Non-Abelian gauge invariance and the infrared approximation

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Two constructions are given of infrared approximations, defined by a nonlocal configuration-space restrictions, which preserve the local, non-Abelian gauge invariance of SU(N) two-dimensional QCD (QCD_2). These continuum infrared methods are used to estimate the quenched order parameter $\langle\bar{\psi}\psi\rangle$ in the strong-coupling, or chiral, limit and are compared to a previous calculation where gauge invariance was not manifest. Both constructions provide results which, in the chiral limit, differ from each other and from the previous estimation by an inessential, multiplicative scaling factor.

I. INTRODUCTION

In the preceding paper[1] some computations of the quenched order parameter $\langle\bar{\psi}\psi\rangle$ in the infrared (IR) approximation were reported for the case of SU(N) two-dimensional QCD, for finite $N(=2,3)$ and in the limit $N\to\infty$. The method used was an extension of earlier gauge-invariant techniques[2,3] in two-dimensional QED (QED_2); and the claim was made that the calculation, while not manifestly gauge invariant in its entirety, could be understood as a trivially scaled version of the output of one that is manifestly gauge invariant. It is the purpose of these remarks to justify that claim by showing how the computations of Ref. 1, here designated as method 1, are related to two other methods which are manifestly gauge invariant. We here discuss the problem of enforcing rigorous gauge invariance in the QCD sense on an IR method of approximation successfully used in QED, where the Abelian nature of the theory posed no problem. Quite apart from justifying the results of Ref. 1, the question is interesting and important in its own right, especially if these IR techniques are going to be applied to other, more complicated physical problems.

The central question can be phrased in the following way. Non-Abelian gauge transformations demand invariance under a local, configuration-space operation. The IR method, on the other hand, limits the magnitude of virtual momenta which enter into the computation of any physical process; and that restriction, local in momentum space, is nonlocal in configuration space. How can local gauge invariance in configuration space be preserved under nonlocal IR restriction?

Two answers to this question will be constructed below, labeled methods 2 and 3. Before that, however, it will be useful and most appropriate to remind the reader of the methods and results of Ref. 1. There, one began with the exact gauge invariance of the Fradkin representation[4] for the logarithm of the fermion determinant $L[A]$ in the background field $A_\mu^a(z)$, and approximated the functional form of $L[A]$ (the "multipole approximation") in a rigorously gauge-invariant way, using a nontrivial generalization of a similar, Abelian technique. The other half of the IR method, however, is the decomposition of A_μ, or of $F_{\mu\nu}$, into parts containing "soft" and "hard" momenta, and it is here that the real problem of maintaining gauge invariance arises when one retains only the soft parts of every interaction.

The physical, gauge-invariant quantity desired in this problem is

$$\langle\bar{\psi}\psi\rangle=-\frac{\partial}{\partial m}\ln\langle 0_+\,|\,0_-\rangle\Big/\int d^2x_E\ , \tag{1.1}$$

where the vacuum-to-vacuum amplitude is given by (with all quantities in Euclidean space)

$$\langle 0_+\,|\,0_-\rangle=\int d[A]\delta(\mathcal{F}[A])\det M[A]\times\exp\left[-\tfrac{1}{4}\int F^2+L[A]\right]\ . \tag{1.2}$$

In (1.2) $\det M[A]$ is the measure corresponding to the gauge condition $\mathcal{F}[A]=0$; $\int d[A]$ is typically represented by $\prod_{a,\mu,z}\int dA_\mu^a(z_i)$;

$$L[A]=\mathrm{Tr}\ln(1-ig\gamma\cdot AS_c)=\mathrm{Tr}(G_c^{-1}[A]G_c[0])\ ;$$

and $G_c[A]$ is the causal quark propagator in the presence of the background field A, with $G_c[0]=S_c$.

The only properties of the measure which we assume[5] are (i) it is gauge invariant, corresponding to an invariant value of (1.2), and (ii) in any axial gauge, or in the coordinate gauge $x_\mu A_\mu(x)=0$, $\det M[A]$ is a field-independent constant which can be absorbed into the normalization of $\langle 0_+\,|\,0_-\rangle$, with physical quantities such as (1.1) independent of that normalization. The closed-loop functional $L[A]$, where all the dynamics resides, has a familiar representation first given by Schwinger[6] in terms of a proper-time variable s:

$$L[A]=-\frac{1}{2}\int_0^\infty\frac{ds}{s}e^{-ism^2}\mathrm{Tr}(e^{-is[\gamma\cdot(gA+i\partial)]^2})-(g\to 0)\ , \tag{1.3}$$

where, whenever the analytic continuation $s\to -i\tau$ is performed, τ will be called the proper time [even though its dimensions are those of (time)^2]. The exact representation given by Fradkin,[7] most useful in a variety of problems where an IR approximation is contemplated,[8] rewrites (1.3) in terms of a Gaussian functional integral over a four-vector $\phi_\mu(s)$, in the form

$$L[A]=-\frac{1}{2}\int_0^\infty \frac{ds}{s}e^{-ism^2}\int d^2x\,N(s)\int d[\phi]\exp\left[\frac{i}{4}\int_0^s ds'\phi^2(s')\right]\delta\left[\int_0^s ds'\phi(s')\right]\mathrm{Tr}U(s)\ , \tag{1.4}$$

with

$$U(s)=\left[\exp\left\{-ig\int_0^s ds'\left[\phi_\mu(s')A_\mu\left[x-\int_0^{s'}\phi\right]-i\sigma_{\mu\nu}F_{\mu\nu}\left[x-\int_0^{s'}\phi\right]\right]\right\}\right]_+ ,$$

$$N(s)^{-1}=\int d[\phi]\exp\left[\frac{i}{4}\int_0^s ds'\phi^2(s')\right],\quad \sigma_{\mu\nu}=\tfrac{1}{4}[\gamma_\mu,\gamma_\nu],\quad A_\mu(z)=\lambda^a A_\mu^a(z)\ ,$$

$$F_{\mu\nu}(z)=\lambda^a F_{\mu\nu}^a(z)=\lambda^a(\partial_\mu A_\nu^a-\partial_\nu A_\mu^a+gf_{abc}A_\mu^b A_\nu^c)\ .$$

