\delta g^{(2)ab}$ is dropped since it involves the sixth-order operator.

The Jacobian appearing in (16) can be evaluated following Ref. 9. We define

$$\int [d\delta g]e^{-\|\delta g\|^2/2}=1, \quad (18a)$$

$$\int [d\delta\xi]e^{-\|\delta\xi\|^2/2}=1, \quad (18b)$$

and

$$\int [d^2\delta\tau]\exp\left[-\delta\tau_i\delta\tau_i\int d^2\sigma\sqrt{g}/2\right]=\frac{2\pi L^2}{\int d^2\sigma\sqrt{g}}. \quad (18c)$$

A straightforward computation leads to the result[9]

$$J(\tau)=\frac{1}{\tau_2^3}\det{}'(-\Delta)\det{}'\left[-\frac{1}{\tilde{\alpha}_0 T}\Delta+1\right], \quad (19)$$

where we have chosen $B=1/2\tilde{\alpha}_0 T$.

In (19) the first determinant factor is well known and arises from gauge fixing the reparametrization invariance. The second determinant factor arises due to the generalized reparametrization defined in Eqs. (13)–(15). However, it is shown[16] that there is a deeper reason for this determinant. The action for the extrinsic curvature in (1) is invariant under so-called H variations on stationary surfaces. H variations are normal variations in which the normals are in the direction of the mean curvature vector of the world sheet. Fixing the gauge associated with H invariance yields the second determinant factor in (19).

The path integral now takes the form

$$W_{\text{torus}}=T^{d/2}\left[\prod_{\mu}L^{\mu}\right]\int\frac{d^2\tau}{4\pi\tau_2^2}(2\pi\tau_2)^{-(d/2-1)}\left[\det{}'(-\Delta)\det{}'\left[-\frac{1}{\tilde{\alpha}_0 T}\Delta+1\right]\right]^{1-d/2}. \quad (20)$$

The determinant $\det'(-\Delta)$ has been calculated in Ref. 9:

$$\det{}'(-\Delta)=\tau_2^2 e^{-\pi\tau_2/3}|f(e^{2\pi i\tau})|^4, \quad (21)$$

where

$$f(e^{2\pi i\tau})=\prod_{n=1}^{\infty}(1-e^{2\pi i n\tau}). \quad (22)$$

The determinant of $[-(1/\tilde{\alpha}_0 T)\Delta+1]$ is evaluated in the Appendix:

$$\det{}'\left[-\frac{1}{\tilde{\alpha}_0 T}\Delta+1\right]=\exp\left[\frac{\tau_2 a^2}{4\pi}+2\pi\tau_2 I(a)\right]|f(e^{2\pi i W_+})|^4(1-e^{-\tau_2 a})^2, \quad (23)$$

where

$$I(a)=4\int_0^\infty \frac{dy}{1-e^{2\pi(y+a/2\pi)}}y^{1/2}\left[y+\frac{a}{\pi}\right]^{1/2}, \tag{24}$$

$$W_+=n\tau_1+i\tau_2(n^2+a^2/4\pi^2)^{1/2}, \tag{25}$$

and

$$a^2=\tilde{\alpha}_0 TL^2 . \tag{26}$$

$I(a)$ has the limiting values

$$I(a)=-\tfrac{1}{6} \text{ as } a\to 0 ,$$

$$=0 \text{ as } a\to\infty .$$

Substituting (21) and (23) into (20), we find

$$W_{\text{torus}}=T^{d/2}\left[\prod_\mu L^\mu\right]\int \frac{d^2\tau}{4\pi\tau_2^2}(2\pi\tau_2)^{1-d/2}|f(e^{2\pi i\tau})|^{2(2-d)}|f(e^{2\pi i W_+})|^{2(2-d)}(1-e^{-\tau_2 a})^{2-d} \times\exp\left[\left[\frac{d}{2}-1\right]\frac{\pi\tau_2}{3}\left[1-\frac{3a^2}{4\pi^2}-6I(a)\right]\right]. \tag{27}$$

