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Chiral-symmetry breaking in continuum two-dimensional QCD by an infrared method

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Estimates of $\langle \bar{\psi}\psi \rangle$ in the quenched approximation for two-dimensional QCD (QCD_2) are obtained, based upon a continuum, infrared approximation previously developed for QED_2. Nontrivial gauge-invariant extensions are devised for general SU(N), and computations are carried out for finite $N(=2,3)$ and in the large-N limit. Specifically non-Abelian structure appears for finite N, while the large-N limit displays an "effective Abelian" simplification. A nonzero value of $\langle \bar{\psi}\psi \rangle$ is found in the chiral limit for all N; in the limit $N \to \infty$, $\langle \bar{\psi}\psi \rangle \sim -gN^{3/2}$, independently of the value g. (This generalized a previous result of Zhitnitsky, who found a similar combination in the planar limit, with $gN^{1/2}$ fixed.)

I. INTRODUCTION

Introduced by 't Hooft[1] in 1974, two-dimensional QCD (QCD_2) has been extensively studied as a possible source of information about non-Abelian dynamics. In spite of the considerable simplifications found in two dimensions, there is some belief[2] that its dynamical structure is not completely at variance with that of the confining regime of QCD_4. The bosonization of QCD_2 has recently been achieved both by operator[3] and path-integral[4] methods, while there exist several compelling reasons[5,6] to study massless fermionic theories as limits of massive ones; this certainly is also of interest in studies of numerical simulations.

In this paper we consider the possibility of chiral-symmetry breaking (CSB) in QCD_2, by calculating the order parameter $\langle \bar{\psi}\psi \rangle$ in the chiral limit, as the quark mass approaches zero. This limit is relevant to the low-energy behavior of the theory, and will be approached by infrared (IR) techniques previously used in the Abelian case[7,8] and suitably extended to the non-Abelian one. What is evaluated is the sum of all gluons exchanged across a closed quark loop, with the (Euclidean) momenta of the gluons effectively limited to a continuous spectrum of values less than or on the order of the quark mass m. The analysis is carried out in the quenched approximation, in a gauge-invariant way which is, however, not manifestly invariant; but it is separately justified, using an argument that depends on translational invariance. In the following paper we show the equivalence of the calculations presented here with the results of two different, manifestly gauge-invariant formulations of the IR approximation. In this paper we follow a far simpler and more intuitive route, which keeps the formalism to a minimum while yielding specific, numerical answers.

As m vanishes we find that $\langle \bar{\psi}\psi \rangle$ develops a finite, nonzero, N-dependent value, indicative of explicit chiral-symmetry breaking, and that this effect persists even in the limit of large N, with no restriction on the coupling constant g. Our work complements a recent calculation of Zhitnitsky,[9] who found in the planar limit that $\langle \bar{\psi}\psi \rangle_{N\to\infty} = N(g^2N/12\pi)^{1/2}$ with g^2N fixed.

One of the aims of this paper and the one to follow is to illustrate what we feel is the proper way of performing an approximate calculation for any physical, gauge-invariant quantity Q. One starts with a formal but explicitly gauge-invariant representation for Q, and approximates this in some desired and physically motivated way, all the while retaining invariance of Q under the exact gauge group. Then, to perform the calculation, one chooses a gauge, but once that choice of gauge has been made no further approximations are permitted. For us, in this paper, $Q = \langle \bar{\psi}\psi \rangle$, and the physical motivation is the expectation that this order parameter is mostly dependent on low-energy effects. Because g and m have the same dimensions, the ratio g/m is the effective, dimensionless coupling of the theory, so that the chiral and strong-coupling (SC) limits are the same. Indeed, one has all the weight of decades of nonperturbative, eikonal calculations to suggest that SC physics may be correctly described in the continuum when virtual photon/gluon momenta are suitably restricted to be suitably small. But however one may view our motivation (see below), the essential gauge invariance of our computation will be made clear, qualitatively in this paper and explicitly in the one which follows.

Because the basic techniques of the IR method have been discussed in Refs. 7 and 8, we here present in detail only those parts of the analysis which are new and specifically relevant to the gauge group SU(N). It will, however, be useful to most readers to include a brief, qualitative and physical description of the motivation behind the techniques which define the IR method. Our work represents an extension of the exact solutions for the Green's functional $G_c[A] = S_c(1 - ie\gamma\cdot AS_c)^{-1}$ and the closed-fermion-loop functional $L[A] = \mathrm{Tr}\ln(1 - ie\gamma\cdot AS_c)$ which were found by Schwinger in 1951 (Ref. 10), for the special situation where the fields $F_{\mu\nu}$ are constants, independent of space-time coordinates. (Solutions were also found for laser-type, single-frequency fields of arbitrary strength; but those are of no interest here.) Schwinger's solution—a rare example of an exact, SC solution in quantum field theory—has been the starting point of many investigations, e.g., those trying to estimate lepton pair production in intense electric fields.[11]

Ever since the exact, formal, functional solutions for the generating functionals of local quantum field theories were first obtained (by Schwinger, Symanzik, Fradkin; and in the form of path integrals by Feynman), there has been a continuing search for a nonperturbative way of approximating the intricately coupled n-point functions which define any full theory. Applications to meson and nucleon problems, and more recently to questions of quark and gluon confinement have made the invention of workable SC techniques a paramount question. The advent of modern computer technology has provided the means for large-scale computer programs, such as those of lattice gauge theory; and while their results have occasionally been impressive, one continues to wonder if there might not be another, analytic way to obtain estimates of SC quantities, using pencil and paper, directly in the continuum.

The IR method is one recent attempt to do just that. It represents the first step in a systematic approximation procedure which invokes a "perturbation theory" quite different from that of ordinary Feynman graphs, a procedure that is designed to emphasize the coherent, or large-scale aspects of any solution to a SC problem. The method has generalizations to all topics in mathematical physics which may be reached by a Green's-function approach, but such extensions will not be discussed here.

The simplest idea of what one intends to do can be most easily conveyed by considering a function of a single variable, call it $f(t)$, which represents the exact, SC solution to some nonlinear differential equation (DE). Suppose that $f(t)$ has the form of a slowly varying function upon which is superimposed some rapidly varying t dependence, with both the slowly and rapidly varying dependence strongly dependent on the nonlinearity of the DE; one can then expect that a perturbation approximation for the nonlinear part of the DE would generate results completely different from the exact solution.[12] How, then, is one to obtain an analytic, nonmachine sequence of approximations to $f(t)$?

The answer given by the IR method is to generate a systematic set of approximations such that in the simplest, or zeroth approximation the high-frequency fluctuations of $f(t)$ are suppressed, and only in subsequent "higher-order" corrections does one attempt to reproduce the "fine structure" of the exact $f(t)$ (Ref. 13). In the first paper of Ref. 12, dealing with SC approximations to ordered exponentials, this has been accomplished in a somewhat more direct manner than those which follow from the methods of this, and previous papers, in field theory. But the basic idea is the same: if the essential, qualitative physics of a particular problem is represented by relatively slowly varying dependence, rather than by rapid fluctuations, the IR method should be a sensible method of approximation.

The n-point functions of quantum field theory may, in principle, be calculated by a two-step process: one first builds a representation for the various $G_c[A]$ and $L[A]$ entering into the problem; and one then calculates (typically Gaussian) fluctuations over combinations of these quantities, as appropriate to the n-point function under consideration. The first step has traditionally been impossible, except for a perturbative development (in which case the second step is always trivial), and except in certain model approximations.

Among the latter it is useful to focus attention on those compactly described by the word "eikonal," which refers to a collection of approximations to scattering and production processes that may be constructed, from first principles, under the restriction of sufficiently small momentum transfer q to the scattering particles. In any Abelian theory, e.g., one may imagine the scattering of a pair of fermions by the exchange of an infinite number of virtual bosons, each of some virtual momentum k_i, and replete with closed fermion loops inserted in all possible ways among the virtual bosons; then, the amplitude for this scattering process takes the form of an eikonal approximation, with the eikonal function given as the sum over all cross-channel, connected amplitudes.[14] (Graphs of high complexity can be included, such as the "towers" of Cheng and Wu,[15] which originally pointed out the possibility of rising total cross sections.)

More precisely, if p denotes the magnitude of the c.m. momenta of the incident particles, the eikonal model follows under the restriction $q \ll p$. The reason for this is that typical values of k_i are of the order of q; and hence the restriction $q \ll p$ means that the virtual boson momenta are restricted to "soft" values, $k_i \ll p$. All of the relativistic eikonal models of the past few decades may be thought of as IR approximations in which virtual momenta are kept smaller than the asymptotic momenta of the scattering particles. And since the resulting eikonal amplitude contains all powers of the coupling constant, it may be considered as a candidate for a strong-coupling approximation, a point of view which is reinforced by the more than qualitative success of such models in atomic, nuclear, and high-energy physics.

Here, then, is the first hint of how to construct more general, analytic, continuum, SC approximations in field theory: the leading behavior in a SC limit of some function, or symmetry-breaking parameter, or quantity strongly dependent on long-distance configuration-space effects may be obtained by extracting the low-frequency, or IR, part of the relevant, virtual processes. SC by IR extraction is the basic idea of the IR method. Tests of this idea were first carried out[16] in two relatively simple attempts to reproduce the behavior seen in machine calculations of renormalization-group β functions in the large coupling limit; and the results were sufficiently interesting to warrant the application of the IR method to the SC problem of symmetry breaking in QED_2, Refs. 7 and 8, and here, in QCD_2, where one must face the complexities of gauge invariance in a non-Abelian theory.