Here, the λ^a are the Gell-Mann (fundamental representation) matrices of SU(N), satisfying

$$[\lambda^a,\lambda^b]=if_{abc}\lambda^c,\quad \mathrm{tr}[\lambda^a]=0,\quad \mathrm{Tr}[\lambda^a\lambda^b]=2\delta_{ab}\ .$$

An explicit demonstration of the gauge invariance of $\mathrm{tr}U(s)$ under the transformations

$$A_\mu(x)\to A'_\mu(x)=V^\dagger(x)\left[A_\mu V+\frac{i}{g}\partial_\mu V\right],\quad F_{\mu\nu}(x)\to F'_{\mu\nu}(x)=V^\dagger(x)F_{\mu\nu}V,\quad V^\dagger V=1\ , \tag{1.5}$$

was given in Ref. 1. This invariance is independent of the magnitude of $\phi_\mu(s)$, and can be expected to hold for any expansion or regrouping of $\mathrm{tr}U$ in powers of ϕ. In particular, a simple method of generating the approximate form of $L[A]$ corresponding to the retention of quadratic ϕ dependence in (effectively) the logarithm of $U(s)$ was explained in Ref. 1; this produces for a non-Abelian theory the equivalent of the "multipole expansion" in QED, and replace $\mathrm{tr}U$ by the simpler expression

$$\mathrm{tr}U(s)\simeq\mathrm{tr}\left[\exp\left[-i\int_0^g dg'\frac{\partial}{\partial g'}[g'F_{\mu\nu}(g',x)]M_{\mu\nu}(s)\right]\right]_{+(g)}\ , \tag{1.6}$$

where

$$M_{\mu\nu}(s)=\Omega_{\mu\nu}(s)-i\sigma_{\mu\nu}\ ,$$

$$\Omega_{\mu\nu}(s)=\int_0^s ds'\phi_\mu(s')\int_0^{s'}ds''\phi_\nu(s'')\ ,$$

$$F_{\mu\nu}^a(g,x)=\partial_\mu A_\nu^a-\partial_\nu A_\mu^a+igf_{abc}A_\mu^b A_\nu^c\ ,$$

and the symbol $+(g)$ denotes an ordering with respect to the variable coupling g'. Equation (1.6) is strictly gauge invariant under the transformations (1.5), which is to be expected in view of the invariance of the exact (1.4). Equation (1.6) is still a rather complicated object—it is not the same as $\exp[-igF_{\mu\nu}(g,x)M_{\mu\nu}]$—but in two dimensions it will reduce to a relatively simple form in any axial gauge, or in the coordinate gauge, when the time comes to calculate.

The second half of the IR method, the separation into soft and hard parts and the insertion of the soft $F_{\mu\nu}$ into (1.6), must now be accomplished. The method adopted in Ref. 1, here called method 1, was the simplest possible generalization of the Abelian prescription: every color component of $F^a(k)=F_{14}^a(k)$ was rewritten in the form

$$\tilde{F}(k)=e^{-k^2/\mu_c^2}\tilde{F}(k)+(1-e^{-k^2/\mu_c^2})\tilde{F}(k)$$

$$\equiv\tilde{F}_S(k)+\tilde{F}_H(k)\ , \tag{1.7}$$

and an expansion developed in powers of F_H^a. We discuss below, in detail, what is apparently wrong with this procedure, how to correct it, and why the results of Ref. 1 are nevertheless appropriate; but it will first be useful to describe, very briefly, the motivation behind this second part of the IR method.

As in QED, the intuitive idea here is that strong-coupling (SC) limits should be given by the F_S^a dependence only, while corrections to that limit are defined by the sequence of terms of the F_H^a expansion. The upper cutoff μ_c to the virtual-gluon momenta is chosen to be $c/\sqrt{\tau}$, where c is a real, positive constant on the order of unity; this choice of μ_c, in turn, justifies the multipole expansion. All necessary functional integrals can then be evaluated, yielding a function containing all powers of g. In two dimensions g and m have the same dimension, so that the true dimensionless coupling is g/m, and the chiral and SC limits are the same. In the chiral limit of QED_2 this approach[9] agrees very nicely with the known exact answer[10] for $\langle\bar{\psi}\psi\rangle$, even when the approximation of quenching is partially removed.[11] The IR calculations are really estimates, for they depend multiplicatively on the parameter c (in the combination gc). Any exact answer, of course, would be independent of c; but this will not be the case when an approximation is made. (Rather like a constant of integration, or the value of a scattering amplitude at the subtraction point of a once-subtracted dispersion relation, the constant c must be specified in a manner external to our approximate IR computation.) Nevertheless, the agreement between the exact QED_2 chiral result (E is Euler's constant)

$$\langle\bar{\psi}\psi\rangle=-ge^E/2\pi^{3/2}$$

and the corresponding quenched IR estimate

$$\langle\bar{\psi}\psi\rangle=-gc/4\pi^{3/2}$$

is so close, including the same phase-space factors $\pi^{-3/2}$, that the IR method is probably a very reasonable way of estimating SC effects directly in the continuum. To the best of our knowledge, no machine calculation as yet has produced values of $\langle\bar{\psi}\psi\rangle$ for SU(N) QCD$_2$, for all N and arbitrary g, in the chiral limit, as computed in Ref. 1.