Equation (27) is invariant under the transformation $\tau\to\tau+1$. This invariance requires that τ_1 is restricted to the region chosen to be $-\frac{1}{2}<\tau_1<\frac{1}{2}$ and $\tau_2>0$. In the usual string theory the path integral is also invariant under $\tau\to-1/\tau$. This makes the path integral modular invariant where modular transformations are of the form $\tau\to\alpha\tau+\beta/\gamma\tau+\delta$, $\alpha,\beta,\gamma,\delta$ are integers, and $\alpha\delta-\beta\gamma=1$. This requires that τ be chosen to be in the fundamental domain defined by $-\frac{1}{2}<\tau_1<\frac{1}{2}$, $\tau_2>0$, and $|\tau|>1$. The extrinsic curvature term breaks modular invariance of the path integral. Note that (27) is not invariant under $\tau\to-1/\tau$. The free energy $F(\beta)$ for a gas of strings is given by

$$F(\beta)=-\left[\prod_\mu L^\mu\right]^{-1}W_{\text{connected}} . \tag{28}$$

Following Polchinski's procedure we find that

$$F(\beta)=-T^{d/2}\int_0^\infty \frac{d\tau_2}{2\pi\tau_2^2}\int_{-1/2}^{1/2}d\tau_1(2\pi\tau_2)^{1-d/2}|f(e^{2\pi i\tau})|^{2(2-d)}|f(e^{2\pi i W_+})|^{2(2-d)} \times(1-e^{-\tau_2 a})^{2-d}\exp\left[4\pi\tau_2\left[1-\frac{3a^2}{4\pi^2}-6I(a)\right]\frac{d-2}{24}\right]\sum_{\gamma=1}^{\infty}e^{-\gamma^2\beta^2 T/2\tau_2} . \tag{29}$$

In order to understand the content of (29) we compare it with the free energy for a collection of free particles whose energy spectrum is $\omega_k=\sqrt{k^2+m^2}$:

$$F(\beta,m^2)=\frac{1}{\beta}\int\frac{d^{d-1}}{(2\pi)^{d-1}}\ln\left[1-e^{-\beta\omega_k}\right] =\int_0^\infty\frac{ds}{s}(2\pi s)^{-d/2}\sum_{\gamma=1}^{\infty}e^{-m^2 s/2-\gamma^2\beta^2/2s} . \tag{30}$$

In terms of occupation numbers of transverse oscillators $N_{n_i}^{(1)}$, $\tilde{N}_{n_i}^{(1)}$, $N_{\tilde{n}_i}^{(2)}$, $\tilde{N}_{\tilde{n}_i}^{(2)}$, and $N_i^{(3)}$, the spectrum is given by

$$m^2(a)=4\pi T\left[\left[-2+\frac{3a^2}{2\pi^2}+12I(a)\right]\frac{d-2}{24}+\sum_{i=1}^{d-2}\sum_{n=1}^{\infty}n(N_{n_i}^{(1)}+\tilde{N}_{n_i}^{(1)})+\sum_{i=1}^{d-2}\sum_{n=1}^{\infty}\tilde{n}(N_{\tilde{n}_i}^{(2)}+N_{\tilde{n}_i}^{(2)})+\frac{a}{2\pi}\sum_{i=1}^{d-2}N_i^{(3)}\right] \tag{31}$$

subject to the closed-string constraints

$$\sum_{i=1}^{d-2}\sum_{n=1}^{\infty} n(N_{n_i}^{(1)}-\tilde{N}_{n_i}^{(1)})=0\ ,$$
$$\sum_{i=1}^{d-2}\sum_{n=1}^{\infty} n(N_{\tilde{n}_i}^{(2)}-\tilde{N}_{\tilde{n}_i}^{(2)})=0\ , \tag{32}$$

where

$$\tilde{n}=\left[n^2+\frac{a^2}{4\pi^2}\right]^{1/2} . \tag{33}$$

Summing (30) over oscillator spectrum (31) subject to constraints (32) yields exactly the free energy given in (29). The τ_1 integral enforces (32) while $\tau_2=sT$. Thus (31) and (32) replace the usual spectrum for the Nambu-Goto theory. There are several interesting features to take note of.