But what is to be done when the process under consideration, such as the present one, does not contain asymptotic momenta, and the conventional eikonal construction cannot be used? For those situations, one must arrange for the upper cutoff of the virtual momenta to be provided in a natural way, and in particular, in such a way that allows the possibility of systematic corrections. (In the $\langle \bar{\psi}\psi \rangle$ calculations, that scale is effectively provided by the fermion-loop mass; but in other situations, such as an application of these Green's-function techniques to realistic fluids,[17] that upper cutoff can itself be a quantity to be determined dynamically.) In every systematic approximation scheme one must identify an appropriate small parameter with respect to which the sequence of approximations can be developed; and while it may be introduced in an intuitive way, it must enter as part of a formalism, so that corrections to the zeroth approximation can be defined. For this purpose, the proper-time method of Schwinger, extended by the lovely representations of Fradkin,[18] provide the natural framework.

Detailed explanations of these representations may be found in Refs. 7 and 8. Here, we comment only on the motivation for the use of (2.3) below, and on the form of the results which follow. In the situation where the fields $F_{\mu\nu}(z)$ of (2.3) are constants, $L[A]$ reduces immediately (in QED) to the form originally given by Schwinger. In our "lowest-order" approximation, where a continuous range of suitably low frequencies are retained, thereby replacing each component of $F_{\mu\nu}(z)$ by the corresponding form of (2.14), the result is a simple generalization of Schwinger's QED form in the sense that the constant field entering into his expression is here replaced by a field having "soft" frequency components; in QCD, the results are somewhat more complicated, but in fact quite similar. The upper cutoff to the virtual momenta allowed in this initial approximation is, by internal consistency, required to be on the order of the loop mass; larger momenta appear in the systematic development of "higher-order" corrections.

The details of these topics, with particular attention paid to the requirement of gauge invariance under non-Abelian transformations, appear in the text of this and the following paper. In QED, these techniques yielded an $L[A]$ in IR approximation which was really no surprise, for that result of necessity had to reduce to Schwinger's constant field solution when the field $F_{\mu\nu}$ was that most IR of all functions, a constant. In fact, the IR approximation to $L[A]$ for QED_4 can simply be obtained by writing Schwinger's constant-field solution and replacing his $F_{\mu\nu}$ by one containing only soft components. Precisely this approximation to $L[A]$ has just been used[19] to discuss the new e^+e^- resonances found in the collision of heavy ions of large electric charge, in a SC application to QED_4.

The arrangement of these remarks is as follows. In the next section we set up the calculation and discuss as much gauge invariance as is necessary for the numerical computations. In Sec. III the detailed calculation of $\langle\bar{\psi}\psi\rangle_{\rm IR}$ is discussed for $N_c=2$ and 3, with all technical details confined to the Appendixes. In the next section a discussion of the limit $N_c\to\infty$ is given, with emphasis on the appearance of specifically Abelian and non-Abelian structure. Understanding the latter, in particular the nature of the quantum fluctuations saturating the order parameter, was one of the aims of this paper. Section V contains a brief summary and some relevant comments, while several appropriate appendixes complete this paper. In all that follows we have restricted ourselves to the case of only one quark flavor, since the large-N_c limit to which we shall pay particular attention is flavor blind; then, with $N_f=1$, N_c will be written as N, everywhere.

II. GAUGE INVARIANCE AND THE INFRARED APPROXIMATION

We remind the reader that the IR approximation, defined in Refs. 7 and 8, is supposed to produce an estimate of the leading behavior in the SC limit of any physical quantity appropriately sensitive to low-energy effects. Corrections to this IR estimate may be performed in a systematic way, but will not be discussed here. The exact definition of the order parameter (a Euclidean metric will be used almost everywhere)

$$\langle\bar{\psi}\psi\rangle=-\frac{\partial}{\partial m}\ln\langle 0_+\,|\,0_-\rangle\Big/\int d^2x_E\ , \tag{2.1}$$

may be rewritten in quenched approximation as

$$\langle\bar{\psi}\psi\rangle^Q=-\frac{\partial Q_1}{\partial m}\Big/\int d^2x_E\ , \tag{2.2}$$

where Q_1 denotes a single quark loop with all possible gluons exchanged across the loop:

$$Q_1=e^{\mathcal{D}_A}L[A]\,|_{A=0}\ ,\quad \mathcal{D}_A=-\frac{i}{2}\int\frac{\delta}{\delta A^a_\mu}D^{ab}_{\mu\nu}\frac{\delta}{\delta A^b_\nu}\ ,$$

with $L(A)=\mathrm{Tr}\ln(1-ig\gamma\cdot AS_c)$. Here $\mathcal{D}_A$ is a linkage operator given in terms of the gluon propagator of a particular gauge, chosen when it is time to actually compute; we use this Abelian-type shorthand to represent the effect of functional integration over all appropriate variables, once a specific gauge has been chosen. The more customary but cumbersome functional integral notation will be used as needed, when one discusses strict gauge invariance, in the paper to follow.

These forms follow from the original Lagrangian density

$$\mathcal{L}=-\tfrac{1}{4}(F^a_{\mu\nu})^2-\bar{\psi}[m+\gamma_\mu(\partial_\mu-igA^a_\mu\lambda^a)]\psi\ .$$

We work in the fundamental representation of SU(N), where the Gell-Mann matrices satisfy the relations

$$[\lambda^a,\lambda^b]=2if_{abc}\lambda^c\ ,\quad \{\lambda^a,\lambda^b\}=\frac{4}{N}\delta_{ab}+2d_{abc}\lambda^c\ ,$$

with $\mathrm{tr}[\lambda^a]=0$, $\mathrm{tr}[\lambda^a\lambda^b]=2\delta_{ab}$.

We will use the approximation of quenching, neglecting the fermionic determinant $L(A)$ everywhere except as written in (2.2). In recent papers[8,20] it was estimated that corrections to the order parameter due to other closed fermion loops in QED_2 were on the order of 25–30 %; in non-Abelian situations quenching should be even better, since one expects fermion loops to be suppressed by a factor $1/N$. In effect, the quenched approximation in our IR method appears to correspond to a simple, finite rescaling of the coupling constant, although, because of the nonsmooth limit as m tends to zero, other methods of calculation[21,22] can display different types of divergences in quenched approximation.

As in Refs. 7 and 8 we shall use the Fradkin representation to write an exact representation for the QCD closed-loop functional $L(A)$, in terms of Gaussian fluctuations of a proper-time-dependent vector field $\phi_\mu(s)$:

$$L[A]=-\frac{1}{2}\int_0^\infty \frac{ds}{s}e^{-ism^2}\int d^2x\, N(s)\int d[\phi]\exp\left[\frac{i}{4}\int_0^s ds'\phi^2(s')\right]\delta\left[\int_0^s ds'\phi(s')\right]\mathrm{Tr}[U(s)-1]\ , \tag{2.3}$$

where

$$U(s)=\left[\exp\left\{-ig\int_0^s ds'\left[\phi_\mu(s')A_\mu\left[x-\int_0^{s'}\phi\right]-i\sigma_{\mu\nu}F_{\mu\nu}\left[x-\int_0^{s'}\phi\right]\right]\right\}\right]_+\ , \tag{2.4}$$

and

$$A_\mu(z)=\lambda^a A_\mu^a(z)\ ,\quad \sigma_{\mu\nu}=\tfrac{1}{4}[\gamma_\mu,\gamma_\nu],\quad F_{\mu\nu}(z)=\lambda^a F_{\mu\nu}^a(z)=\lambda^a(\partial_\mu A_\nu^a-\partial_\nu A_\mu^a+gf_{abc}A_\mu^b A_\nu^c)(z)\ ,$$

and the normalization constant $N(s)$ of (2.3) is given by

$$N(s)^{-1}=\int d[\phi]\exp\left[\frac{i}{4}\int_0^s ds'\phi^2(s')\right]\ .$$

Under a subsequent continuation, $s\rightarrow -i\tau$, and τ will then be called the proper time. [It really has the dimension of $(\mathrm{time})^2$.] The trace operation Tr includes a summation over color degrees of freedom, which will be denoted by tr.