In any axial gauge, or in the coordinate gauge, this simplest procedure of (1.7) generates,[1] for the order parameter in the quenched approximation,

$$\langle\bar{\psi}\psi\rangle=-\frac{m}{2\pi}\int_0^\infty\frac{d\tau}{\tau}e^{-\tau m^2}\eta\int d[F]\exp\left[-\tfrac{1}{2}\int F^2\right]\mathrm{tr}\{[g\tau\boldsymbol{\lambda}\cdot\mathbf{F}_S(x)]\coth[g\tau\boldsymbol{\lambda}\cdot\mathbf{F}_S(x)]-1\}\ , \tag{1.8}$$

where

$$\eta^{-1}=\int d(F)\exp\left[-\tfrac{1}{2}\int F^2\right],\quad F_S^a(x)=\int d^2z\,f(x-z)F^a(z)\ ,$$

and

$$f(x-z)=(\mu_c^2/4\pi)\exp[-(x-z)^2\mu_c^2/4]\ .$$

This can be rewritten so as to disentangle the spatial and color variables, in a form somewhat different from that of Ref. 1, but useful for subsequent comparison with methods 2 and 3,

$$\langle\bar{\psi}\psi\rangle=-\frac{m}{2\pi}\int d^Lu\ \mathrm{tr}[(\boldsymbol{\lambda}\cdot\mathbf{u})\coth(\boldsymbol{\lambda}\cdot\mathbf{u})-1]\int\frac{d^LQ}{(2\pi)^L}e^{-i\mathbf{Q}\cdot\mathbf{u}}\int_0^\infty\frac{d\tau}{\tau}e^{-\tau m^2}I[J]\ , \tag{1.9}$$

with

$$I[J]=\eta\int d[F]\exp\left[-\tfrac{1}{2}\int F^2+i\int d^2u\ \mathbf{J}(u)\cdot\mathbf{F}(u)\right]\ , \tag{1.10}$$

where $\mathbf{J}(u)=g\tau f(x-u)\mathbf{Q}$, and $L=N^2-1$.

The functional integral of (1.10) is trivial and yields

$$I[J]=\exp\left[-\tfrac{1}{2}\int d^2u\ \mathbf{J}^2(u)\right]=\exp(-g^2\tau^2\mathbf{Q}^2\mu_c^2/16\pi)\ . \tag{1.11}$$

Then, the $\int dQ$ is easily done, leading to

$$\langle\bar{\psi}\psi\rangle=-\frac{m}{2\pi}(2\pi)^{-L/2}\int_0^\infty\frac{dt}{t}e^{-t}\int d^Lz\,e^{-z^2/2}\mathrm{tr}\{[\rho\sqrt{2t}\,(\boldsymbol{\lambda}\cdot\mathbf{z})]\coth(\rho\sqrt{2t}\,\boldsymbol{\lambda}\cdot\mathbf{z})-1\}\ , \tag{1.12}$$

where $\rho=gc/4m\sqrt{\pi}$, and some obvious variable rescalings have been used. The tr operation on the color coordinates may be written as

$$\sum_l\{[\rho z\sqrt{2t}\,\xi_l(\hat{\mathbf{z}})]\coth[\rho z\sqrt{2t}\,\xi_l(\hat{\mathbf{z}})]-1\}\ , \tag{1.13}$$

where $\xi_l(\hat{\mathbf{z}})$ is one of the N eigenvalues of $\boldsymbol{\lambda}\cdot\hat{\mathbf{z}}$, with $\hat{\mathbf{z}}=\mathbf{z}/z$, $z=+(\mathbf{z}^2)^{1/2}$. In the SC/chiral limit where $\rho\gg1$, (1.12) can be brought into the relatively simple form

$$\begin{aligned}\langle\bar{\psi}\psi\rangle\,|_{g/m\to\infty}&\to-\frac{m\rho}{2\pi}(2\pi)^{-L/2}\sqrt{2}\int_0^\infty\frac{dt}{\sqrt{t}}e^{-t}\int d^Lz\,e^{-z^2/2}\,|z|\sum_l|\xi_l(\hat{\mathbf{z}})|\\&=-\left[gc\,\Gamma\left[\frac{L+1}{2}\right]\Big/4\pi m\,\Gamma\left[\frac{L}{2}\right]\right]\sum_l\langle\,|\xi_l(\hat{\mathbf{z}})|\,\rangle\ ,\end{aligned} \tag{1.14}$$

where $\langle\,|\xi_l|\,\rangle$ denotes an average over the solid angle Ω_L of an ($L=N^2-1$)-dimensional sphere,

$$\langle\,|\xi_l(\hat{\mathbf{z}})|\,\rangle=\int d\Omega_L\,|\xi_l(\hat{\mathbf{z}})|/\Omega_L\ .$$

This is not a particularly easy expression to evaluate, in comparison to the more powerful Mehta-Dyson methods used in Ref. 1; but its advantage is that it allows a straightforward comparison with the output of methods 2 and 3. From (1.14) one can also write down simple upper and lower bounds on the SC limit of $\langle\bar{\psi}\psi\rangle$, as indicated in the Appendix.