From (31) we see that there are only transverse modes in the mass spectrum. The longitudinal modes of $N_{n_i}^{(1)}$ and $\tilde{N}_{n_i}^{(1)}$ have been canceled by the factor $\det'(-\Delta)$ in the Jacobian (19) while those of $N_{\tilde{n}_i}^{(2)}$, $N_{\tilde{n}_i}^{(2)}$, and $N_i^{(3)}$ by the factor $\det'[-(1/\alpha_0 T)\Delta+1]$ in (19) which comes from the term $\delta g_{ab}^{(2)}$ in (13).

In Eq. (31) the number operators $N_{n_i}^{(1)}$ and $N_{\tilde{n}_i}^{(2)}$ represent independent right movers while $N_{\tilde{n}_i}^{(1)}$ and $\tilde{N}_{\tilde{n}_i}^{(2)}$ are the corresponding left movers. These are commuting operators. $N_i^{(3)}$ arises due to zero-point fluctuations of the smooth string. The constraints (32) are the usual ones: namely, the number of left-moving degrees of freedom coincide with those of the right-moving degrees of freedom. Consider first the tachyonic state. The $(\text{mass})^2$ for this state in the presence of extrinsic curvature is given by

$$m^2(a)=\left[\frac{d-2}{24}\right]4\pi T\left[-2+\frac{3a^2}{2\pi^2}+12I(a)\right] . \tag{34}$$

As $a\to 0$ and $I(a)\to-\frac{1}{6}$ and $m^2(a)\ =[(d-2)/24](-16\pi T)$ is just twice the mass square of the tachyon of the Nambu-Goto theory. As $a\to 0$, the extrinsic curvature term dominates and in this case the theory has a tachyon. For nonvanishing a, clearly, there exists a critical value of a for which $m^2(a_c)=0$. This implies that the smooth strings can be free of tachyons for $a\ge a_c$. It should be remarked here that our expressions for the path integral and the free energy are not valid for $a^2\to\infty$. The reason for this lies in our regularization scheme. As explained in the Appendix, it is valid as long as a is finite.

Turning now to the excited states, we note that there are two sets of oscillator states described by $N^{(1)}$, $\tilde{N}^{(1)}$, $N^{(2)}$, and $\tilde{N}^{(2)}$. This is reminiscent of the situation in classical nonrelativistic stiff strings.[17] In the nonrelativistic case the equations of motion are fourth order in space and each normal mode, for a clamped string, is twofold degenerate. According to (31) we see that for the quantum relativistic stiff string, this degeneracy is lifted as n and $\tilde{n}$ have different values. It can be seen from (31) that there are no zero-mass spin-2 states in the excitation spectrum of the smooth strings. This agrees with the fact that smooth string theory does not have modular invariance.[11] Finally, we note that the operator $N_i^{(3)}$ denotes the zero-point excitation of smooth strings.

APPENDIX

We wish to evaluate $\det'[-(1/\alpha_0 T)\Delta+1]$. The metric and its inverse are

$$g_{ab}=\begin{bmatrix}1 & \tau_1\\ \tau_1 & |\tau|^2\end{bmatrix}$$

and

$$g^{ab}=\frac{1}{\tau_2^2}\begin{bmatrix}|\tau|^2 & -\tau_1\\ -\tau_1 & 1\end{bmatrix} . \tag{A1}$$

So the operator is

$$\Delta=\frac{1}{\sqrt{g}}\partial_a(g^{ab}\sqrt{g}\,\partial_b)$$
$$=g^{ab}\partial_a\partial_b=\frac{1}{\tau_2^2}(\,|\tau|^2\partial_1^2+\partial_2^2-2\tau_1\partial_1\partial_2) . \tag{A2}$$

The eigenfunctions are

$$U_{mn}(\sigma^1,\sigma^2)=\exp\left[2\pi i\left[\frac{m\sigma^1}{L_1}+\frac{n\sigma^2}{L_2}\right]\right] . \tag{A3}$$