Before considering any approximation to the $L(A)$ of (2.3) we first show, explicitly, that $\mathrm{tr}U(s)$ is invariant under the general, position-dependent gauge transformations generated by the unitary operator $V(x)=\exp[i\lambda_a v_a(x)]$:

$$A_\mu(z)\rightarrow A'_\mu(z)=V^\dagger(z)\left[A_\mu(z)+\frac{i}{g}\partial_\mu\right]V(z)\ ,\quad F_{\mu\nu}(z)\rightarrow F'_{\mu\nu}(z)=V^\dagger(z)F_{\mu\nu}(z)V(z)\ .$$

Under such a transformation, $U(s)\rightarrow U'(s)$, where

$$U'(s)=\left[\exp\left\{-ig\int_0^s ds'\left[\phi_\mu(s')V^\dagger\left[x-\int_0^{s'}\phi\right]\left[A_\mu+\frac{i}{g}\partial_\mu\right]V-i\sigma_{\mu\nu}V^\dagger\left[x-\int_0^{s'}\phi\right]F_{\mu\nu}V\right]\right\}\right]_+\ , \tag{2.5}$$

and where, for compactness, in writing (2.5) and similar expressions, dependence on the common variable $(x-\int_0^{s'}\phi)$ is exhibited in only the first member of the product. To understand the relation between U and U', it is useful to consider the differential equation for U':

$$\frac{\partial U'}{\partial s}=-ig\left[\phi_\mu(s)V^\dagger\left[x-\int_0^s\phi\right]\left[A_\mu+\frac{i}{g}\partial_\mu\right]V-i\sigma_{\mu\nu}V^\dagger\left[x-\int_0^s\phi\right]F_{\mu\nu}V\right]U'\ . \tag{2.6}$$

Setting $U'(s)=V^\dagger(s)W(s)$, substituting into (2.6), and comparing with the corresponding equation for U, yields, with $V(s)\equiv V(x-\int_0^s\phi)$,

$$\frac{\partial W}{\partial s}=-ig\left[\phi_\mu(s)A_\mu\left[x-\int_0^s\phi\right]-i\sigma_{\mu\nu}F_{\mu\nu}\left[x-\int_0^s\phi\right]\right]W\ , \tag{2.7}$$

where we have made the replacements

$$\varphi_\mu(s)V^\dagger(s)\partial_\mu V=-V^\dagger(s)\frac{\partial V}{\partial s}$$

and

$$\frac{\partial V^\dagger}{\partial s}V=-V^\dagger\frac{\partial V}{\partial s}\ .$$

If one takes into account the initial condition $W(0)=V(0)$, in comparison with (2.4) one can write the solution of (2.7) as $W(s)=U(s)V(0)$; so that, finally,

$$U'(s)=V^\dagger(s)U(s)V(0)\ . \tag{2.8}$$

But $V^\dagger(s)=V^\dagger(0)$ if the (closed-loop) condition $\int_0^s\phi_\mu=0$ is satisfied, as required by the representation (2.3). Hence $\mathrm{tr}U=\mathrm{tr}U'$, and (2.3) with (2.4), has been explicitly shown to be gauge invariant. This is not a surprise, of course, for the Fradkin representation is exact; but an understanding of just how gauge invariance works before any approximations are made is a useful preliminary step for understanding the gauge structure of the approximations we are about to perform.

In Refs. 7 and 8 the IR method for QED, the lowest-order approximation of a systematic expansion in powers of a "hard" (i.e., nonsoft) interaction, was introduced by the following sequence of steps.

(i) The corresponding $L(A)$ was rewritten (by means of an elementary integration by parts on an s' variable) so that its complete dependence upon $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ was made explicit. In (Euclidean) momentum space, every component of $F_{\mu\nu}$ was rewritten in the form

$$\tilde{F}(k)\equiv e^{-k^2/\mu_c^2}\tilde{F}(k)+(1-e^{-k^2/\mu_c^2})\tilde{F}(u)$$
$$\equiv\tilde{F}_S(k)+\tilde{F}_H(k)\ ,$$

where μ_c is a cutoff parameter specifying the upper limit of the "soft" momenta included, and will be specified in a manner that permits all steps of an estimation to be performed. An expansion is developed in powers of F_H, of which only the lowest-order terms are retained; effectively, $\tilde{F}(k)$ is replaced everywhere by $\tilde{F}_S=\exp(-k^2/\mu_c^2)\tilde{F}(k)$.

(ii) Even with this step, the forms entering into $L(A)$ are too complicated to permit the Gaussian fluctuations over ϕ_μ to be performed; but with a judicious choice of μ_c one could argue that a multipole expansion could be made of the phase factor $\exp(-ik\int_0^{s'}\phi\,ds'')$, in the integrand of the Fourier representation of $F_S(x-\int_0^{s'}\phi\,ds'')$, with the leading term given by replacing this factor by unity. [The essential reason is that ϕ scales as $1/\sqrt{\tau}$, so that $O(k\int_0^{s'}ds''\phi)\lesssim\mu_c\tau/\sqrt{\tau}$. Hence if μ_c is chosen as $c/\sqrt{\tau}$, where $c\lesssim 1$, the multipole expansion is sensible. Because τ itself scales as m^{-2}, this effectively corresponds to $\mu_c\sim m$; and indeed, one can simply choose $\mu_c=cm$. There is one reason for preferring the choice $\mu_c=c/\sqrt{\tau}$, for the functional averaging over the photon fields then produces as an internal step an exponentiated result whose form of scaling dependence on τ is exact; and this is later reflected in the fact that our results, both quenched and nonquenched for $\langle\bar{\psi}\psi\rangle$, contain the correct form of phase-space dependence, a factor proportional to $\pi^{-3/2}$.]

(iii) After the replacement $\exp(-ik\int_0^{s'}\phi\,ds'')\to 1$, one is left with an $F_S(x)$ dependent on τ, via the μ_c dependence; but one which is independent of ϕ. Hence the Gaussian functional integration over ϕ can be carried through and a form obtained for $L_{\rm IR}(F)$ in terms of an integration over the proper time. The phrase "IR approximation" then denoted the result of keeping the leading terms of two expansions, the multipole expansion, and that obtained by the replacement of F by F_S; and corrections can be obtained in a systematic way.

One would like to proceed in much the same fashion for QCD, but there are two distinct and fundamental differences associated with its non-Abelian gauge transformations which must first be resolved. First, what is required here is invariance under local, configuration space operations; but the IR approximation begins with a local restriction in momentum space. How can the IR approximation, which is then nonlocal in configuration space, be compatible with invariance under local, configuration space transformations? Second, certain exponential factors of QED_2 now appear as ordered exponentials; but before the Gaussian fluctuations of the Fradkin representation can be computed one must have in hand the appropriate and *explicit* ϕ dependence.

We shall show in the following paper that the first point can be answered by embedding the simplified method of calculation used in this paper into two different, manifestly gauge-invariant formulations; the only difference between the three results for the order parameter will be a rescaling of the parameter c by an amount of ~ 1, an effect of no practical importance. The reason this rescaling is of no real importance is that the value of c must be specified by some method external to this computation. Rather like a constant of integration, or a subtraction constant in a dispersion relation, c should be chosen by comparison with a known result. For example, in the chiral limit of QED_2, one knows (from bosonization) the exact result: $\langle\bar{\psi}\psi\rangle|_{(m/g)\to 0}=-ge^{\gamma}/2\pi^{3/2}$, where γ is Euler's constant. In comparison, the quenched IR method yields $\langle\bar{\psi}\psi\rangle_{(m/g)\to 0}=-gc/4\pi^{3/2}$, with $c/2$ effectively replacing e^{γ}. One may then use the IR method to calculate other, c-dependent Green's functions, and the choice $c/2=e^{\gamma}$ should give a decent approximation to those functions. Of course the *exact* answer, of which the IR method is just the first approximation, is independent of c; but any finite correction will depend on c. In the present SU(N) case we can do the same thing, by choosing c (at least in the large-N, chiral limit) to agree with the only known answer, Zhitnitsky's, for $\langle\bar{\psi}\psi\rangle$. The essence of the IR method is that $c\sim 1$, so that our calculations are really only estimations; but even with this drawback, it is interesting to be able to obtain the form and the order of magnitude of a SC quantity directly in the continuum.

The simplified method of calculation used here, as defined by (2.14) and (2.15) below, makes the same division into soft and hard parts as used in the QED papers of Refs. 7 and 8. Although it is not manifestly gauge invariant, it is in fact invariant, as we show in the argument following (2.15). In the following paper, two different, manifestly gauge-invariant constructions of the IR method are outlined, and their results are shown to be equivalent to those of the simpler calculations of this paper.

The resolution of the second point, necessary for any approximate calculation, is fortunately a good deal simpler. In QED it was possible to perform a rearrangement of the factor

$$\exp\left[-ig\int_0^s ds'\phi_\mu(s')A_\mu\left[x-\int_0^{s'}\phi\right]\right]$$

of (2.4), replacing it by its exact equivalent

$$\exp\left[-ig\int_0^s ds'\phi_\mu(s')\int_0^{s'}ds''\phi_\nu(s'')\times\int_0^1 d\lambda\,\lambda F_{\mu\nu}\left[x-\lambda\int_0^{s'}\phi\right]\right]\ .$$

In so doing one neglects an exponential factor of form

$$\exp\left[-ig\int_0^s ds'\,\phi_\mu(s')A_\mu(x)\right]$$

because of the closed-loop restriction $\int_0^s ds'\phi=0$. In QCD, however, this factor lies inside an order exponential (ordered in s'), and terms proportional to $\int_0^s ds'\phi_\mu(s')$ cannot be discarded until the ordering process has been completed; it is this manipulation which, e.g., generates the $[A_\mu, A_\nu]$ contribution to the non-Abelian $F_{\mu\nu}$. One needs a simple way of extracting from the exact form (2.4) explicit dependence on the A_μ and/or $F_{\mu\nu}$; and for this one can use the following argument.

Rewrite the $U(s)$ of (2.4) as U_0^s (adapting a notation and a technique used long ago by Symanzik in his studies of generating functionals in quantum field theory), where s and 0 refer, respectively, to the upper and lower limits of the integral $\int ds'$. We omit, for the moment, the $\sigma \cdot F$ dependence of (2.4), which will simply be added into the final result. Calculate the variation of U_0^s with respect to a small change in coupling g,

$$\frac{\partial U_0^s}{\partial g} = -i\int_0^s ds' U_{s'}^s \phi_\mu(s') A_\mu\left[x - \int_0^{s'}\phi\right] U_0^{s'} , \quad (2.9)$$

assuming for simplicity that A does not depend on g. (A generalization of this construction for the case of arbitrary dependence of A and F on g is left as an exercise for the interested reader.) If so, F is linear in g; and subsequent gauge transformations should be effected by unitary operators $V(x)$ which are independent of g, in order to preserve the form of the g dependence of F.