The only difficulty with the prescription (1.7) of method 1 is that one loses the manifest gauge invariance maintained through (1.6), where $L[A]$, which we now call $L[F]$, is invariant. For if $F_{\mu\nu}$ transforms under (1.5) by means of the unitary operator

$$V(x)=\exp\left[i\sum_a \lambda^a v^a(x)\right],$$

and if $\tilde{F}^a_S(k)=\exp(-k^2/\mu_c^2)\tilde{F}^a(k)$, then the transformation (1.5) generates a nonlocal transformation of $F^a_S(x)$, and the manifest, local gauge invariance of (1.6) is not sufficient to undo the nonlocality. In detail, suppose that an infinitesimal transformation $V=1+i\boldsymbol{\lambda}\cdot\delta\mathbf{v}$ is performed on

$$F(z)=\sum_a \lambda^a F^a(z)=\boldsymbol{\lambda}\cdot\mathbf{F}(z) .$$

The change $\delta F^a_S(x)$ is then given by

$$\sum_{bc} f_{abc}\int d^2z\, f(x-z)\delta\mathbf{v}^b(z)F^c(z)\equiv\delta u^a(x) .$$

But because of the invariance of (1.6), one could always adjoin to V a gauge transformation $W(x)$, inside $L[F_S]$,

$$L[W^\dagger(x)F_S(x)W(x)]\equiv L[F_S(x)] .$$

In infinitesimal form, this would generate a total change in $F_S(x)$ of an amount

$$\delta F^a_S(x)=\sum_{bc} f_{abc}\delta W^b(x)F^c_S(x)+\delta u^a(x) ;$$

and if δW^a could be chosen so that this δF^a_S vanishes, invariance under the V transformation of (1.5) would be manifest. Unfortunately, such a choice of δW^a is not always possible, for the vanishing of δF^a_S requires that $\sum_a F^a_S(x)\delta u^a(x)=0$ as well, a condition which need not at all be satisfied. The difficulty is easily traced to the form of such a transformation, which rotates $\mathbf{F}(y)$ in color space, in a manner dependent on configuration coordinates, but leaves unchanged the magnitude of $F(y)$: if $\boldsymbol{\lambda}\cdot\mathbf{F}\to\boldsymbol{\lambda}\cdot\mathbf{F}'=V^\dagger\boldsymbol{\lambda}\cdot\mathbf{F}V$, then $F'^2=F^2$.

It may well be that method 1 possesses a gauge invariance which is simply not manifest—in two dimensions, e.g., it generates a result that is independent of the choice of axial gauge—but it is unsettling not to be able to ascertain at a glance whether or not this is true. (A rough argument was given in Ref. 1 to suggest that translational invariance could be invoked to show that non-gauge-invariant components are suppressed in a final step of the calculation.)

In the next two sections we outline two methods for the construction of manifestly gauge-invariant IR approximations, which should have application beyond the simple estimates of $\langle\bar{\psi}\psi\rangle$. These two methods, labeled 2 and 3, are quite different; but they have as output, in the chiral limit of $\langle\bar{\psi}\psi\rangle$, the feature that the multiplicative constant c (appearing in the combination gc) is merely scaled, for all N, by a numerical factor on the order of unity, in comparison with the results of Ref. 1. In this sense the calculations of Ref. 1 can be considered as "embedded" within an estimation that is manifestly independent of gauge. Based on (1.14), or on the results of methods 2 and 3, we mention in a brief Appendix simple upper and lower bounds for $\langle\bar{\psi}\psi\rangle$, for all N.

II. METHOD 2

It was noted above that $\hat{\mathbf{F}}=\mathbf{F}/F$ is rotated under a gauge transformation, while $F=+\sqrt{\mathbf{F}^2}$ is left unchanged. This suggests a different way of defining the soft/hard separation,

$$\begin{aligned}&\mathbf{F}_S(x)\equiv\hat{\mathbf{F}}(x)F_S(x) ,\\ &F_S(x)=\int d^2z\, f(x-z)F(z) ;\end{aligned}\tag{2.1}$$

that is, $\hat{\mathbf{F}}$ is untouched while the soft/hard decomposition refers only to the magnitude F, $F(x)=F_S(x)+F_H(x)$, with

$$F_S(x)=\int d^2z\, f(x-z)F(z) .$$

It immediately follows that $\boldsymbol{\lambda}\cdot\mathbf{F}_S$ is transformed in exactly the same local way as $\boldsymbol{\lambda}\cdot\mathbf{F}$ under the gauge transformation (1.5), and hence the original, manifest gauge invariance of $L[\lambda F]$ is preserved for $L[\lambda F_S]$.

There is, however, a price to pay for this simple, if somewhat asymmetric, resolution of our problem. When a gauge is chosen, and the computation attempted, one finds that instead of performing a functional integral (Fl) over a vector in color space, one is faced with the same Fl over the magnitude of that color-space vector. That is, in method 1 one required the elementary Fl

$$\begin{aligned}I[J]=\eta\int d[F]\exp&\left[-\tfrac{1}{2}\int d^2u\, F^2\right.\\&\left.+i\int \mathbf{J}(u)\cdot\mathbf{F}(u)d^2u\right]\\=\exp&\left[-\tfrac{1}{2}\int d^2u\, \mathbf{J}^2(u)\right],\end{aligned}\tag{2.2}$$

with the definition of $\mathbf{J}(u)$ given following (1.9). For the present method 2 calculation, one will need

$$\eta\int d[F]\exp\left[-\tfrac{1}{2}\int d^2u\, F^2+i\int d^2u\, J(u)F(u)\right],\tag{2.3}$$

with the magnitude $J(u)$ defined as $g\tau\mathbf{Q}\cdot\hat{\mathbf{F}}(x)f(x-u)$. This seemingly small difference, however, harbors a singularity which must be understood and removed. The variable x, incidentally, is that of the fermion loop; and because, by translational invariance, it drops out of the final answer, we have omitted numerator and denominator factors of $\int d^2x$ in these expressions, as well as in those of the previous section.