Therefore

$$\det'\left[-\frac{1}{\tilde{\alpha}_0 T}\Delta+1\right]=\det'\left[-\frac{1}{\tilde{\alpha}_0 T\tau_2^2}[(\tau_1^2+\tau_2^2)\partial_1^2+\partial_2^2-2\tau_1\partial_1\partial_2]+1\right]$$
$$=\sum_{m,n}\left\{-\frac{1}{\tilde{\alpha}_0 T\tau_2^2}\left[\left[\frac{2\pi i m}{L_1}\right]^2(\tau_1^2+\tau_2^2)+\left[\frac{2\pi i n}{L_2}\right]^2+2\tau_1(2\pi i)^2\frac{m\cdot n}{L_1\cdot L_2}\right]+1\right\}$$
$$=\sum_{m,n}\frac{4\pi^2}{\tau_2^2\tilde{\alpha}_0 T}\left[\frac{m^2}{L_1^2}\tau_2^2+\left[\frac{n}{L_2}-\frac{\tau_1 m}{L_1}\right]^2+\frac{\tilde{\alpha}_0 T\tau_2^2}{4\pi^2}\right] \tag{A4}$$

and

$$\ln\det'\left[-\frac{1}{\tilde{\alpha}_0 T}\Delta+1\right]=\sum_{m,n}\ln\left\{\frac{4\pi^2}{\tau_2^2\tilde{\alpha}_0 T}\left[\frac{m^2}{L_1^2}\tau_2^2+\left[\frac{n}{L_2}-\frac{\tau_1 m}{L_1}\right]^2+\frac{\tilde{\alpha}_0 T\tau_2^2}{4\pi^2}\right]\right\} . \tag{A5}$$

We use the ζ-function regularization without introducing any scale to find the finite part without additional subtraction. Thus, for finite $\tilde{\alpha}_0 T$, define

$$\ln\det'\left[-\frac{1}{\tilde{\alpha}_0 T}\Delta+1\right]=-\lim_{s\to 0}\frac{d}{ds}\sum_{m,n}\left\{\frac{4\pi^2}{\tau_2^2\tilde{\alpha}_0 T}\left[\frac{m^2}{L_1^2}\tau_2^2+\left[\frac{n}{L_2}-\frac{\tau_1 m}{L_1}\right]^2+\frac{\tilde{\alpha}_0 T\tau_2^2}{4\pi^2}\right]\right\}^{-s}$$
$$=-\lim_{s\to 0}\frac{d}{ds}\left[\frac{4\pi^2}{\tau_2^2\tilde{\alpha}_0 T L_1^2}\right]^{-s}\sum_{m,n}\left[\frac{m^2\tau_2^2 L_2^2}{L_1^2}\left[n-\frac{\tau_1 L_2}{L_1}m\right]^2+\frac{\tilde{\alpha}_0 T L_2^2\tau_2^2}{4\pi^2}\right]^{-s}. \tag{A6}$$

The sum over n is converted into an integral using the Sommerfeld-Watson transformation[9]

$$\ln\det'\left[-\frac{1}{\tilde{\alpha}_0 T}\Delta+1\right]=-\lim_{s\to 0}\frac{d}{ds}\left[\left[\frac{4\pi^2}{\tilde{\tau}_2^2 a^2}\right]^{-s}\int_c dz\sum_m\frac{e^{i\pi z}}{2i\sin\pi z}\left[m^2\tilde{\tau}_2^2+(z-m\tilde{\tau}_1)^2+\frac{a^2\tilde{\tau}_2^2}{4\pi^2}\right]^{-s}+\text{H.c.}\right]$$
$$+\lim_{s\to 0}\frac{d}{ds}\left[\left[\frac{4\pi^2}{\tilde{\tau}_2^2 a^2}\right]^{-s}\int_c\frac{dz}{2}\sum_m\left[m^2\tilde{\tau}_2^2+(z-m\tilde{\tau}_1)^2+\frac{a^2\tilde{\tau}_2^2}{4\pi^2}\right]^{-s}+\text{H.c.}\right], \tag{A7}$$

where we have set $\tilde{\tau}=\tau L_2/L_1$ and $a^2=\tilde{\alpha}_0 T L_1^2$. The contour passes above the real axis, from $+\infty+i\epsilon$ to $-\infty+i\epsilon$. The first term in large square brackets converges at $s=0$ and gives

$$2\sum_{m=1}^{\infty}\ln|1-e^{2\pi i W_+}|^2+2\ln(1-e^{-\tau_2 a}) \tag{A8}$$

with

$$W_+=m\tilde{\tau}_1+i\tilde{\tau}_2(m^2+a^2/4\pi^2)^{1/2}\,. \tag{A9}$$

The second term converges for $s>1$. It reduces to

$$\lim_{s\to 0}\frac{d}{ds}\left[\frac{4\pi^2}{\tilde{\tau}_2^2 a^2}\right]^{-s}\frac{\sin\pi s}{\cos\pi s}\frac{\Gamma^2(1-s)}{\Gamma(2-2s)}(2\tilde{\tau}_2)^{1-2s}\sum_m\frac{1}{(m^2+a^2/4\pi^2)^{s-1/2}}\,. \tag{A10}$$