Performing an integration by parts with respect to the s' variable allows one to rewrite (2.9) as

$$\frac{\partial U_0^s}{\partial g} = -iQ(g,s)U_0^s , \quad (2.10)$$

with

$$Q(g,s) = \int_0^s ds'\left\{\phi_\mu(s')A_\mu\left[x - \int_0^{s'}\phi\right] + ig\phi_\nu(s')\int_0^{s'} ds''\phi_\mu(s'')U_{s'}^s\left[A_\mu\left[x - \int_0^{s''}\phi\right], A_\nu\left[x - \int_0^{s'}\phi\right]\right]U_{s'}^{s\dagger}\right\} .$$

In the subsequent IR expansion we are going to retain only a quadratic exponential dependence on the ϕ, and the form that this approximation will take can be simply inferred by retaining only quadratic ϕ dependence in the $Q(g,s)$ of (2.10):

$$Q(g,s) \simeq \int_0^s ds'\phi_\mu(s')A_\mu(x) - \tfrac{1}{2}\Omega_{\mu\nu}(s)(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + 2gf_{abc}A_\mu^b A_\nu^c)\lambda^a ,$$

where

$$\Omega_{\mu\nu}(s) = \int_0^s ds'\phi_\mu(s')\int_0^{s'} ds''\phi_\nu(s'') .$$

(With ϕ_μ replaced by dx_μ/ds, $\Omega_{\mu\nu}$ is just the projection of the area of a Wilson loop onto the μ,ν plane.)

Integration of (2.10) is given in terms of an exponential ordered in the variable g', not s',

$$U_0^s(g) = \left[\exp\left[-i\int_0^g dg' Q(g',s)\right]\right]_{+(g)} , \quad (2.11)$$

and hence the condition $\int_0^s ds'\phi = 0$ can be freely implemented, leaving

$$U_0^s(g) = \left[\exp\left[-i\int_0^g dg'\frac{\partial}{\partial g'}[\Omega_{\mu\nu}(s)g'\lambda^a F_{\mu\nu}^a(g')]\right]\right]_{+(g)} , \quad (2.12)$$

where the $F_{\mu\nu}(g)$ written in (2.12) is the exact field strength, with its proper (linear) dependence on g. Because (2.12) is ordered in g' it is still a very complicated object; it is not the same thing as the ordinary exponential $\exp(-ig\Omega_{\mu\nu}\lambda^a F_{\mu\nu}^a)$. Nevertheless, gauge invariance of the trace of (2.12) is immediate for any unitary $V(s)$ (independent of g). That this is true is really no surprise, for the $L(A)$ was shown above to be rigorously invariant, independently of the size of ϕ; and this invariance will persist under any expansion, or regrouping in powers of ϕ.

Our gauge-invariant form for the IR approximation to the $\mathrm{tr}U(s)$ of (2.5) is then given by

$$\mathrm{tr}U(s) = \mathrm{tr}\left[\exp\left[-i\int_0^g dg'[\Omega_{\mu\nu}(s) - is\sigma_{\mu\nu}]\right] \times \frac{\partial}{\partial g'}[g'F_{\mu\nu}^a(g',x)\lambda^a]\right]_{+(g)} , \quad (2.13)$$

where we have included the simplest IR approximation to the $\sigma \cdot F$ factor of (2.4). The fact that (2.13) is a complicated object as it stands is quite irrelevant to the gauge invariance of our procedure. After we have resolved item (i) in a gauge-invariant way, we are, in two dimensions, finished with simplifications or modifications; when we choose a gauge within which to perform calculations, we shall make no further approximations.

We must now define just how the F components of (2.13) are to be replaced by their soft approximations. The immediate generalization of (i), written for each component of $F_{\mu\nu}^a$,

$$\tilde{F}_s^a(k) = e^{-k^2/\mu_c^2}\tilde{F}^a(k) , \quad F_s^a(x) = \int d^2z\, f(x-z)F^a(z) , \quad (2.14)$$

with

$$f(x-z) = \frac{\mu_c^2}{4\pi}e^{-(x-z)^2\mu_c^2/4} , \quad (2.15)$$

will be used here, for simplicity, even though this procedure is not manifestly gauge invariant. It generates a result in which all noninvariant pieces drop away, leaving a gauge-invariant quantity, as can be seen from the following argument.

The behavior in configuration space of F_S, as given by (2.14), depends on the relative size of the Fourier components k of F compared to μ_c. For distances $x<\mu_c^{-1}$, F_S is essentially given by an average of F over a sphere of radius $(\mu_c)^{-1}$; while for large x, that is, $x>\mu_c^{-1}$, it is essentially the same as that F which would be constructed by keeping only frequency components $k<\mu_c$. But since larger frequencies are not going to enter the fluctuation calculations, one may suppose that for such large x, F_S is equivalent to F. One can then imagine calculating quantum effects in a configuration-space volume of finite radius R from which a hole of radius $(\mu_c)^{-1}$ has been excluded. If, for such large x, F_S is equivalent to F, then the gauge invariance of (2.13) will be preserved for $R>x>(\mu_c)^{-1}$. Finally, using translational invariance, one passes to the limit $R\to\infty$ before any other parameter limit (such as $m\to 0$) is taken, and the results for $\langle\bar\psi\psi\rangle$ will be exactly the results obtained in this paper. By this qualitative argument, it is not that gauge invariance is manifestly preserved by the choice (2.14) as that the gauge-variant pieces of the amplitude generated by (2.14) are excluded from the final result.

An analogous fact is well known in perturbative QCD_4 where the planar graphs' leading contributions [the so-called leading-logarithmic approximation[23] (LLA)] can be shown to be gauge invariant.

III. FINITE-N CALCULATIONS

In this section we outline the salient points of our estimation of $\langle\bar\psi\psi\rangle$ in quenched approximation.

We adopt (2.14), where each $F^a_{\mu\nu}$ is replaced by its soft part, and write

$$\mathrm{tr}U(s)=\mathrm{tr}\left[\exp\left[-i\int_0^g dg'[\Omega_{\mu\nu}(s)-is\sigma_{\mu\nu}]\frac{\partial}{\partial g'}[g'F^s_{\mu\nu}(g',x)]\right]\right]_{+(g)} \tag{3.1}$$

and

$$F^s_{\mu\nu}(g',x)=\int d^2z\,f(x-z)F_{\mu\nu}(g',z)\ ,$$

with the function f specified in Eq. (2.15). The next step is the explicit evaluation of the Q_1 of (2.2),

$$Q_1^s=e^{\mathcal{D}_A}L^s[A]\,|_{A=0}\ ,$$

which involves the evaluation of

$$e^{\mathcal{D}_A}\mathrm{tr}U(s)\,|_{A=0}\ . \tag{3.2}$$

Indeed, the dependence of $U(s)$ upon the $F_{\mu\nu}$ fields only suggests that one pass to a field-strength formulation of the theory. After the recent work of Durand and Mendel[24] it is difficult to avoid using the coordinate gauge specified by the condition

$$x_\mu A_\mu(x)=0\ , \tag{3.3}$$

and in which one has a unique reconstruction of the potentials from the field without constraints on the latter in the sense of the Bianchi-indentity restrictions appearing in higher dimensions. One has, simply,

$$A_\mu(x)=\int_0^1\alpha\,d\alpha\,x_\nu F_{\mu\nu}(\alpha x)\ . \tag{3.4}$$

In addition this gauge enjoys the following properties, in common with axial gauges.

In two dimensions, $F_{\mu\nu}(g',x)=F_{\mu\nu}(x)$ independently of g', so that the g'-ordered exponential of (3.1) becomes an ordinary exponential. Taking advantage of these properties one can now write

$$e^{\mathcal{D}_A}\mathrm{tr}U(s)\,|_{A=0}=\exp\left[-\frac{i}{2}\int\frac{\delta}{\delta F^a}(\delta^{ab})\frac{\delta}{\delta F^b}\right]\times\mathrm{tr}(e^{-igM_{\mu\nu}F^s_{\mu\nu}})\Big|_{F=0}\ , \tag{3.5}$$

where we use

$$M_{\mu\nu}\equiv\Omega_{\mu\nu}-is\sigma_{\mu\nu}\ , \tag{3.6}$$

a two-by-two matrix in the external μ,ν space-time indices. We do not find any evidence of the lack of translational invariance using this gauge, as alluded to elsewhere.[25] Had we carried out the calculation in an axial gauge, e.g., $A_1(x)=0$ we would have found the same structure exactly:

$$e^{\mathcal{D}_A}\mathrm{tr}U(s)\,|_{A=0}=\exp\left[-\frac{i}{2}\int\frac{\delta}{\delta A_4^a}(\delta^{ab}\partial_1^{-2})\frac{\delta}{\delta A_+^b}\right]\times\mathrm{tr}(e^{-2iM_{14}\partial_1A_4^a(x)})\Big|_{A\to 0}\ , \tag{3.7}$$

with the above properties holding in that gauge as well. As demonstrated in Appendix C the expressions (3.5) and (3.7) are identical, and can be recast into an equivalent form independent of the spatial coordinate x; one has

$$e^{\mathcal{D}_A}\mathrm{tr}U(s)\,|_{A=0}=\sum_{\{+,-\}}\exp\left[-\frac{i}{2}\int\frac{\delta}{\delta F_a}(\delta^{ab})\frac{\delta}{\delta F_b}\right]\times\mathrm{tr}(e^{-i\sqrt{iq}\,\Lambda_\pm F})\Big|_{F=0}\ , \tag{3.8}$$

with

$$q=\int d^2z\,f^2(x-z)=\frac{i}{8\pi}\mu_c^2 \tag{3.9}$$

and

$$\Lambda_\pm=g(\Omega_{14}\pm s)\ .$$

The trace in the right-hand side (RHS) of Eq. (3.8) sums only upon color indices. It is the relatively simple structure of two dimensions and of the gauge choice which permits one to pass from the complicated object defined in (2.13) to the explicit forms of this section. In (3.8) we can observe that any spatial coordinate dependence has disappeared; this is an expression of translational invariance.