Imagine subdividing all of configuration space into a dense set of very small volumes Δ, centered at the points x_i. The Fl volume element

$$d[F]=\prod_{x_i,a} dF^a(x_i)$$

may also be written as

$$\prod_{x_i} d\Omega_i F_i^{L-1}dF_i ,$$

where $d\Omega_i$ denotes an element of solid angle in an $(L=N^2-1)$-dimensional vector space, and F_i represents

the magnitude of the vector $\mathbf{F}(x_i)$. The exponential factor

$$-\tfrac{1}{2}\int F^2 d^2u = -\tfrac{1}{2}\Delta\sum_i F_i^2$$

is independent of solid angle. Denote the x coordinate of $J(u)$ by x_j; then, for that cell $i=j$, one must evaluate

$$\int d\Omega_j \exp[ig\tau F_S(x_j)\mathbf{Q}\cdot\hat{\mathbf{F}}_j] ,$$

where

$$F_S(x_j)=\int d^2u f(x_j-u)F(u) .$$

This solid-angle integration will always generate some function of $(g\tau F_S(x_j))^2\mathbf{Q}^2$, which can be rewritten as

$$\int d\Omega \exp[ig\tau Q F_S(x_j)\hat{\mathcal{J}}\cdot\hat{\mathbf{v}}] ,$$

where $Q=+\sqrt{\mathbf{Q}^2}$, $\hat{\mathbf{v}}$ is a fixed but arbitrary unit vector, and $\hat{\mathcal{J}}$ is the unit vector which ranges over all the solid angle Ω. Multiplying and dividing by a factor of Ω, and returning to the continuous limit, one can then replace (2.3) by

$$\frac{1}{\Omega}\int d\Omega\,\eta\int d[F]\exp\left[-\tfrac{1}{2}\int F^2+i\int J(u)F(u)\right] , \tag{2.4}$$

where, now, $J(u)=g\tau Q\hat{\mathcal{J}}\cdot\hat{\mathbf{v}}f(x-u)$. Other than the single $\hat{\mathcal{J}}$ vector associated with $\int d\Omega$, the field dependence is on magnitudes only.

To calculate the Fl of (2.4), return to the dense set of discrete points u_i, cancel from numerator and denominator all factors of $\int d\Omega_i$, and rescale each F_i magnitude variable by $F_i=G_i/\sqrt{\Delta}$; one is then left with a product over all i of the ratio

$$\int_0^\infty dG_i G_i^{L-1}e^{-G_i^2/2+i\sqrt{\Delta}J_iG_i}\Big/\int_0^\infty dG_i G_i^{L-1}e^{-G_i^2/2} , \tag{2.5}$$

where, for the moment, we suppress the final integration over $\int d\Omega$. Anticipating the limit $\Delta\to 0$, one may expand each factor $\exp(i\sqrt{\Delta}J_iG_i)$, retaining the first two nonvanishing, Δ-dependent terms:

$$1+iJ_i\sqrt{2\Delta}\frac{\Gamma\left[\frac{L+1}{2}\right]}{\Gamma\left[\frac{L}{2}\right]}-\frac{L}{2}\Delta J_i^2+\cdots . \tag{2.6}$$

Because we are dealing with magnitudes, the $\sqrt{2\Delta}$ term of (2.6) does not vanish, as it would by symmetry if we were calculating a method 1 Fl over $\exp(i\mathbf{J}\cdot\mathbf{F})$. Its contribution here is that of an infinite phase factor, since the product over all such points u_i is equivalent to

$$\prod_i \exp\left[i\Delta\sqrt{2/\Delta}\frac{\Gamma\left[\frac{L+1}{2}\right]}{\Gamma\left[\frac{L}{2}\right]}J_i-C\Delta J_i^2\right]\to\exp\left[-C\int d^2u\,J^2(u)+i\left[\frac{2}{\Delta}\right]^{1/2}\frac{\Gamma\left[\frac{L+1}{2}\right]}{\Gamma\left[\frac{L}{2}\right]}\int d^2u\,J(u)\right] , \tag{2.7}$$

with $C=L/2-[\Gamma((L+1)/2)/\Gamma(L/2)]^2$. This is clearly unacceptable for nonzero $\int d^2u\,J(u)$, as is the case here.

One way of avoiding this trouble is to require a redefinition of the Fl which removes the offending imaginary part of (2.6); that is, one retains only the real part of each cell's integration,

$$\int d[F]\to\int \mathrm{Re}\,d[F]\cdots\equiv\prod_i \mathrm{Re}\left[\int d^LF_i\cdots\right] .$$

This sort of prescription has been used, with some success, in quite another context[12] (where it was necessary to remove successive errors made in a white-noise Gaussian Fl over an approximation to an ordered exponential). It can be defined here as a kind of "renormalization" prescription, defined to give an unambiguous, gauge-invariant meaning to the Fl over magnitudes only; and it provides, directly from (2.6), the result

$$\exp\left[-\frac{L}{2}\int d^2u\,J^2(u)\right]$$
$$=\exp[-g^2\tau^2Q^2(\hat{\mathcal{J}}\cdot\hat{\mathbf{v}})^2\mu_c^2L/16\pi] . \tag{2.8}$$

This differs from the corresponding method 1 result by the inclusion of the factor $L(\hat{\mathcal{J}}\cdot\hat{\mathbf{v}})^2$. In the chiral limit, in comparison with (1.14), one sees that the $\langle\bar{\psi}\psi\rangle$ following from (2.8) is proportional to the additional, constant, multiplicative factor ζ, with

$$\zeta=(\sqrt{L}/\Omega)\int d\Omega\,|\,\hat{\mathcal{J}}\cdot\hat{\mathbf{v}}\,|$$
$$=(L/\pi)^{1/2}\Gamma(L/2)/\Gamma((L+1)/2) .$$

For large N this is just a constant independent of N, $\zeta\to(2/\pi)^{1/2}$. The result of this method 2 calculation is then just an extra factor ζ multiplying the gc dependence of the result of method 1. Since the IR estimate does not specify c, and the multiplicative factor ζ is of order unity, the two computations are effectively equivalent. When one is not in the SC limit the relation between the two results is more complicated than multi-

plicative; but, then, the IR estimates should not be taken seriously.