We use the Sommerfeld-Watson transformation once again to convert the m sum to an integral

$$\sum_m\frac{1}{(m^2+a^2/4\pi^2)^{s-1/2}}=\left[\int_c dz\frac{e^{i\pi z}}{2i\sin\pi z}\left[z^2+\frac{a^2}{4\pi^2}\right]^{1/2-s}+\text{H.c.}\right]-\left[\int\frac{dz}{2}\left[z^2+\frac{a^2}{4\pi^2}\right]^{1/2-s}+\text{H.c.}\right]. \tag{A11}$$

The contour passes above the real axis from $+\infty+i\epsilon$ to $-\infty+i\epsilon$. The first term in large square brackets converges at $s=0$ and gives

$$-4\int_0^\infty\frac{dy}{1-e^{2\pi(y+a/2\pi)}}y^{1/2-s}\left[y+\frac{a}{\pi}\right]^{1/2-s}\sin\pi(s-\tfrac{1}{2})\,. \tag{A12}$$

The second term converges for $s>1$. Its value is

$$\frac{2\sin^2\pi(s-\frac{1}{2})}{(a/\pi)^{2s-2}\sin\pi(2s-2)}\frac{\Gamma^2(\frac{3}{2}-s)}{\Gamma(3-2s)}\,. \tag{A13}$$

Substituting (A11), (A12), and (A13) into (A10), we find

$$(\text{A10})=8\pi\tilde{\tau}_2\int_0^\infty\frac{dy}{1-e^{2\pi(y+a/2\pi)}}y^{1/2}\left[y+\frac{a}{\pi}\right]^{1/2}+\frac{\tilde{\tau}_2 a^2}{4\pi}\,. \tag{A14}$$

Finally, we get

$$\ln\det\left[-\frac{1}{\tilde{\alpha}_0 T}\Delta+1\right]=2\sum_{n=1}^{\infty}\ln|1-e^{2\pi i W_+}|^2+2\ln(1-e^{-\tilde{\tau}_2 a})+8\pi\tilde{\tau}_2\int_0^\infty\frac{dy}{1-e^{2\pi(y+a/2\pi)}}y^{1/2}\left[y+\frac{a}{\pi}\right]^{1/2}+\frac{\tilde{\tau}_2 a^2}{4\pi} \tag{A15}$$

or

$$\det'\left[-\frac{1}{\tilde{\alpha}_0 T}\Delta+1\right]=e^{\tilde{\tau}_2[a^2/4\pi+2\pi I(a)]}|f(e^{2\pi i W_+})|^4(1-e^{-\tilde{\tau}_2 a})^2 \tag{A16}$$

with

$$I(a)=4\int_0^\infty \frac{dy}{1-e^{2\pi(y+a/2\pi)}}y^{1/2}\left[y+\frac{a}{\pi}\right]^{1/2}, \qquad \text{(A17)}$$

$$W_+=n\tilde{\tau}_1+i\tilde{\tau}_2(n^2+a^2/4\pi^2)^{1/2},$$

and

$$\tilde{\tau}=\frac{L_2\tau}{L_1}.$$

It should be noted that the ζ-function regularization (A6) is valid only for finite $\alpha_0 T$. As $\tilde{\alpha}_0 T$ tends to infinity

$$\lim_{a\to\infty} \ln \det'\left[-\frac{1}{\tilde{\alpha}_0 T}\Delta+1\right]\to \ln(1)=0 .$$

Therefore the extrinsic curvature term does not contribute to the path integral.

From (A15) or (A16), we see that by redefining $\tilde{\tau}=\tau L_2/L_1$, $\det[-(1/\tilde{\alpha}_0 T)\Delta+1]$ depends only on a which just contains L_1 as a parameter. So we see that the path integral [see (20)–(26)] depends only on the physical length of the string taken to be L_1.

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