Even though the problem has been given a much simpler form, the computation of (3.8) is not at all trivial. In particular, a direct summation of its series expansion is hopeless as soon as $N>3$. But omitting inessential constants, for clarity, one can show (Appendix A) that (3.8) admits the following series expansion:

$$\exp\left[-\frac{i}{2}\int\frac{\delta}{\delta F^a}\frac{\delta}{\delta F^a}\right]e^{(F_c\lambda_c)}\Bigg|_{F=0}=\sum_{n=0}^{\infty}\frac{1}{2n!n!2^n}\sum_{\{b_1,\dots,b_{2n}\}}\sum_{P\in S_{2n}}\prod_{i=1}^{n}\{\lambda_{b_{2i-1}},\lambda_{b_{2i}}\}\delta^{P(b_{2i-1})P(b_{2i})},\tag{3.10}$$

in which the reader will have recognized the explicit form of Wick's theorem, where $\{b_1,b_2,\dots,b_{2n}\}$ is a $2n$-plet of indices chosen among the (N^2-1) possibilities.

S_{2n} is the group of the permutations (the P's) of $2n$ elements. This structure suggests passing, by the aid of a path-integral treatment, to the so-called Mehta-Dyson representation.[26] The following set of remarks is useful in order to make this application possible.

(i) We first change to the functional integral version of (3.8); that is,

$$\exp\left[-\frac{i}{2}\int\left[\frac{\delta}{\delta F}\right]^2\right]\mathrm{tr}(e^{(F^c\lambda^c)})\Bigg|_{F=0}=\left[\int d[G]\exp\left[\frac{i}{2}\int G^2\right]\right]^{-1}\times\int d[G]\exp\left[\frac{i}{2}\int G^2\right]\mathrm{tr}(e^{(\lambda G)}),\tag{3.11}$$

where G is a (N^2-1)-component field.

(ii) For any N, a Gell-Mann-type basis of SU(N) can be constructed explicitly, with the normalization

$$\mathrm{tr}[\lambda^a\lambda^b]=2\delta^{ab},\tag{3.12}$$

and one therefore has, for SU(N), $2G^2=\mathrm{Tr}(G^a\lambda^a)^2$.

(iii) We pass from the measure $d(G)$ to the measure $d(H)(\mathrm{tr}H)$, where H is a $N\times N$ Hermitian matrix, and

$$d[H]=d\rho_1\cdots d\rho_N\times\prod_{1\le i<j\le N}(\rho_i-\rho_j)^2dp_1\cdots dp_{N^2-N}f(p).\tag{3.13}$$

The ρ_i's are the N eigenvalues of H, whereas the P_n are $N(N-1)$ angular variables and $f(p)$ is an unspecified function of these variables whose integral cancels out in the normalization.

Now the exploitation of the Mehta-Dyson method is lengthy but straightforward (Appendix B), and leads to the following result:

$$e^{\mathcal{D}_a}\mathrm{tr}U(s)\,|_{A=0}=\exp\left[-\frac{iq}{4N}(\Lambda_\pm)^2\right]\times\int_{-\infty}^{+\infty}d\theta\, e^{\theta\sqrt{iq}(\Lambda_\pm)}\sigma_N(\theta)\tag{3.14}$$

with

$$\sigma_N(\theta)=\sum_{n=0}^{N-1}\varphi_n^2(\theta),\tag{3.15}$$

where

$$\varphi_n(\theta)=(2^n n!\sqrt{\pi})^{-1/2}e^{-\theta^2/2}H_n(\theta)\tag{3.16}$$

are the normalized oscillator wave functions, and $H_n(\theta)$ the nth Hermite polynomial.

The series (3.15) entails few terms for $N=2,3$ and hence can be used directly. [This allows one to check that (3.14) is correct (Appendix B).]

But alternatively, by making use of the relations

$$\int_{+\infty}^{+\infty}dx\, e^{-(x-y)^2}H_m(x)H_n(x)=2^n\sqrt{\pi}m!y^{n-m}\times L_n^{n-m}(-2y^2)\tag{3.17}$$

and

$$\sum_{m=0}^{n}L_m^\alpha(x)=L_n^{\alpha+1}(x),\tag{3.18}$$

relating Hermite and Laguerre polynomials, a general expression can be worked out for $\langle\bar\psi\psi\rangle$ valid for any N (Appendix B),

$$\langle\bar\psi\psi\rangle=\frac{m}{\pi}L_{N-1}^1\left[\frac{2N}{1-N}\frac{\partial}{\partial\zeta}\right]\times\left\{\ln\left[\frac{1}{\rho}\left[\frac{N}{\zeta(N-1)}\right]^{1/2}\right]-\psi\left[\frac{1}{\rho}\left[\frac{N}{\zeta(N-1)}\right]^{1/2}\right]-\frac{\rho}{2}\left[\frac{\zeta(N-1)}{N}\right]^{1/2}\right\}\Bigg|_{\rho=1},\tag{3.19}$$

where ψ is the Euler ψ function and ρ is an effective coupling constant previously encountered in similar QED_2 studies:

$$\rho=\frac{gc}{m\sqrt{8\pi}}.\tag{3.20}$$

With

$$L^1_{N-1}(x)=\sum_0^{N-1}(-1)^m\frac{N!}{(N-m-1)!(m+1)!}\frac{x^m}{m!}\ , \tag{3.21}$$

the relation (3.19) can be used to calculate the chiral limit of $\langle\bar\psi\psi\rangle$ for any N. There $\rho\to\infty$, and the quantity in curly brackets in (3.19), is first replaced by the expression

$$\ln\left[\frac{1}{\rho}\left[\frac{N}{\zeta(N-1)}\right]^{1/2}\right]+\gamma-\frac{\rho}{2}\left[\frac{\zeta(N-1)}{N}\right]^{1/2} . \tag{3.22}$$

Acting upon (3.22) by

$$L^1_{N-1}\left[\frac{2N}{1-N}\frac{\partial}{\partial\zeta}\right]$$

and setting $\zeta=1$ at the end of the computation, one finds

$$\lim_{m\to 0}\langle\bar\psi\psi\rangle=-\frac{1}{2}\frac{gc}{(2\pi)^{3/2}}\Sigma(N)\ , \tag{3.23}$$

with $\Sigma(N)$ the series

$$\Sigma(N)=\left[1-\frac{1}{N}\right]^{1/2}\sum_{p=0}^{N-1}(-1)^p\frac{N!(2p)!}{(N-p-1)!(p+1)!(p!)^2}\times\left[\frac{N}{N-1}\right]^p\frac{1}{2^p}\frac{1}{1-2p} . \tag{3.24}$$

These expressions (3.23) and (3.24) are valid for any N as long as the chiral limit $m=0$ is taken first.

For SU(2) and SU(3) the series for $\Sigma(N)$ is readily evaluated:

$$\begin{aligned}N=2:&\ \ \langle\bar\psi\psi\rangle=-\frac{gc}{2\pi^{3/2}}\ ,\\ N=3:&\ \ \langle\bar\psi\psi\rangle=-\frac{gc}{2\pi^{3/2}}\frac{51}{16\sqrt{3}}\ .\end{aligned} \tag{3.25}$$

As the soft/hard separation depends on the numerical constant c assumed to be on the order of unity but otherwise undetermined by our analysis, a more meaningful way of representing these order parameters is to compare them to the quenched QED_2 estimate, in this chiral limit, which was also proportional to a similar constant c. Taking both constants to be the same, one can express the QCD_2 order parameter in terms of that of QED_2; and thus the predictions (3.25) can be translated into pure numbers:

$$\begin{aligned}N=2:&\ \ \langle\bar\psi\psi\rangle/\langle\bar\psi\psi\rangle_{QED}=2\ ,\\ N=3:&\ \ \langle\bar\psi\psi\rangle/\langle\bar\psi\psi\rangle_{QED}\simeq 4\ .\end{aligned} \tag{3.26}$$

Of course, there is no reason why the constants c need be chosen the same in different theories, and (3.26) is just the simplest way of representing these results. To the best of our knowledge, these quantities have not yet been machine calculated, and so our results represent predictions. Again, we emphasize that our results are only estimates, because of the need to fix c. If the result of our nonquenched QED calculations[8] is any guide, we would expect the magnitudes of these order parameters to be diminished by a factor on the order of or less than 25% (the smaller as N grows) when the quenched approximation is removed.

IV. THE LARGE-N BEHAVIOR

In the limit of large N, QCD is thought to simplify considerably[14] and this can be clearly seen in our calculations. In order to see this, one might be tempted to investigate the large-N behavior of $\Sigma(N)$ in Eq. (3.24). But indeed, not only is the series $\Sigma(N)$ difficult to handle, but this way of proceeding would prevent us from gaining interesting insights into the dynamics.