While perfectly gauge invariant, this way of introducing the soft-hard decomposition is asymmetric, treating $\hat{\mathbf{F}}$ differently from $|\mathbf{F}|$. Further, the necessary FI over magnitudes will develop an infinite phase factor, depending on the cell size Δ used in its evaluation; and a special (although simple and gauge-invariant) method of evaluation must be adopted to ensure a finite, real result. It would therefore be useful to have available an alternate method of calculation, one which avoids those unpleasant features of method 2.

III. METHOD 3

The method sketched in this section for the construction of a gauge-invariant IR approximation is intrinsically different from that of method 2 in the sense that it will involve transformations of the same, basic field $A(x)$ appearing everywhere in the exact FI's, rather than the previous formulation which replaces F by F_S in $L[F]$ only, but one which will involve the soft Fourier components of the $A(x)$ in a very definite way.

We return to the expression for $\langle 0+|0-\rangle$ of (1.2), and note that there is another way of writing the functional volume element

$$d[A]=\prod_{a,\mu,x_i} dA_\mu^a(x_i)$$

in terms of momentum-space components $\tilde{A}_\mu^a(k)$, if one is careful to include a summation over the real and imaginary parts[13] of each $\tilde{A}_\mu^a$ (which, for ease of notation we henceforth suppress, as we do all color and Lorentz indices). For clarity, imagine a finite space-time volume of quantization given in terms of a denumerably infinite number of momentum modes $k_n=2\pi n/L$, where L, here, denotes the length of each side of the quantization volume. One can then characterize the variables by $\prod_n d\tilde{A}_n$. The total number of modes summed upon cannot be limited—even if they are to be divided into soft n_s and hard n_h modes, they must all be included—but one *can* invent a method for suppressing dependence on the n_h modes sitting in the gauge-invariant coefficient functionals.

To this end, note that $L[A]$ is itself given as a sum over all proper-time values, $L[A]=\int d\tau L[\tau;A]$, which we now think of as a summation over neighboring values of a discrete proper time τ_i, $L[A]=\Delta\tau\sum_i L_i[A]$. This suggests rewriting (1.2) as a product over all "different worlds" each characterized by its own τ_i,

$$\langle 0_+|0_-\rangle=\prod_i\left[\prod_n\int d\tilde{A}_n\mathcal{G}_i[A]\right], \tag{3.1}$$

where

$$\mathcal{G}_i[A]=\delta(\mathcal{F}(A))\det M[A]\times\exp\left[-\tfrac{1}{4}\mathrm{tr}\int F^2[A]+\Delta\tau L_i[A]\right].$$

A division into soft and hard (Euclidean) momentum components can now be defined by the τ_i-dependent restrictions

$$n_s:\ k_n^2<\mu_i^2,\quad n_h:\ k_n^2>\mu_i^2\ , \tag{3.2}$$

where $\mu_i^2=\mu_c(\tau_i)^2=c^2/\tau_i$. Note that there is a "sharp" distinction between soft and hard momenta, for each τ_i and that the decomposition (3.2) is made simultaneously everywhere in the coefficient functional $\mathcal{G}_i[A]$.

We now rewrite $\mathcal{G}_i[A]$ as $\mathcal{G}_i[\tilde{A}_{ns},\tilde{A}_{nh}]$, using the notation of (3.2), and imagine that a Fourier transform of $\mathcal{G}_i$ can be rewritten for all modes, soft and hard,

$$\mathcal{G}_i(\tilde{A}_{n_s},\tilde{A}_{n_h}]=\prod_{n_s}\int\frac{d\rho_{n_s}}{2\pi}e^{i\rho_{n_s}\tilde{A}_{n_s}}\prod_{n_h}\int\frac{d\rho_{n_h}}{2\pi}e^{i\rho_{n_h}\tilde{A}_{n_h}}\times\tilde{\mathcal{G}}_i[\rho_{n_s},\rho_{n_h}]. \tag{3.3}$$

The functional soft/hard approximation may be introduced by rewriting the products

$$\prod_{n_s n_h}e^{i\rho_{n_s}\tilde{A}_{n_s}+i\rho_{n_h}\tilde{A}_{n_h}}$$

as

$$\prod_{n_s,n_h}e^{i\rho_{n_s}\tilde{A}_{n_s}+i\epsilon\rho_{n_h}\tilde{A}_{n_h}},$$

or as

$$\exp\left[i\sum_{n_s}\rho_{n_s}\tilde{A}_{n_s}\right]\left\{1+\epsilon\left[\exp\left[i\sum_{n_h}\rho_{n_h}\tilde{A}_{n_h}-1\right]\right]\right\},$$

or by any other convenient method (convenient, that is, for the calculation of corrections), and then expanding in powers of ϵ. (As in the Abelian calculations and as in methods 1 and 2, this is supposed to be a sensible approximation only in a SC limit, with the leading behavior of that limit estimated by the $\epsilon=0$ terms, and corrections defined by the subsequent terms in the ϵ expansion.) The leading, or purely soft dependence of (3.3) is then given by

$$\prod_{n_s}\int\frac{d\rho_{n_s}}{2\pi}e^{i\rho_{n_s}\tilde{A}_{n_s}}\prod_{n_h}\int\frac{d\rho_{n_h}}{2\pi}\tilde{\mathcal{G}}_i[\rho_{n_s},\rho_{n_h}]\equiv\mathcal{G}_i[\tilde{A}_{n_s},0]\ . \tag{3.4}$$

Inserting (3.4) into (1.2) yields

$$\langle 0_+|0_-\rangle_s=\prod_i\left[\prod_{n_s}\int d\tilde{A}_{n_s}\prod_{n_h}\int d\tilde{A}_{n_h}\mathcal{G}_i[\tilde{A}_{n_s},0]\right]=\prod_i\langle 0_0|0_-\rangle_i\ , \tag{3.5}$$

where all the multiplying coefficient functionals depend only on soft momentum components. In any axial

gauge, in two dimensions, it is clear that the "hard" part of the FI variables will provide an infinite but unimportant multiplicative constant, which cancels out of the computation. When we pass to the limit of an infinite configuration-space volume, there will be an infinite number of the soft k_{n_s} modes; the configuration-space $A(x)$ variables, which we now call $A_{\tau_i}(x)$, may be imagined to be given in terms of the nondenumerably infinite spread of k frequencies within the upper cutoff μ_i. We rewrite (3.5) in terms of configuration-space variables as $\mathcal{G}[A_{\tau_i}]$.

If we suppose that each unitary $V(x)$ is limited to maximum frequency components $\mu_i/3$, then any gauge transformation of form (1.5) will produce an $A_{\tau_i}(x)$ in the same τ_i world. But even if that restriction is dropped, and V is considered to have arbitrary frequency dependence, there will be no change in the $\mathcal{G}[A_{\tau_i}]$ entering into the computation of $\langle 0_+ | 0_- \rangle$, for that $\mathcal{G}[A_{\tau_i}]$ is invariant; by definition and construction it is independent of V. Each $\langle 0_+ | 0_- \rangle_i$ maintains a manifest gauge invariance under the transformations (1.5) for arbitrary V, and the sum of such $\langle 0_+ | 0_- \rangle_i$ over all τ_i is similarly invariant.

The construction of $\langle 0_+ | 0_- \rangle$ in our two-dimensional problem is now quite simple. From (1.1) and (3.5) we have

$$\langle \bar{\psi}\psi \rangle = -\left[1\Big/\int d^2x\right]\sum_i \Delta\tau\left[\int d[A]\mathcal{M}[A_{\tau_i}]\Big/\int d[A]\mathcal{D}[A_{\tau_i}]\right], \tag{3.6}$$

with

$$\mathcal{M}[A_{\tau_i}] = \delta(\mathcal{F}[A_{\tau_i}])\det M[A_{\tau_i}]\frac{\partial L_i[A_{\tau_i}]}{\partial m}\exp\left[-\tfrac{1}{4}\int \mathrm{tr}F_i^2 + \Delta\tau L_i A_{\tau_i}\right],$$

and

$$\mathcal{D}[A_{\tau_i}] = \delta(\mathcal{F}[A_{\tau_i}])\det M[A_{\tau_i}]\exp\left[-\tfrac{1}{4}\int \mathrm{tr}F_i^2 + \Delta\tau L_i[A_{\tau_i}]\right].$$

In quenched approximation one omits the $\exp(L_i[A_{\tau_i}])$ factors, observes that the "hard" $\int dA_{nh}$ components cancel away for each τ_i, and then passes to the limit of continuous τ_i. Again choosing an axial gauge, or the coordinate gauge, and with the same analysis which led to (1.11), one finds that one must perform the FI

$$\prod_{n_s;a}\int d\tilde{F}^a_{n_s}\exp\left[-\tfrac{1}{2}|\tilde{F}^a_{n_s}|^2 + \frac{i}{2}(\tilde{J}^a_{n_s}\tilde{F}^{*a}_{n_s} + \text{c.c.})\right]\Big/\int d\tilde{F}^a_{n_s}\exp(-\tfrac{1}{2}|\tilde{F}^a_{n_s}|^2), \tag{3.7}$$

where

$$\int d\tilde{F}^a_{n_s} = \int dR^a_{n_s}\int dI^a_{n_s},$$

$$\tilde{F}^a_n = R^a_n + iI^a_n, \quad \tilde{J}^a_n = \frac{g\tau}{\sqrt{LT}}Q^a e^{-ik_n\cdot x}$$

[that is, $\mathbf{J}(u)$ is here given by $g\tau\mathbf{Q}\delta(u-x)$]. For simplicity, the space-time volume LT has been kept finite.

The result of (3.7) is just $\exp(-\tfrac{1}{2}|J^a_{n_s}|^2)$, so that summing over all soft modes in the infinite-volume limit produces

$$\exp\left[-\frac{g^2\tau^2Q^2}{2}\int_0^{\mu_c^2(\tau)}\frac{d^2k}{(2\pi)^2}\right] = \exp[-g^2\tau^2Q^2\mu_c^2(\tau)/8\pi], \tag{3.8}$$

of the same form as the corresponding step of method 1. The only difference is an additional factor of 2 in the exponent of (3.8), so that the product gc appears to be larger than that of method 1 by a factor of $\sqrt{2}$. Again, we find the result of a manifestly gauge-invariant estimation to be just a trivial rescaling of the result of method 1.