We will instead resort to the so-called "semicircle approximation" introduced by Wigner;[26] that is, perform the following replacement in Eq. (3.14):

$$\sigma_N(\theta)\,|_{N\gg 1}\to\begin{cases}\frac{1}{\pi}\sqrt{2N-\theta^2} & \text{if } \theta^2<2N\ ,\\ 0\ \ \text{if } \theta^2>2N\ .\end{cases} \tag{4.1}$$

With this limit, the full expression for the order parameter reads

$$\langle\bar\psi\psi\rangle=-\frac{m}{2\pi}\int_{-\infty}^{+\infty}\frac{d\alpha}{2\sqrt{\pi}}e^{-\alpha^2/4}\int_0^\infty\frac{ds}{s}e^{-ism^2}N(s)\int d^2x\int d[\phi]\exp\left[\frac{i}{4}\int_0^s\phi^2\right]\int\frac{d^2p}{(2\pi)^L}\exp\left[ip\cdot\int_0^s\phi\right]\times\sum_{\{+,-\}}\exp\left[\alpha\left[\frac{-iq}{4N}\right]^{1/2}\Lambda_\pm\right]\int_{-\sqrt{2N}}^{+\sqrt{2N}}d\theta\, e^{\theta\sqrt{iq}(\Lambda_\pm)}\frac{1}{\pi}\sqrt{2N-\theta^2}\ . \tag{4.2}$$

A subsidiary integration (over α) has been introduced in order to get quadratic dependences only upon the ϕ_μ fields; then the two quadratures over ϕ_μ and p_ν can be carried out exactly as in the Abelian case, where one found a result of the form

$$\langle\bar\psi\psi\rangle_{QED}=-\frac{m}{2\pi}\int d\alpha(\ )\int_0^\infty\frac{du}{u}e^{-u}[(\alpha\rho\sqrt{u})\coth(\alpha\rho\sqrt{u})-\Lambda]\ . \tag{4.3}$$

For the present case of SU(N) QCD_2 the result becomes

$$\langle\bar\psi\psi\rangle=-\frac{Nm}{\pi}\int_{-\infty}^{+\infty}\frac{d\alpha}{2\sqrt{\pi}}e^{-\alpha^2/4}\int_0^\infty\frac{du}{u}e^{-u}\int_{-1}^{+1}d\theta\sqrt{1-\theta^2}[(\rho_N Z\sqrt{u})\coth(\rho_N Z\sqrt{u})-1]\ , \tag{4.4}$$

where in both (4.3) and (4.4) we have passed to the proper time τ by means of the continuation ($s \to -i\tau$), and then to the dimensionless variable $u = \tau m^2$.

But here, in (4.4), one has

$$\rho_N = \tfrac{1}{2}\rho\sqrt{N} \tag{4.5}$$

and $Z = \theta\sqrt{8} + i\alpha/N$; that is, a qualitatively new feature appears, related to the complex nature of the argument of the function ($x\coth x - 1$). Thus Eq. (4.4) appears as generalizing (5.3) to the non-Abelian case, for large enough N so that one can use the replacement specified in (4.1). But at finite N, such as $N=2$ or 3, the forms (4.4) do not entirely express the order parameter; and a new term enters the integrand, which then exactly reads

$$\left[a_N + b_N \left[\frac{\partial}{\partial\alpha}\right]^2\right][(\cdots\alpha\cdots)\coth(\cdots\alpha\cdots) - 1] , \tag{4.6}$$

and is therefore no longer proportional to the ($x\coth x - 1$) function of the Abelian case (4.3), for the new (b_N) part does not vanish. In the chiral limit, $m=0$, one effectively ends up with the expression (Appendix C)

$$\langle\bar{\psi}\psi\rangle = -\frac{1}{4\pi}\frac{gc}{\sqrt{2\pi}}\left[1 - \frac{1}{N}\right]^{1/2}(a_N + b_N/4) , \tag{4.7}$$

which can be used to derive the previous finite-N results.

Returning to (4.4) we write, for short,

$$\begin{aligned} h(\rho_N Z\sqrt{u}) &\equiv [(\rho_N Z\sqrt{u})\coth(\rho_N Z\sqrt{u}) - 1] , \\ h(\rho_N Z\sqrt{u}) &= \mathrm{Re}h + i\,\mathrm{Im}h . \end{aligned} \tag{4.8}$$

Then, upon performing the θ and α integrations, only the real part of h survives, whereas Imh vanishes because it is odd in both variables (Appendix C); that is, one is left with

$$\mathrm{Re}h(\rho_N Z\sqrt{u}) = -1 + \frac{Z_1\cosh Z_1\sinh Z_1}{\sinh^2 Z_1 + \sin^2 Z_2} + \frac{Z_2\cos Z_2\sin Z_2}{\sinh^2 Z_1 + \sin^2 Z_2} , \tag{4.9}$$

with Z_1 and Z_2 the real variables:

$$Z_1 = \theta\rho_N\sqrt{8u} , \quad Z_2 = \frac{\alpha}{N}\rho_N\sqrt{u} .$$

At this stage, some physical interpretations can be read off from (4.9).

(i) First, one can easily check that the integral (4.4) with $h(\rho_N Z\sqrt{u})$ replaced by the RHS of (4.9) does exist.

(ii) We started from $N \gg 1$; let it now tend to infinity and one simply finds

$$\lim_{N\to\infty} \mathrm{Re}h(\rho_N Z\sqrt{u}) = h(Z_1) . \tag{4.10}$$

This is obviously an *Abelian-type result,* as can be seen by comparison with (4.3). Moreover, referring to the analysis of Ref. 8, it forces the following conclusion: Quantum *(chromo)magnetic fluctuations saturate* the CSB phenomenon in the large-N limit.

(iii) Once N has been supposed large enough for (4.1) to hold, then the sequence in which one considers first the $N=\infty$ limit or the chiral one is irrelevant; i.e., they commute; and (4.10) is obtained in the chiral limit, also at $N \gg 1$ fixed. We will return to the interpretation of this fact below, in Sec. V.

(iv) We had three parameters as input. The dynamics leave us with two basic combinations of them, which therefore are the relevant parameters in term of which to describe the SU(N)-QCD$_2$ vacuum as probed by $\langle\bar{\psi}\psi\rangle$. They are

$$P_1 = \rho_N \sim \sqrt{N g^2}/m \tag{4.11}$$

and

$$P_2 = \rho_n/N \sim g/\sqrt{N m^2} .$$

Then, typical non-Abelian effects do appear linked to this new parameter (P_2); they are the $+\sin^2(P_2\alpha\sqrt{u})$ in the second term on the RHS of (4.9) and, more clearly, the third term on the RHS of (4.9), which is

$$(\alpha P_2\sqrt{u})\cos(\alpha P_2\sqrt{u})\sin(\alpha P_2\sqrt{u}) \times[\sinh^2(\theta P_1\sqrt{8u}) + \sin^2(\alpha P_2\sqrt{u})]^{-1} ,$$

and can be traced back to what may be thought of as small, effective chromoelectric fluctuations of the F_{14}^a. *At finite N this is one* rather striking difference between Abelian and non-Abelian physics.

One can now proceed to the evaluation of the large-N behavior of $\langle\bar{\psi}\psi\rangle$. In whatever order one considers the limits $N\to\infty$, $m\to 0$, P_1 tends to infinity. Then, previous experience[8] shows that the replacement

$$h(P_1\theta\sqrt{u}) \to |P_1\theta\sqrt{u}| \tag{4.12}$$

preserves entirely the definiteness and leading behavior of type (4.3) integrals.

Gathering Eqs. (4.12), (4.10), and (4.4) one eventually finds

$$\lim_{P_1\sim\sqrt{g^2N}/m\to\infty} \langle\bar{\psi}\psi\rangle = -\frac{gc}{6\pi^3}N^{3/2} , \tag{4.13}$$

where the symbol $P_1 = \infty$ stands for both large-N and chiral limits, while expressing their commutivity once N is large enough. Because of the IR separation point c and of the replacement (4.12) [(4.1) can be shown to be quite accurate] one should pay no particular attention to the constant $6\pi^3$ in the denominator of the RHS of (4.13), other than to observe that the large-N limit changes the effective phase space entering into the computation of $\langle\bar{\psi}\psi\rangle$. (This can be observed also in the results of Ref. 9.) In our calculation there has been no restriction placed on the value of the coupling g. But if one wanted to restrict g^2 to be of the order $(N)^{-1}$ the result (4.13) would be Zhitnitsky's[9] (modulo a factor c).

We end this section by discussing one aspect of the result (4.13). In the Abelian case, one simply used the differential statement

$$\exp\left[-\frac{i}{2}\int\left[\frac{\delta}{\delta F}\right]^2\right]\exp\left[i\int Ff\right]\Bigg|_{F=0} = \exp\left[\frac{i}{2}\int f^2\right] .$$

The most natural extension of it to a non-Abelian situation where one has (N^2-1) generators for SU(N) would clearly be the replacement

$$\exp\left[-\frac{i}{2}\int\left[\frac{\delta}{\delta f_a}\right]^2\right]\exp\left[i\lambda^c\int F^c f\right]\Bigg|_{F\to 0} = \exp\left[\frac{i}{2}G_1\int f^2\right] ,$$

with $G_1=\sum_{a=1}^{N^2-1}(\lambda_a)^2$ the first Casimir operator of SU(N).

But this exactly corresponds to assuming commutivity in the series expansion (3.10), and hence can be considered as the Abelian part of the much more complicated LHS.

Then one can follow every step of the previous calculation, using $\exp(iG_1)$ instead of the RHS of (3.14) (restoring, of course, the q and $\Lambda_\pm$ dependences), and the result comes out to be

$$\langle\bar\psi\psi\rangle_{\rm QCD}\sim\left[\frac{N(N^2-1)}{2}\right]^{1/2}\langle\bar\psi\psi\rangle_{\rm QED} ,$$

which is just equivalent to (4.13). This is still another way to look at the Abelian character of the large-N limit.

V. SUMMARY

Chiral-symmetry breaking (CSB) in a two-dimensional massive SU(N)-QCD theory is a low-energy phenomenon for which the infrared techniques here developed have appeared most useful, in gaining information from both a qualitative and quantitative point of view. Quenching was used as a description consistent with our infrared analysis, and the estimations of Sec. III should be comparable with future numerical simulations with or without quenching.

Some interesting properties were seen while investigating the large-N limit of the order parameter.

Of the three input parameters, m, g, and N, two basic combinations of them appear:

$$P_1\sim g\sqrt{N}/m,\quad P_2\sim g/m\sqrt{N} .$$

P_1 appears to control the leading large-N behavior of $\langle\bar\psi\psi\rangle$ in a typically Abelian way, and the fluctuations saturating the order parameter are chromomagnetic in character. This corresponds to a great simplification of the non-Abelian dynamics, generally expected in such a limit. We observe that because of the very form of its dependence upon N and m, P_1 renders irrelevant the order in which one considers first the large-N limit or the chiral one. This constitutes an additional simplification of the large-N limit that we interpret below.

P_2 breaks this commutivity of the two limits and suggests a more complicated substructure of the SU(N)-QCD_2 vacuum as probed by $\langle\bar\psi\psi\rangle$ which can be dependent on the manner in which one approaches the point $N=0$, $m=0$. The combination P_2 appears related to typical non-Abelian effects which one can think of as induced by small chromoelectric fluctuations of the $F^a_{\mu\nu}$ fields. These always remain subleading effects.

Our large-N estimate of the order parameter in the chiral limit comes out to be $\langle\bar\psi\psi\rangle\sim -gcN^{3/2}$ irrespective of the magnitude of g, which is left a free parameter [as it perhaps should be in any (super)renormalizable quantum theory of *elementary* fields]. Thus by supplying for g a restriction of the type $g^2\sim 1/N$, one recovers the recent result of Ref. 9, which was obtained in the large-N limit with (g^2N) fixed. Our result is somewhat more general, and suggests that the CSB continues to happen at larger values of the coupling constant than those restricted by the planar-graph condition.

In a planar gauge, one has a result of the form

$$\langle\bar\psi\psi\rangle=-N(g^2N)^{1/2}[{\rm const}+O(1/N)+\cdots] ,$$

where the leading-N contribution (the const) is carried by planar graphs (ladders). In our case, the functional operations involve all kinds of graphs. As our calculation can be viewed as carried out in a planar gauge also [cf. Eqs. (3.5) and (3.7)], we have, in symbolical form,

$$\langle\bar\psi\psi\rangle\sim\Sigma(\text{planar graphs})+\Sigma(\text{nonplanar graphs})$$
$$\sim\Sigma(\text{most IR parts}) ,$$

and the fact that both results coincide shows that the leading infrared contributions are carried by the planar graphs. In our calculation, this ambivalence of planar graphs with respect to both IR and leading $(1/N)$ contributions gets nicely translated into the commutivity of the two limits, $m\to 0$ and $N\to\infty$. Here again, the same property is well known in four-dimensional QCD (Ref. 23). Though not discussed in the preceding sections, we mention here the form of the subleading corrections to the chiral leading limit; this, for any N, is based upon the subleading terms of the series expansion (3.24), and one finds the following sequence of orders:

$$\langle\bar\psi\psi\rangle=O(g)\Sigma(N)+O(Nm)+O(m)$$
$$+O(m^2\sqrt{N}/g)+\cdots .$$

We end this section by noting that the $m=0$ chiral limit is really not smooth, in either the Abelian or non-Abelian cases. This is why, by first restricting the gluon's maximum momenta to be roughly on the order of cm and then, second, by letting m tend to zero, it is possible to find a nonzero result in our method of calculation. The theoretical reason is most easily transparent on the bosonized versions of the theories, where one can see that a nonzero mass m defines a sine-Gordon interaction, which completely disappears when m is exactly zero. Our result is, in addition, finite in the quenched approximation, and (effectively) gauge invariant (modulo an unimportant rescaling of the product gc).

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APPENDIX A

We here examine how one can pass from (3.5) and (3.7) to (3.8) where all space-time dependence has disappeared.

We will omit the $M_{\mu\nu}$ matrix for a while, and restore it at the end. We thus simply focus on the expression (3.7) (RHS):

$$\exp\left[-\frac{i}{2}\int \frac{\delta}{\delta A_4^a}(\partial_1)^{-2}\frac{\delta}{\delta A_4^a}\right]\operatorname{tr}(e^{-2i\partial_1 A_4^s(x)})\bigg|_{A=0} \tag{A1}$$

with $A_4^{s,b_i}(x)$ given by

$$A_4^{s,b_i}(x)=\int d^2z\, f(x-z)A_4^{b_i}(x)$$
$$=\int \frac{d^2k}{(2\pi)^2}e^{ik\cdot x-i\alpha k^2}\tilde{A}_4^{b_i}(k),\quad \alpha\to -i\mu_c^{-2}\ .$$

In the axial gauge $[A_1^a(x)=0]$ in which we will do the calculation, there is a term in the series expansion of (A1) which will typically read

$$\int d^2x\int d^2y\int \frac{d^2k}{(2\pi)^2}\left[\frac{\delta}{\delta A_4^a(x)}\frac{e^{ik\cdot(x-y)}}{k_1^2}\frac{\delta}{\delta A_4^a(y)}\right]$$
$$\times\left[\lambda^b\partial_1\int\frac{d^2k'}{(2\pi)^2}e^{-i\alpha k'^2+ik'\cdot x}\tilde{A}_4^b(k')\right]^2 . \tag{A2}$$

Now with

$$\frac{\delta}{\delta A_4^a(b)}\tilde{A}_4^b(k)=e^{ik\cdot y}\delta_{ab}$$

and

$$\int d^2y\, e^{iy\cdot(k-k')}=(2\pi)^2\delta^{(2)}(k-k')\ ,$$

and upon performing the y_j, x_j, kb_i, kb_i integrations, one sees that this factor just amounts to

$$g_{b_ib_j'}\int\frac{d^2k}{(2\pi)^2}e^{-i\alpha k^2-i\alpha(-k)^2}(ik_1)(-ik_1)/k_1^2\ ,$$

and will be factored out, as a constant, n times as j varies from 1 to n. Thus we get an overall

$$\chi^n \quad\text{with } \chi=\int\frac{d^2k}{(2\pi)^2}e^{-2i\alpha k^2}\ ,$$

and any space-time dependence is removed.

As there exists no mixing between the $\{\mu,\nu\}$ and $\{i,j\}$ degrees of freedom, the full trace Tr is given the meaning of

$$\mathrm{Tr}=\operatorname*{tr}_{\{\mu,\nu\}}\otimes\operatorname*{tr}_{\{i,j\}}\ .$$

Thus, denoting by $\mathbb{A}$ the $(N_c\times N_c)$ matrix $[\partial_1 A_0^{s,c}(x)\lambda^c]$, one has

$$\operatorname*{tr}_{\{\mu,\nu\}} e^{-ig(\Omega_{14}\mathbb{1}+is\gamma_5)\mathbb{A}}=e^{-ig\Omega_{14}\mathbb{A}}\operatorname*{tr}_{\{\mu,\nu\}}e^{gs\gamma_5\mathbb{A}}$$
$$=2e^{-ig\Omega_{14}\mathbb{A}}\left[\frac{e^{-igs\mathbb{A}}+e^{igs\mathbb{A}}}{2}\right]$$
$$=\sum_{\{+,-\}}e^{-ig\mathbb{A}\Lambda_\pm}$$

with

$$\Lambda_\pm=\int_0^s ds'\phi_1(s')\int_0^{s'}ds''\phi_4(s'')\pm s\ .$$

This completes the proof for the passage from Eqs. (3.5) and (3.7) to Eq. (3.8).

Relative to Eq. (3.10)

Equation (3.10) is basically obtained by first replacing any $\lambda^{b_i}\lambda^{b_{i+1}}$ by $\frac{1}{2}\{\lambda^{b_i},\lambda^{b_{i+1}}\}$ (one sums upon the b_i's), and then by observing that the functional differentiations will pair any two elements of the $2n$-plet $(b_1,b_2,\ldots,b_{2n})$ into a Kronecker delta in all possible ways.

Thus the reader will have noted that, modulo the correspondence

$$\delta^{P(b_{2i-1})P(b_{2i})}\tfrac{1}{2}\{\lambda_{b_{2i-1}},\lambda_{b_{2i}}\}\to\tfrac{1}{2}G[P(x_{2i-1})-P(x_{2i})]\ ,$$

$b_j\to x_j$, a space-time point, Eq. (3.10) expresses nothing but the explicit forms of Wick's theorem for a scalar field.

From Eq. (3.10) and the use of the Mehta-Dyson method, one can deduce an arithmetic identity which is presumably very difficult to establish directly.