IV. SUMMARY

We have shown how the estimation of $\langle \bar{\psi}\psi \rangle$ in quenched, IR approximation of Ref. 1 is related to two other, gauge-invariant methods of calculation. In each case the invariant computations are just a multiplicative factor away from the result of method 1; and since the IR construction cannot specify the constant c (other than the requirement that it be of order unity), these are all equivalent estimates. In the Appendix we mention simple bounds on the chiral limit of these $\langle \bar{\psi}\psi \rangle$, for all N, using the form (1.14), which are then valid for the gauge-invariant methods as well.

In this paper we have found two different ways of performing IR approximations for $\langle \bar{\psi}\psi \rangle$ in a gauge-invariant manner, and these two approaches are quite different from each other and from that of Ref. 1. In the latter case, faced with the difficulty of assuring gauge invariance of $\mathrm{tr}\{F^2\}$ and of $L[F_S\{F\}]$ simultaneously, the simplest choice was made for F_S, so that $L[F_S\{F\}]$ was not invariant under the transformation which left $\mathrm{tr}\{F^2\}$ invariant. This situation is rectified in method 2, using an asymmetric definition of $F_S\{F\}$ such that $L[F_S\{F\}]$ is invariant under the same unitary transformation as is

$\mathrm{tr}\{F^2\}$; but the price one pays for that manifest invariance is the necessity of redefining the FI expressing $\langle 0_+ | 0_- \rangle$, in order to avoid an infinite phase factor. Method 3, on the other hand, arranges the soft/hard separation so that $F_S = F_S(\tau)$, and all the F dependence of each τ-dependent FI is expressed in terms of the same $F_S(\tau)$; each FI is manifestly invariant, as is their sum over all proper-time values, under the full gauge transformations (1.5). In contrast with method 2, the resulting is simple and finite and yields a result which differs from that of Ref. 1 by a constant factor of $\sqrt{2}$.

There are undoubtedly other, perhaps simpler ways of introducing a gauge-invariant IR approximation, which could be useful for other physical problems. It is interesting to see just how little difference the preservation of manifest gauge invariance makes. This is undoubtedly due to the situation, illustrated by an independent argument in Ref. 1, that the calculation performed there really is gauge invariant, if not manifestly so, with its noninvariant pieces canceling away (in a manner made possible by translational invariance). Even so, it is remarkable to see how little difference exists between the results of all of these computations. Such questions will be of some practical concern, as the IR approach to SC turns to the continuum estimation of physical effects more complicated than that of $\langle \bar{\psi}\psi \rangle$.

Note added in proof. Instead of the asymmetric scheme of method 2 where gauge invariance under local transformations of the fields F is maintained, or that of method 3 where invariance is preserved under local transformations of the F_s, for realistic problems of QCD_4 there should be another way, in which quarks do not play an essential role. One may ask the question: in the absence of a quark mass, where can one find the scale parameter with respect to which an IR approximation may be defined? One possible answer appears to be dimensional transmutation, in the context of a field formalism,[5] and work on this approach is under way.

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APPENDIX

The result of Ref. 1 for the quenched, IR $\langle \bar{\psi}\psi \rangle$ in the chiral limit was expressed in (1.14) as proportional to the factor

$$\left[\sqrt{2} \frac{\Gamma\left[\frac{L+1}{2}\right]}{\Gamma\left[\frac{L}{2}\right]} \right] \sum_l \langle \, | \xi_l(\hat{\mathbf{z}}) | \, \rangle \ , \tag{A1}$$

where the N real eigenvalues $\xi_l(\hat{\mathbf{z}})$ are obtained from the diagonalization of $(\boldsymbol{\lambda}\cdot\hat{\mathbf{z}})$, with $\hat{\mathbf{z}}$ a unit vector in the $(L = N^2 - 1)$-dimensional space whose solid angle is Ω_L:

$$\langle \, | \xi_l(\hat{\mathbf{z}}) | \, \rangle = \frac{1}{\Omega_L} \int d\Omega_L \, | \xi_l(\hat{\mathbf{z}}) | \ . \tag{A2}$$

Because

$$\mathrm{tr}\{(\boldsymbol{\lambda}\cdot\hat{\mathbf{z}})^2\} = 2 = \sum_l \xi_l^2(\hat{\mathbf{z}})$$

and

$$\left[\sum_l | \xi_l | \right]^2 \geq \sum_l \xi_l^2 \ ,$$

one has the immediate lower bound

$$\sum_l \langle \, | \xi_l(\hat{\mathbf{z}}) | \, \rangle \geq \sqrt{2} \ . \tag{A3}$$

An upper bound is easily found, assuming that the solid angle average $\langle \, | \xi_l(\hat{\mathbf{z}}) | \, \rangle$ is independent of l, an expectation that is certainly in agreement with the form of the Mehta-Dyson method.[1] One uses the inequality, for any l,

$$\langle \, | \xi_l |^2 \rangle \geq \langle \, | \xi_l | \, \rangle^2 \ ,$$

so that, summing over all l,

$$2 \geq \sum_l \langle \, | \xi_l | \, \rangle^2 = N \langle \, | \xi_1 | \, \rangle^2 \ .$$

It then follows that $\langle \, | \xi_1 | \, \rangle \leq \sqrt{2/N}$, or that

$$\sum_l \langle \, | \xi_l | \, \rangle \leq \sqrt{2N} \ . \tag{A4}$$

Substituting (A3) and (A4) into (A1), one finds, to within an unimportant multiplicative constant, that

$$gN \leq (-\langle \bar{\psi}\psi \rangle) \leq gN^{3/2} \ . \tag{A5}$$

The explicit calculations of Ref. 1 show that the actual result tends, in the large-N limit, to the upper bound of (A5).

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