Understanding a sum over repeated indices

$$\delta^{\alpha\alpha}=\sum_{\alpha=1}^{L}1=L\ ,$$

one can prove the identity

$$\sum_{P\in S_{2N}}\delta^{P(b_1)P(b_2)}\delta^{P(b_3)P(b_4)}\cdots\delta^{P(b_{2n-1})P(b_{2n})}\delta^{b_1b_2}\delta^{b_3b_4}\cdots\delta^{b_{2n-1}b_{2n}}=2^n n!(2n-2+\delta^{b_ib_i})!!\ , \tag{A3}$$

which may find fruitful applications in other related problems.

APPENDIX B

We here mention the steps leading to Eq. (3.14). The expression we have to cope with (written in symbolical form) is

$$\mathcal{N}\int d[H]e^{i\,\mathrm{tr}(H^2)-\sqrt{iq}(\Lambda_\pm)H}\delta(\mathrm{tr}H)\equiv J\ , \tag{B1}$$

where

$$\mathcal{N}^{-1}=\int d[H]e^{i\,\mathrm{tr}(H^2)}\delta(\mathrm{tr}H)$$

is a fairly complicated object. But the point is that we have only to calculate its trace over color degrees of freedom $I=\mathrm{Tr}J$:

$$I=\mathcal{N}\int d[H]\delta(\mathrm{tr}H)e^{i\,\mathrm{tr}(H^2)}\mathrm{tr}(e^{-\sqrt{iq}H\Lambda_\pm})\ , \tag{B2}$$

so that I is entirely expressible in terms of the eigenvalues of H, the angular variables disappearing in the normalization:

$$I=\mathcal{N}\int_{-\infty}^{+\infty}d\lambda\prod_i^N d\xi_k\prod_{i<j}(\xi_i-\xi_j)^2\exp\left[i\sum_k\xi_k^2\right]\times\exp\left[i\lambda\sum_\sigma\xi_\sigma\right]\sum_\gamma e^{-\sqrt{iq}\,\xi_\gamma\Lambda_\pm} \tag{B3}$$

where $\mathcal{N}^{-1}$ is the same expression as its multiplying factor in (B1), except for the final factor $[\sum_\gamma\exp(-\sqrt{iq}\,\xi_\gamma\Lambda_\pm)]$, and where a representation for the $\delta(\mathrm{Tr}H)$ has been introduced.

Then, using the standard definitions

$$P_N(\xi_1,\ldots,\xi_N)\equiv C_N\exp\left[-\sum_\alpha\xi_\alpha^2\right]\prod_{1\le i<j\le N}(\xi_i-\xi_j)^2\ ,$$

where

$$C_N^{-1}=2^{-N(N-1)/2}N!\pi^{N/2}\left[\sum_{k=0}^{N-1}k!\right]\ ,$$

and the property that

$$\sigma_N(\xi_1)\equiv N\int d\xi_2\cdots d\xi_N P_N(\xi_1,\ldots,\xi_N)\ ,$$

$$\int d\xi_1\sigma_N(\xi_1)=1$$

[which is conserved by Wigner's semicircle approximation (4.1)], one straightforwardly can derive Eq. (3.14), using a convenient change of variables

$$\xi_\alpha=\sqrt{i}\,\theta_\alpha-\lambda/2\ . \tag{B4}$$

About Eq. (3.19)

By using Hermite and Laguerre relations (3.12) and (3.18) one can write

$$\begin{aligned}e^{\mathcal{D}_A}\mathrm{tr}U(s)\,|_0&=\sum_{\{+,-\}}NL_{N-1}^1\left[\frac{2N}{1-N}\frac{\partial}{\partial\zeta}\right]\exp[\tfrac14\Lambda_\pm^2(iq)(1-1/N)\zeta]\Big|_{\zeta=1}\\&=(2\sqrt{\pi})^{-1}L_{N-1}^1\left[\frac{2N}{1-N}\frac{\partial}{\partial\zeta}\right]\sum_{\{+,-\}}\int_{-\infty}^{+\infty}d\alpha\,e^{-\alpha^2/4}\exp\left\{\alpha\Lambda_\pm\left[\left[1-\frac1N\right]\frac{iq}{4}\zeta\right]^{1/2}\right\}\Bigg|_{\zeta=1}\ .\end{aligned} \tag{B5}$$

Using this, one can express the order parameter as

$$\begin{aligned}\langle\bar\psi\psi\rangle=&-\frac{im}{\sqrt{2\pi}}L_{N-1}^1\left[\frac{2N}{1-N}\frac{\partial}{\partial\zeta}\right]\int_{-\infty}^{+\infty}d\alpha\,e^{-\alpha^2/4}\int_0^\infty ds\,e^{-ism^2}\sum_{\{+,-\}}N(s)\int d(\phi)\exp\left[\frac{i}{4}\int_0^s ds'\phi^2(s')\right]\\&\times\delta\left[\int_0^s\phi(s')ds'\right]\exp\left\{\alpha\Lambda_\pm\left[\left[1-\frac1N\right]\frac{iq}{4}\zeta\right]^{1/2}\right\}\Bigg|_{\zeta=1}-(g\to0)\ .\end{aligned} \tag{B6}$$

In this form, the integrations can be carried out as in the Abelian case, yielding Eq. (3.19).

APPENDIX C

Equation (4.1). This form of the finite-N result can be obtained in the following manner.

One starts from Eq. (3.14):

$$I_N=\exp\left[-\frac{iq}{4N}\Lambda_\pm^2\right]\int_{-\infty}^{+\infty}d\theta\,e^{\theta\sqrt{iq}\Lambda_\pm}\sigma_N(\theta)\ ,$$

and for $N_c=2$ and 3 replaced $\sigma_N(\theta)$ by its relevant expression; the θ summation is easily carried out and yields, respectively,

$$I_2=\exp\left[\frac{iq}{8}\Lambda_\pm^2\right]\left[2+i\frac{q}{2}\Lambda_\pm^2\right]\quad\text{for SU(2)}\ , \tag{C1}$$

$$I_3=\exp\left[\frac{iq}{3}\Lambda_\pm^2\right](\tfrac{23}{8}+\tfrac32 iq\Lambda_\pm^2)\quad\text{for SU(3)}\ . \tag{C2}$$

Now by introducing a representation which leaves only a linear $\Lambda_\pm$ dependence, that is,

$$\exp\left[iq\frac{N-1}{4N}\Lambda_\pm^2\right] = \int_{-\infty}^{+\infty}\frac{d\alpha}{2\sqrt{\pi}}\exp\left[-\frac{\alpha^2}{4}+\alpha\Lambda_\pm\left[iq\frac{N-1}{4N}\right]^{1/2}\right],$$

one sees that $\langle\bar\psi\psi\rangle$ can be written as

$$-m\int_{-\infty}^{+\infty}\frac{d\alpha}{2\sqrt{\pi}}e^{-\alpha^2/4}\left[a_N+b_N\left[\frac{\partial}{\partial\alpha}\right]^2\right]\times\int_0^\infty\frac{d\tau}{\tau}e^{-\tau m^2}J(\tau,\alpha,N)\ , \quad \text{(C3)}$$

which is Eq. (4.6), with

$$J(\alpha,\tau,N)=\frac{1}{2\pi}(Q\coth Q-1)\ ,$$

$$Q=\frac{\alpha g c}{2}\left[\frac{\tau(N-1)}{8\pi N}\right]^{1/2},$$

and where the a_N and b_N coefficients are to be identified from (C1) and (C2) for SU(2) and SU(3), respectively. Then the $b_N(\partial/\partial\alpha)^2$ term can be integrated by parts, leading to the following intermediate step:

$$\langle\bar\psi\psi\rangle=-\frac{m}{\pi}\left[a_N-\frac{b_N}{4}p\frac{\partial}{\partial p}\right]j(p)\ , \quad \text{(C4)}$$

where

$$j(p)=\int_0^\infty\frac{du}{u}e^{-up}(u\coth u-1) \quad \text{(C5)}$$

and

$$p^{-1}=Q/\alpha\sqrt{\tau}=\frac{gc}{2m}\left[\frac{N-1}{8\pi N}\right]^{1/2}.$$

In its turn (C4) leads to Eq. (4.7).

It is worth noting here that the simple SU(2) formula

$$e^{i\sigma\cdot F}=\cos|F|+i\frac{\sigma\cdot F}{|F|}\sin|F| \quad \text{(C6)}$$

allows one to check the correctness of Eq. (3.14), because

$$\mathrm{tr}\,e^{(\delta/\delta F)^2}e^{i\sigma\cdot F}|_{F=0}=\mathrm{tr}\left[e^{(\delta/\delta F)^2}\left[\cos|F|+i\frac{\sigma\cdot F}{|F|}\sin|F|\right]\right]\Bigg|_{F=0}$$
$$=e^{i/8}(2+i/2)\rightarrow\exp\left[\frac{iq}{8}\Lambda_\pm^2\right]\left[2+\frac{i}{2}(q\Lambda_\pm^2)\right]$$

of Eq. (C1).

We give here the expression for $\mathrm{Im}h(\rho_N Z\sqrt{u})$. With $Z_1=(\theta\rho_N\sqrt{8u})$, $Z_2=[(\alpha/N)\rho_N\sqrt{u}]$, one has

$$\mathrm{Im}h(\rho_n Z\sqrt{u})=[-Z_1\sin(Z_2)\cos(Z_2)+Z_2\cosh(Z_1)\sinh(Z_1)][\sinh^2(Z_1)+\sin^2(Z_2)]^{-1}\ ,$$

from which we see the oddness in Z_1 (or θ) and in Z_2 (or α).

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